Sharif Quantum Information Group

An introduction to Quantum Error Correction-II

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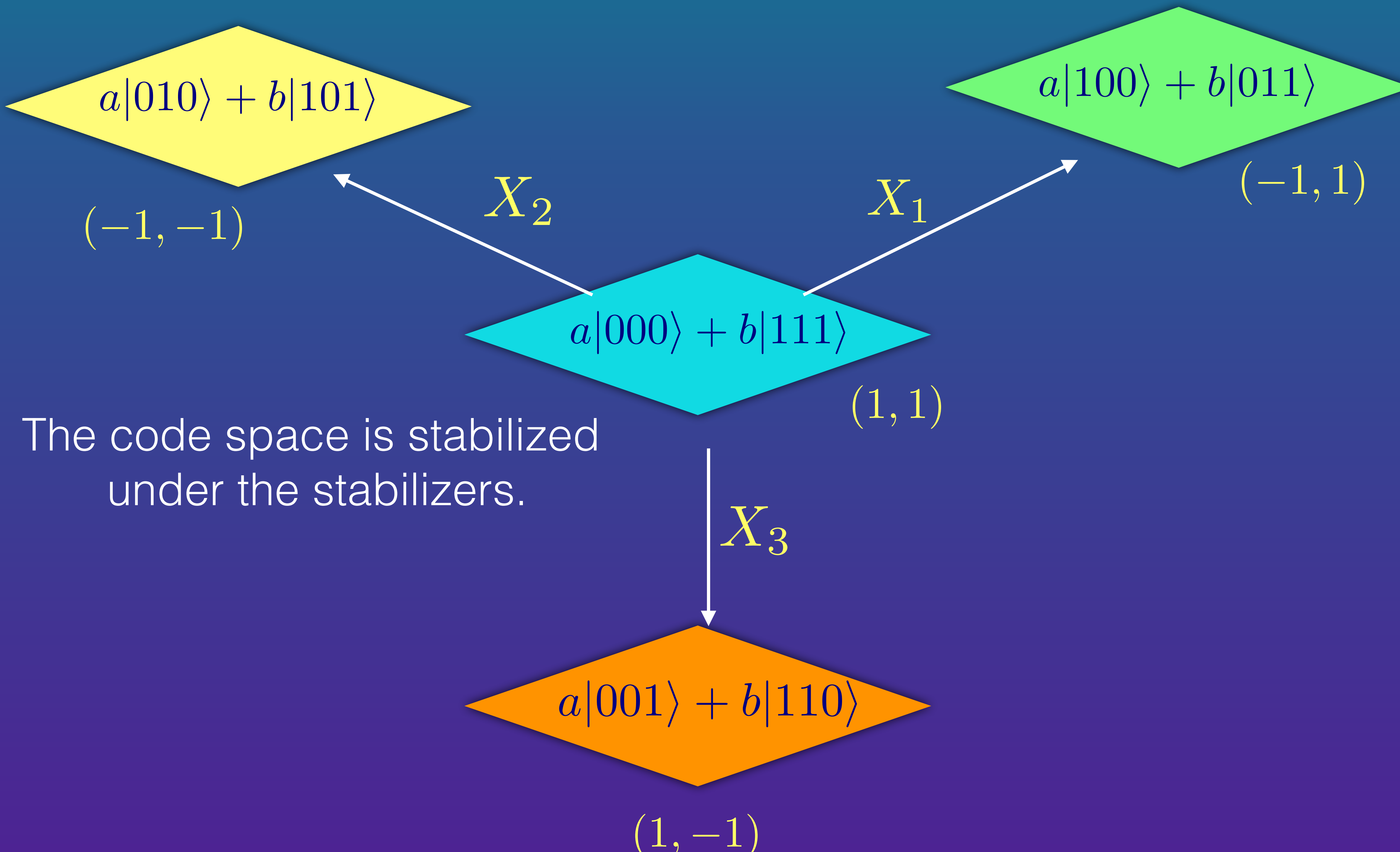
The Stabilizer Formalism

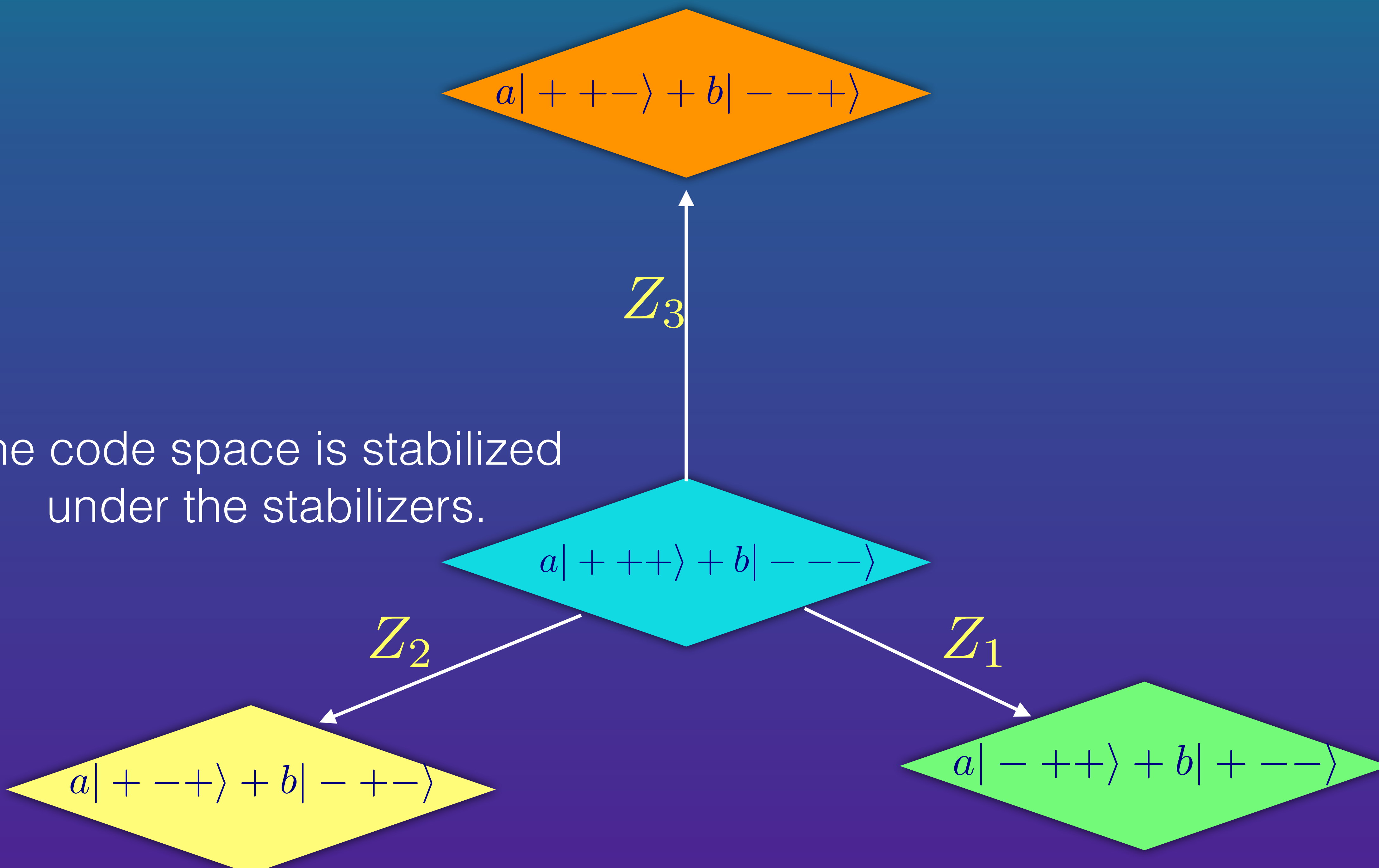
An example: Instead of writing the code and then finding the syndromes,
we determine the syndromes first and define the code space
as the subspace which is stable under these syndromes.

Stabilizers $Z_1 Z_2$ $Z_2 Z_3$



The code space is stabilized under the stabilizers.

Stabilizers X_1X_2 X_2X_3



The code space is stabilized under the stabilizers.

The Pauli Group

$$\{I \quad X \quad Y \quad Z\}$$

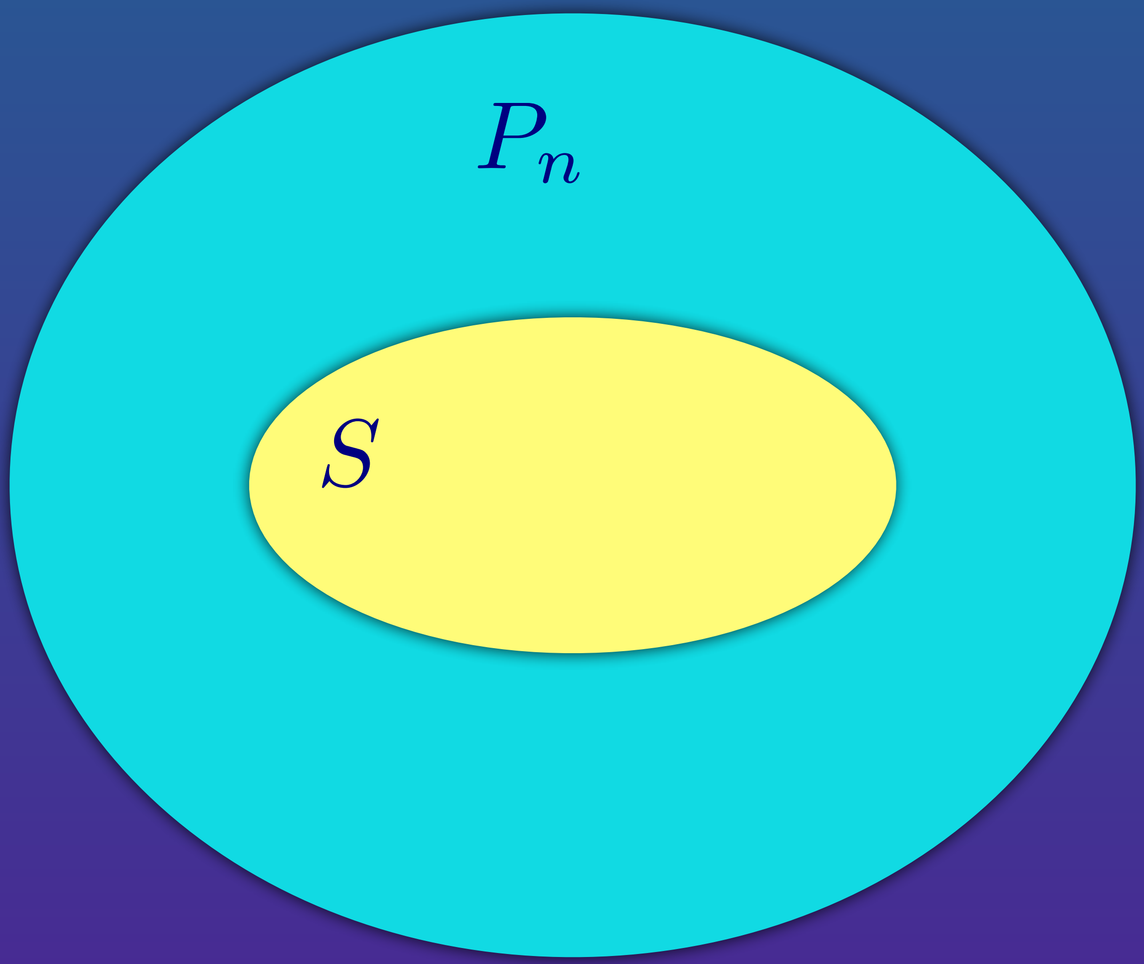
$$\{I \quad X \quad Y \quad Z\ , -I \quad iX \quad iY \quad iZ\}$$

$$P_1 = \{\pm I \quad \pm X \quad \pm Y \quad \pm Z\ , \pm iI \quad \pm iX \quad \pm iY \quad \pm iZ\}$$

$$dim(P_n) = 4 \times 4^n = 4^{n+1}$$

$$S = \{ s_1 \quad s_2 \quad s_3 \quad \cdots \quad s_{\mathcal{N}} \}$$

A stabilizer subgroup is any subgroup of the
Pauli group which is Abelian and does not contain -I.



$$s_i s_j = s_j s_i$$

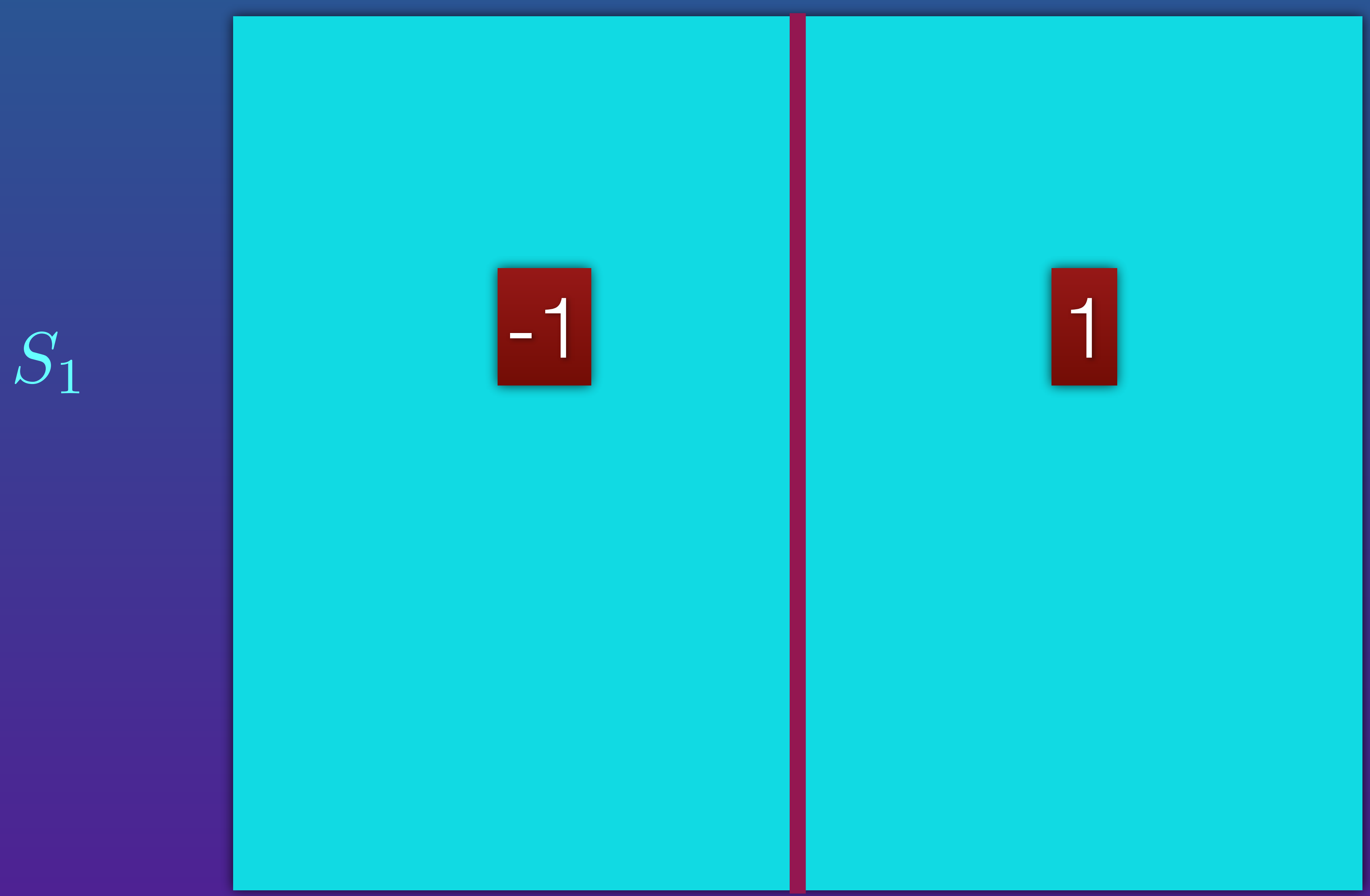
S is generated by these elements.

$$\langle s_1 \quad s_2 \quad \cdots \quad s_k \rangle$$

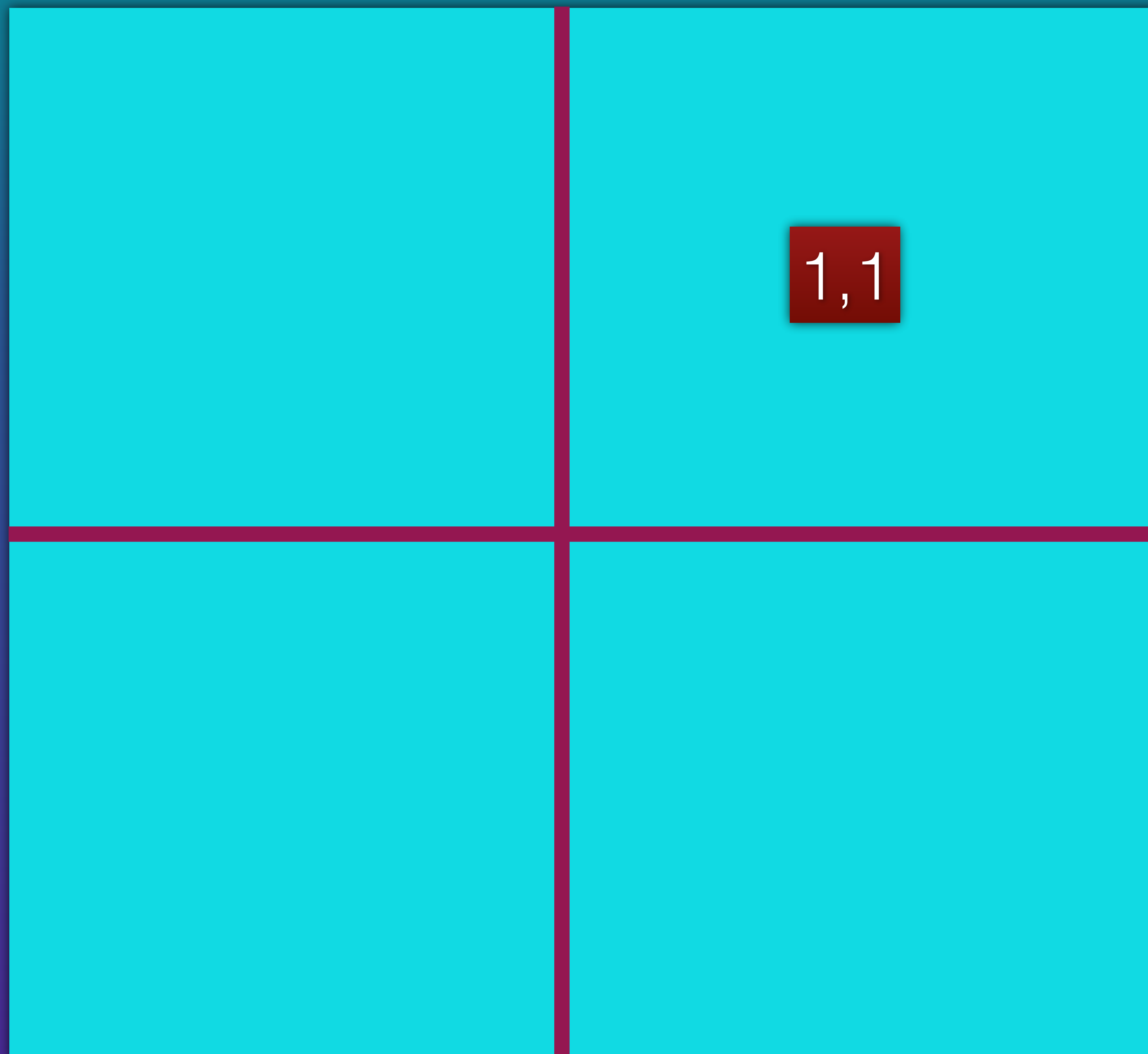
$$C_S = \{ |\psi\rangle, \qquad s_k |\psi\rangle = |\psi\rangle \}$$

$dim(C_S)$

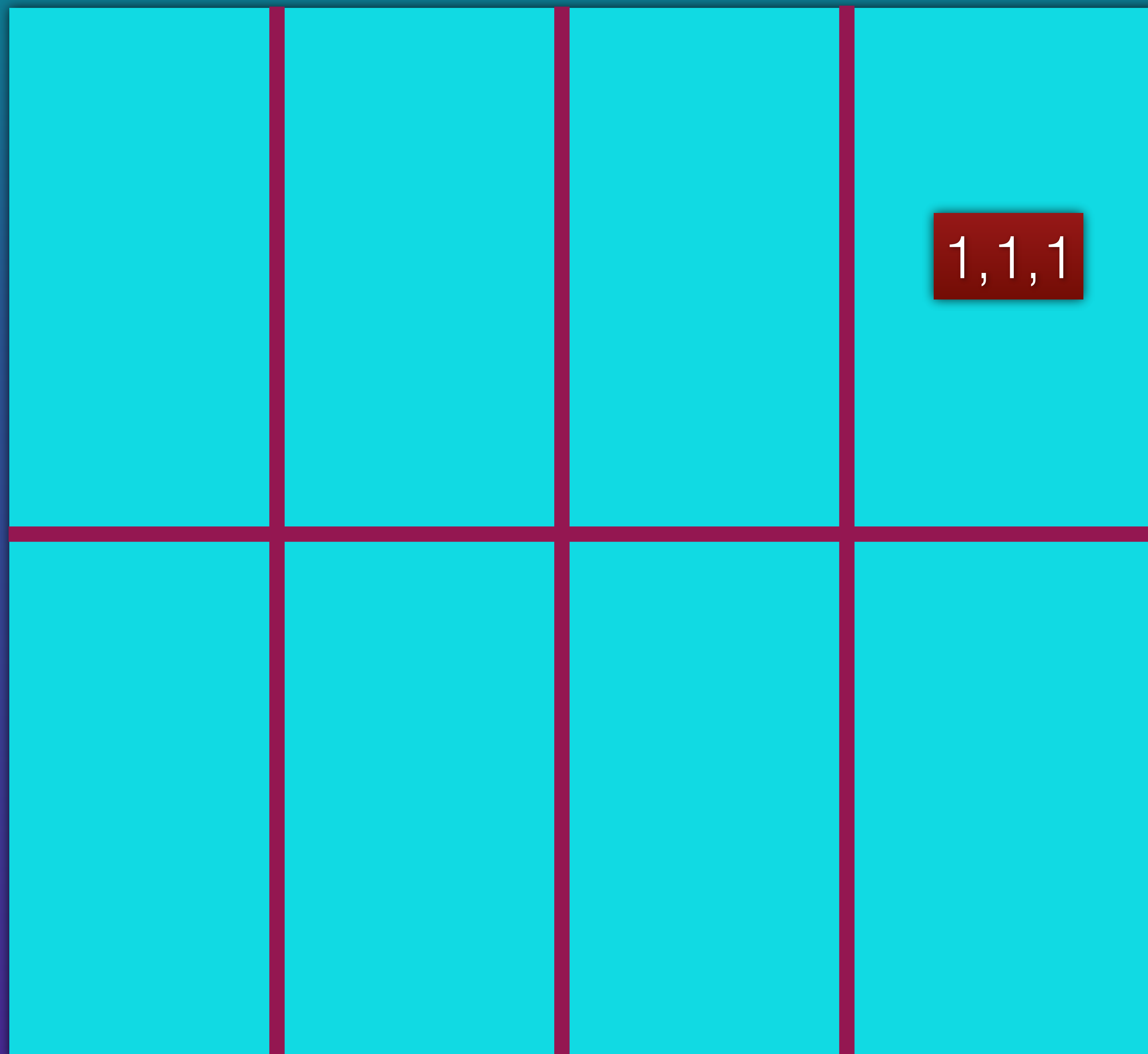
Since S should not contain -I, the square of any of its elements is I. Therefore any element splits the space into two equal halves.



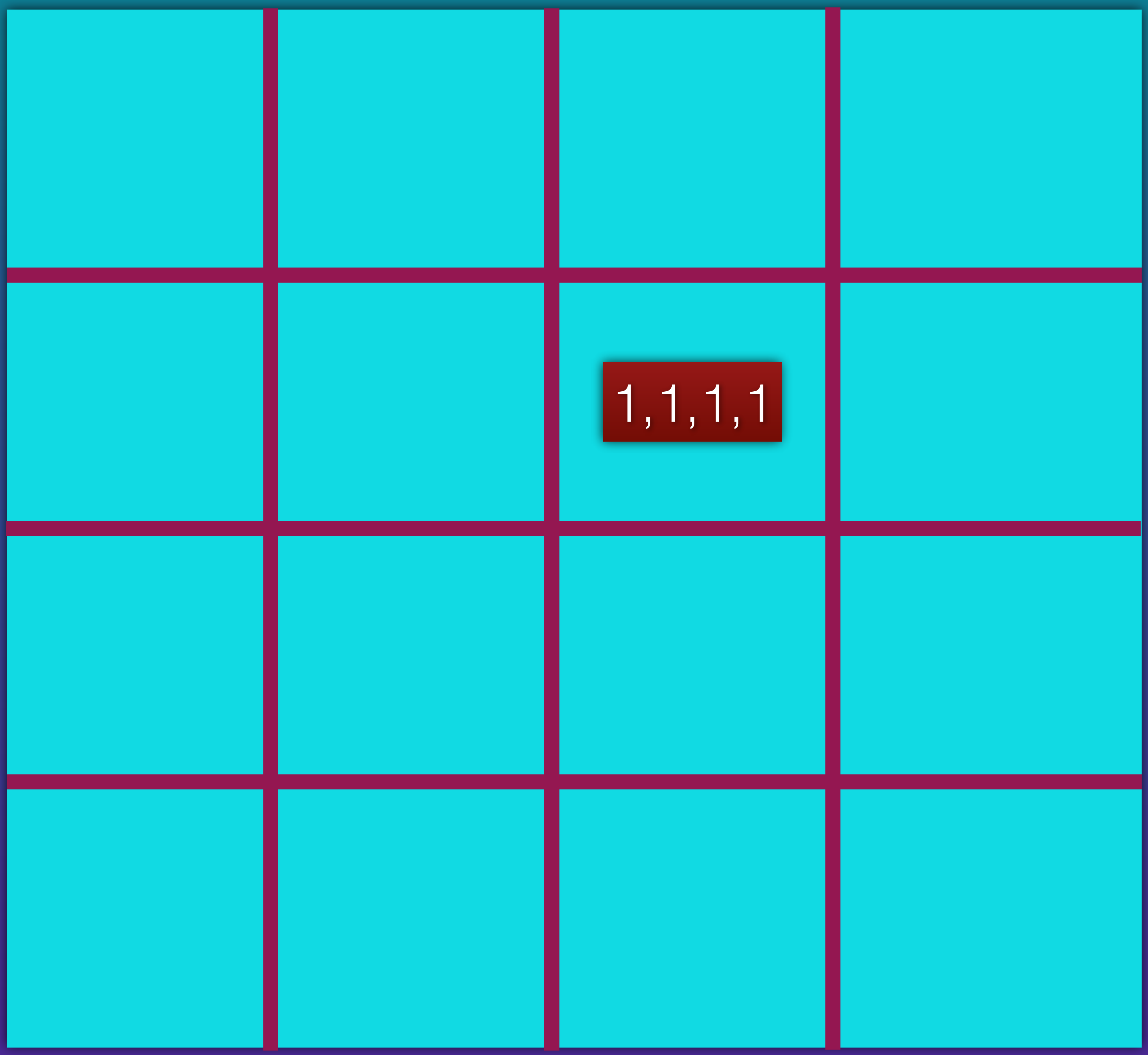
S_2



S_3



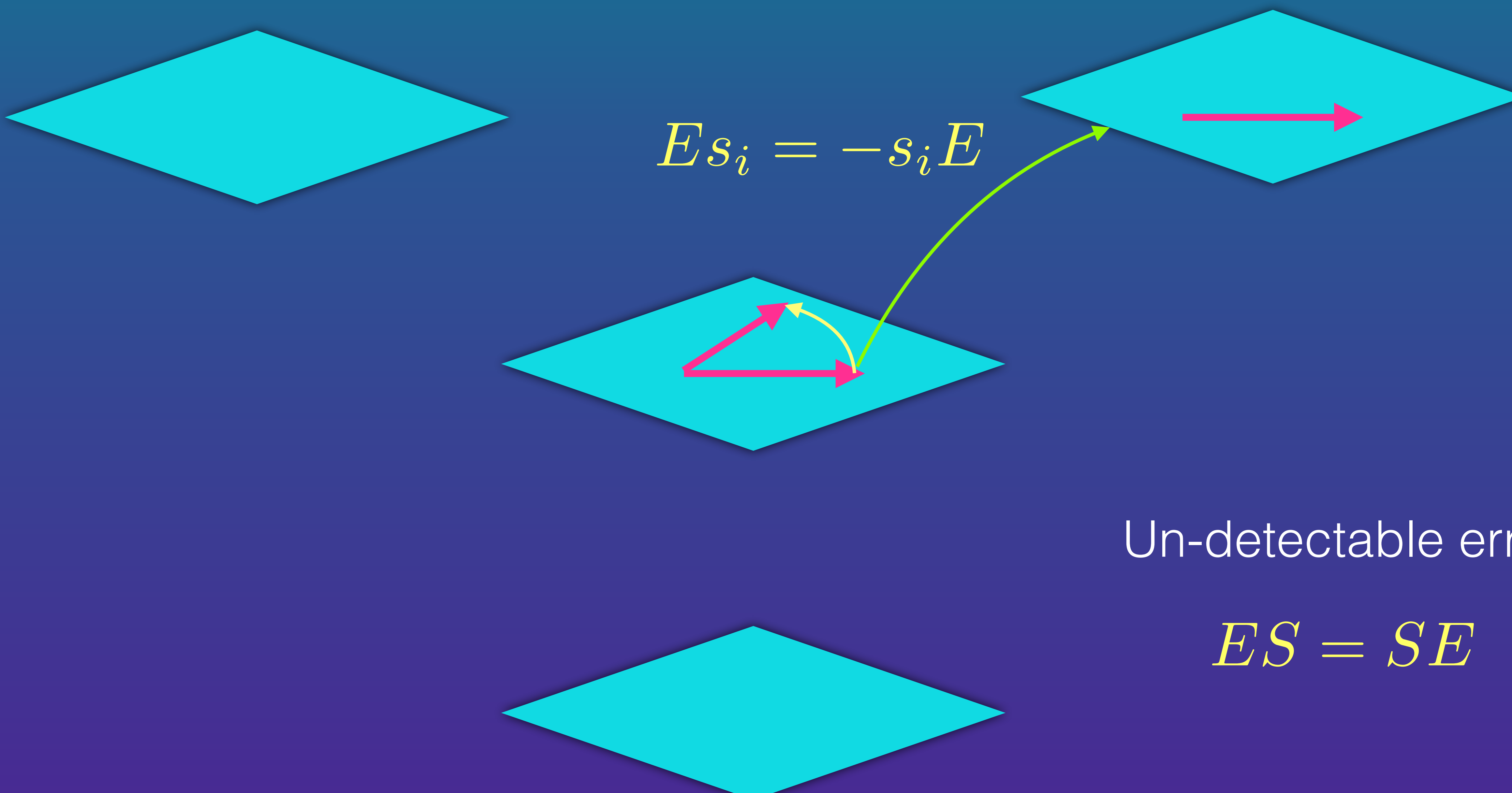
S_4



k is the number of generators of S.

$$dim(C_S) = \frac{2^n}{2^k} = 2^{n-k}$$

Detectable error



$$Es_i = -s_i E$$

Un-detectable error

$$ES = SE$$

Undetectable errors move the state within the code space.

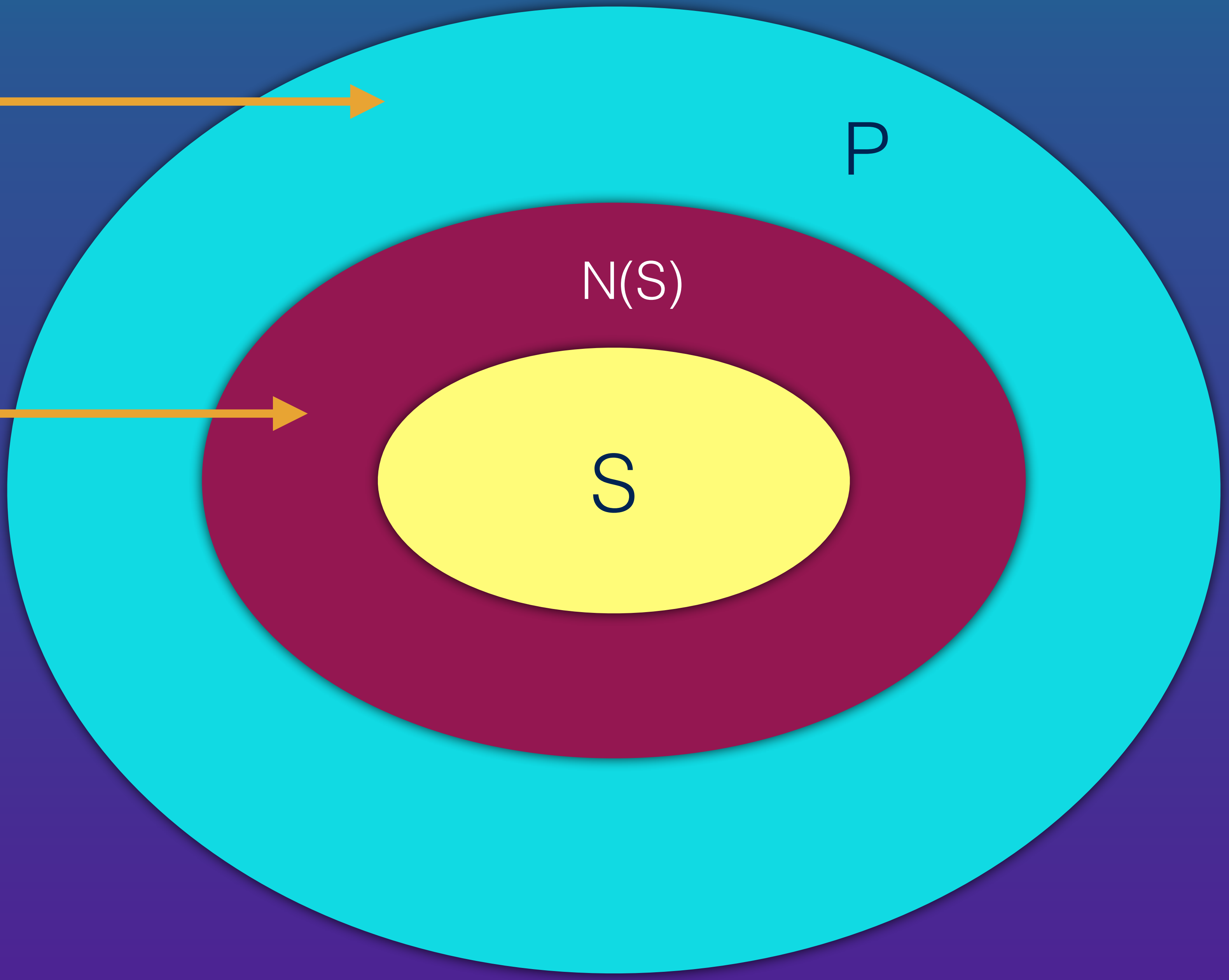
Therefore they act as logical gates.

Normalizer of S

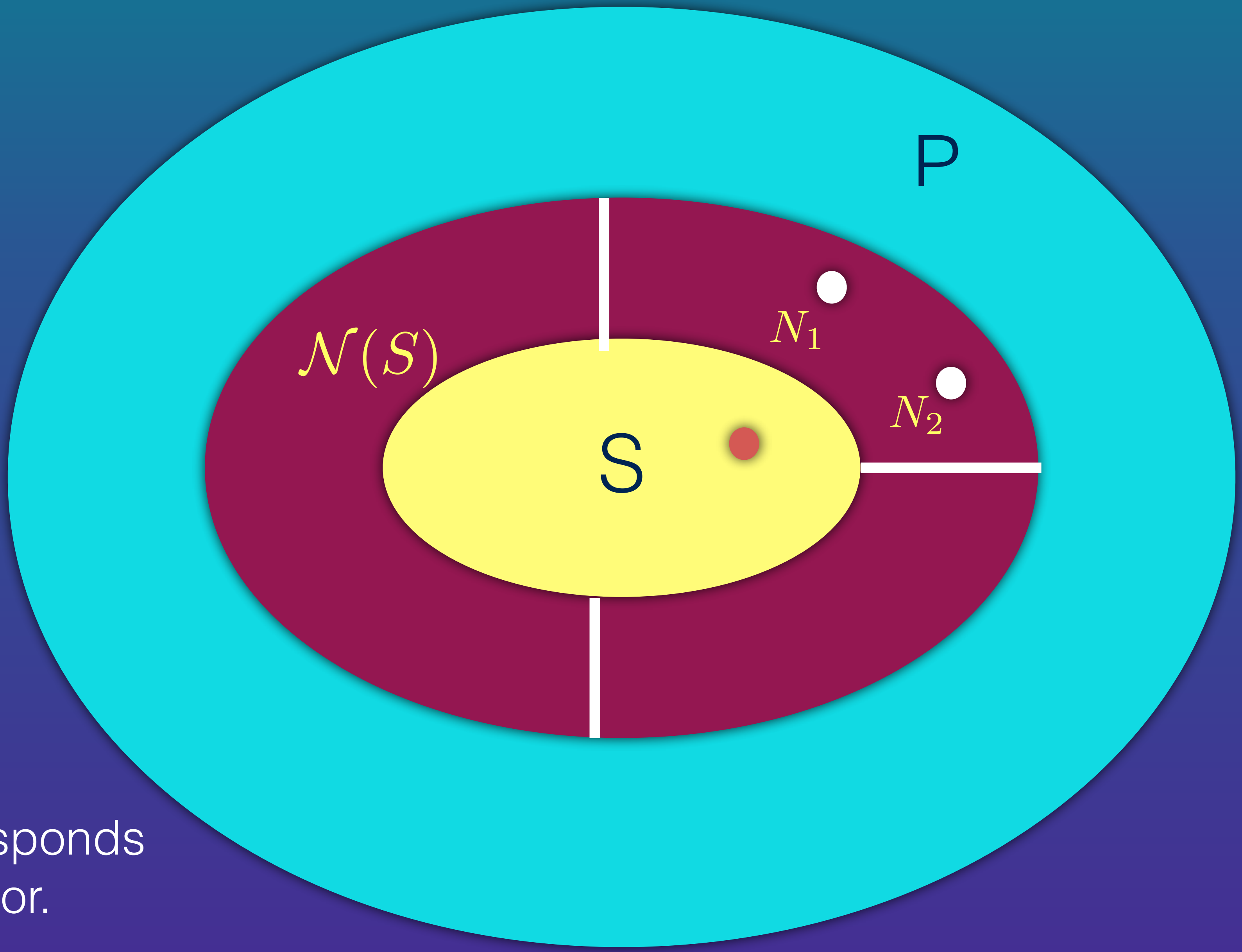
$$N(S) = \{e \in P| \;\; eS = Se\}$$

Detectable errors

Logical gates



$$N_1 \sim N_2 \qquad N_1 = N_2\ s$$

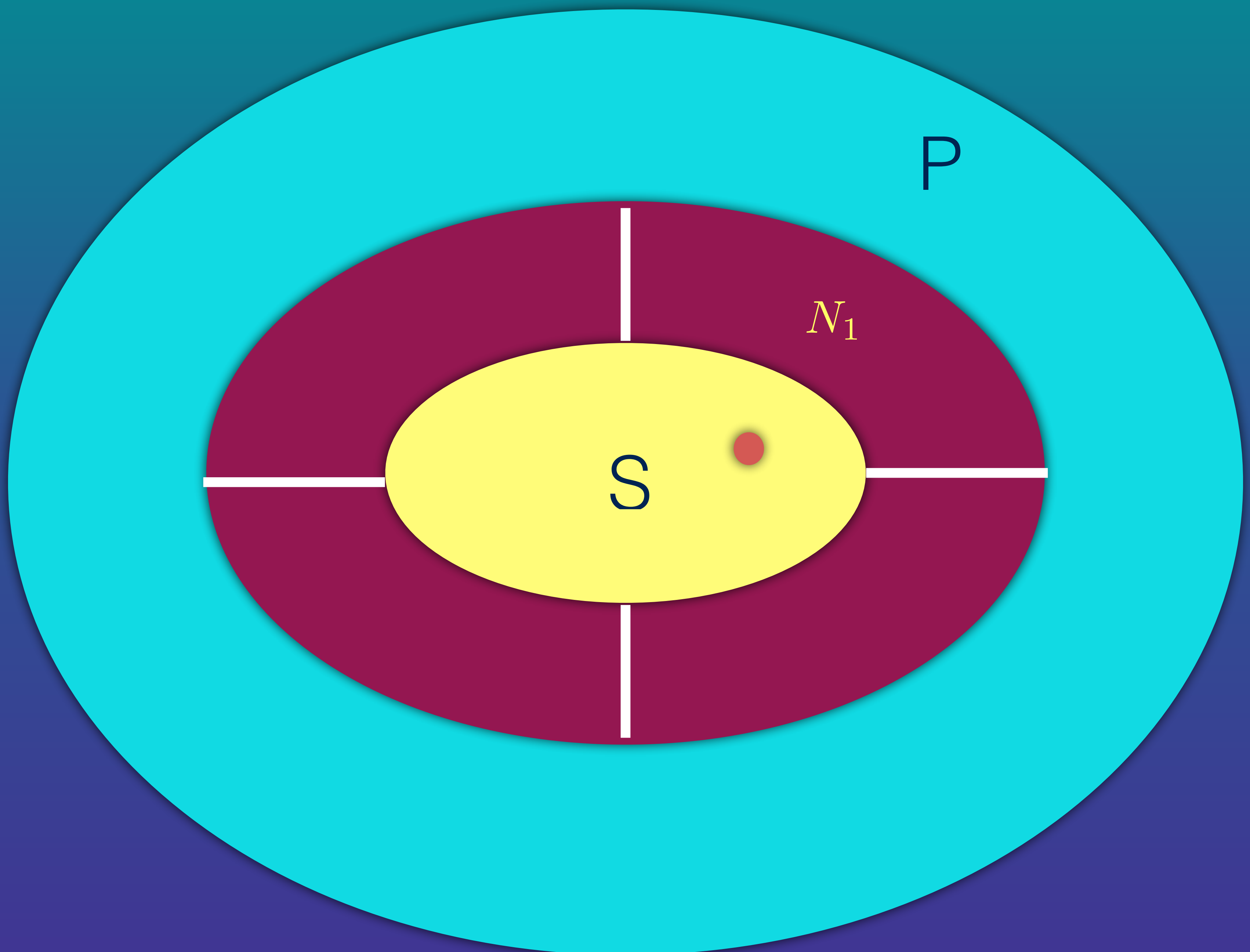


Each coset of S in N(S) corresponds to a single logical operator.

$$N_1|\psi\rangle = N_2 S|\psi\rangle$$

$$= N_2|\psi\rangle$$

$$\overline{Z} = \{Z_1\ Z_2\ Z_3\ Z_1 Z_2 Z_3\}$$



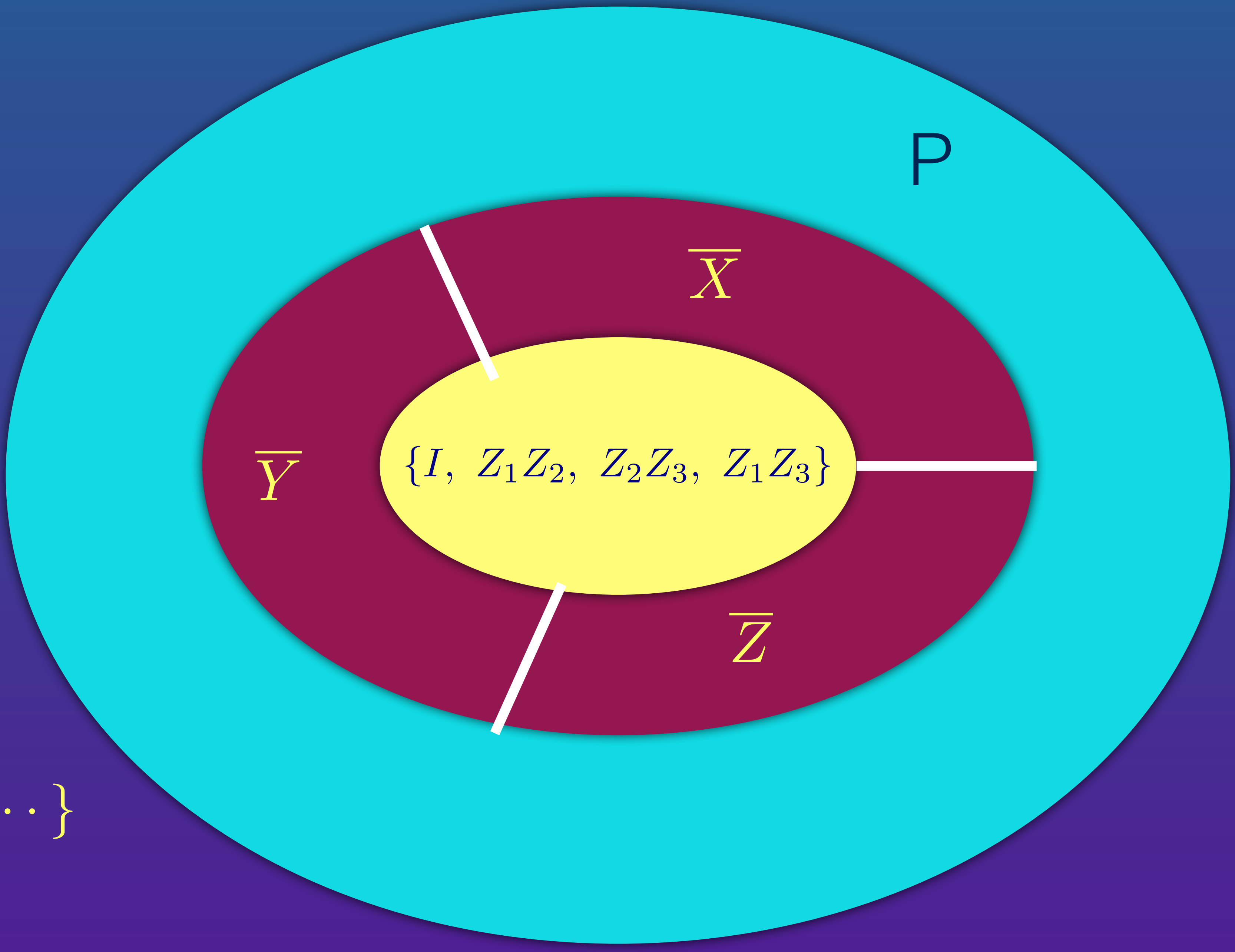
$$[N_1] = \{N_1 \quad N_1 s_1 \quad N_1 s_2 \quad \cdots\} = N_1 S$$

In the simple repetition code, these are the logical operators.

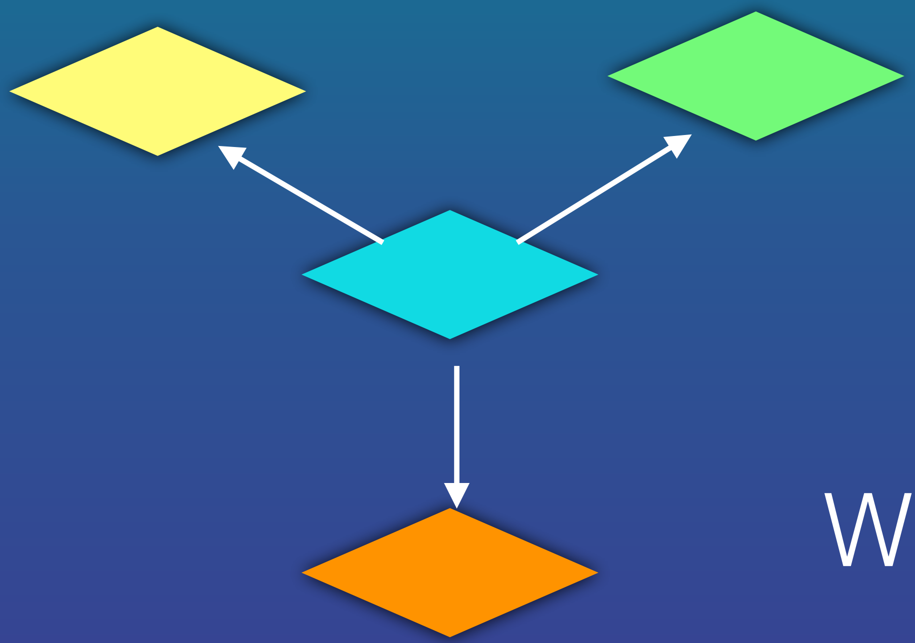
$$\overline{X} = \{X_1X_2X_3, \cdots\}$$

$$\overline{Z} = \{Z_1, Z_2, Z_3, \cdots\}$$

$$\overline{Y} = \{iY_1X_2X_3,\ iX_1Y_2X_3,\ \cdots\}$$



The shortest code for 1 qubit?



$$I, X_1, Y_1, Z_1 \cdots\cdots X_n, Y_n, Z_n$$

$$1 + 3n$$

We need 1+3n orthogonal subspaces.

$$2 \times (3n + 1) \leq 2^n$$

Dimension of the space of n-qubits.

$$n_{min} = 5$$

5 Qubit Code

A possible set of stabilizers

$$s_1 = Z\ Z\ X\ X\ X$$

$$s_2 = X\ Z\ Z\ X\ X$$

$$s_3 = X\ X\ Z\ Z\ X$$

$$s_4 = X\ X\ X\ Z\ Z$$

5 Qubit Code

But not all syndromes are not different!

$$s_1 = Z\ Z\ X\ X\ X$$

$$s_2 = X\ Z\ Z\ X\ X$$

$$s_3 = X\ X\ Z\ Z\ X$$

$$s_4 = X\ X\ X\ Z\ Z$$

The 5 Qubit Code

Now all the syndromes are different

$$s_1 = Z\ I\ Z\ X\ X$$

$$s_2 = I\ Z\ X\ X\ Z$$

$$s_3 = Z\ X\ X\ Z\ I$$

$$s_4 = X\ X\ Z\ I\ Z$$

CSS Codes

Why we don't use the ideas of classical linear codes

to invent quantum codes?

How?

How?

Classical coding.

$$w = \sum_i \alpha_i g_i$$

$$\alpha_i = 0\ ,\ 1$$

$$g_i = (0\ 1\ 0\ 1\ 1\ 0\ 1)$$

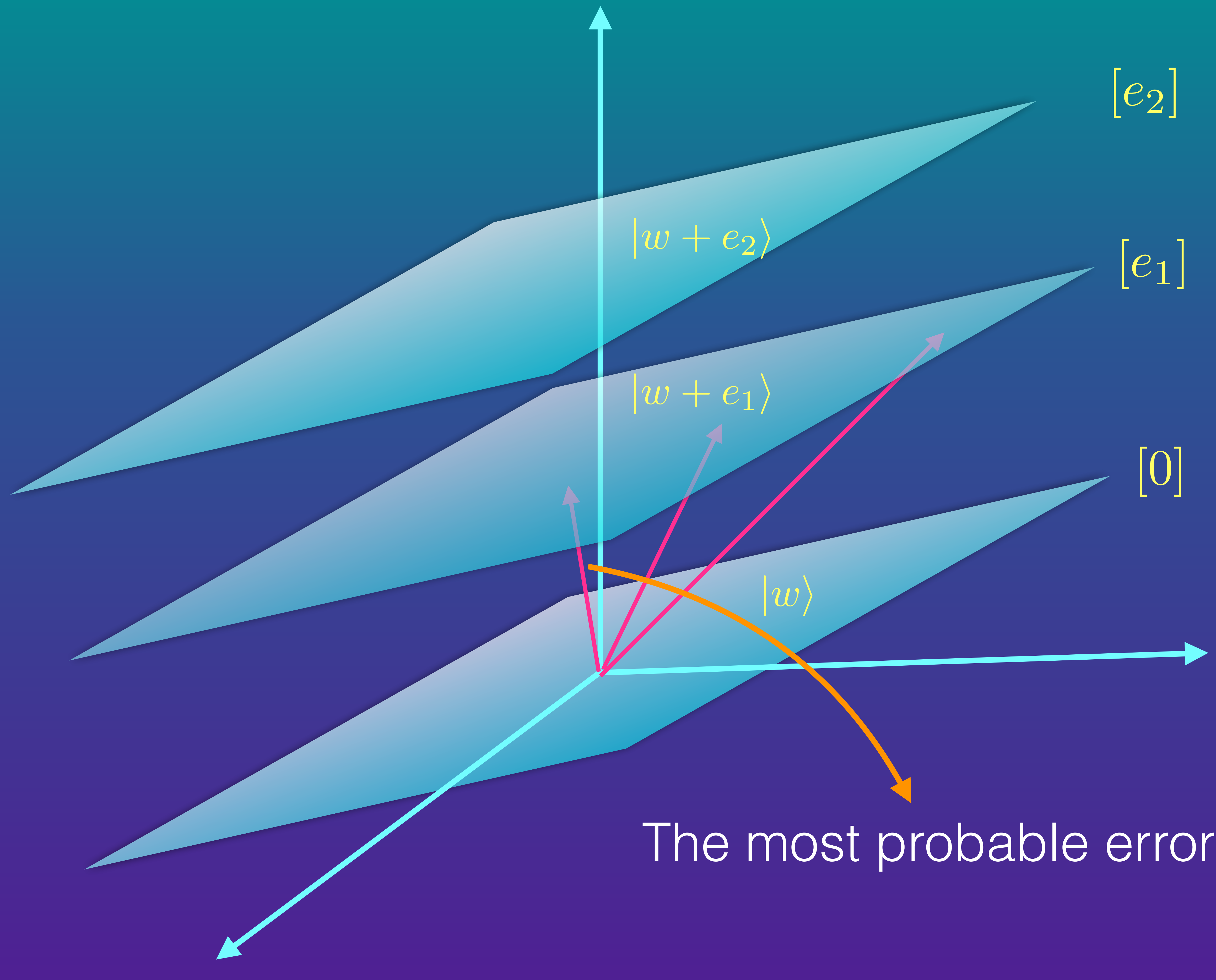
Quantum coding.

$$|w\rangle = \sum_i \alpha_i |g_i\rangle$$

$$\alpha_i \in \text{Complex numbers}$$

This code corrects the same set of bit-flip errors.

$$|g_i\rangle = |0\ 1\ 0\ 1\ 1\ 0\ 1\rangle$$


$[e_2]$
$|w+e_2\rangle$
$[e_1]$
$|w+e_1\rangle$
$[0]$
$|w\rangle$
The most probable error

$$\text{Classical Error}: \quad w \longrightarrow w + e$$

$$e = (1\ 1\ 0\ 0\ 0)$$

$$\text{Quantum (bit Flip) error}: \quad |w\rangle \longrightarrow |w + e\rangle$$

$$X^e = X\ X\ I\ I\ I$$

We can use this technique for bit flip errors.

But how should we combat phase flip errors?

1-We first find the stabilizers of this code.

2-Then we enlarge the set of stabilizers to correct also phase flip errors and finally

3- We determine the states which are stabilized by these stabilizers.

What are the stabilizers of this code?

We know that: $GH^T = 0$

This means: $g_{i1}h_{j1} + \cdots g_{in}h_{jn} = 0$

Or: $(-1)^{g_{i1}h_{j1}+\cdots g_{in}h_{jn}} = 1$

Or: $(-1)^{g_{i1}h_{j1}+\cdots g_{in}h_{jn}}|g_i\rangle = |g_i\rangle$

Where: $|g_i\rangle = |g_{i1}, g_{i2}, \cdots g_{in}\rangle$

From:

$$(-1)^{g_{i1}h_{j1}+\cdots g_{in}h_{jn}}|g_i\rangle = |g_i\rangle$$

$$|g_i\rangle = |g_{i1}, g_{i2}, \cdots g_{in}\rangle$$

$$Z^a|s\rangle = (-1)^{as}|s\rangle$$

$$Z^{h_{i1}}Z^{h_{i2}}\ldots Z^{h_{in}}|g_i\rangle = |g_i\rangle$$

Therefore we have found one set of stabilizers:

$$S_i = Z^{h_{i1}}Z^{h_{i2}}\ldots Z^{h_{in}}$$

Now let us imagine a new set of stabilizers:

$$S'_j = X^{h'_{j1}} X^{h'_{j2}} \dots X^{h'_{jn}}$$

Since:

$$X^a Z^b = (-1)^{ab} Z^b X^a$$

$$S_i S'_j = (-1)^{h_{i1}h'_{j1} + h_{i2}h'_{j2} + \cdots h_{in}h'_{jn}} S'_j S_i$$

Therefore:

$$S_i S'_j = S'_j S_i$$

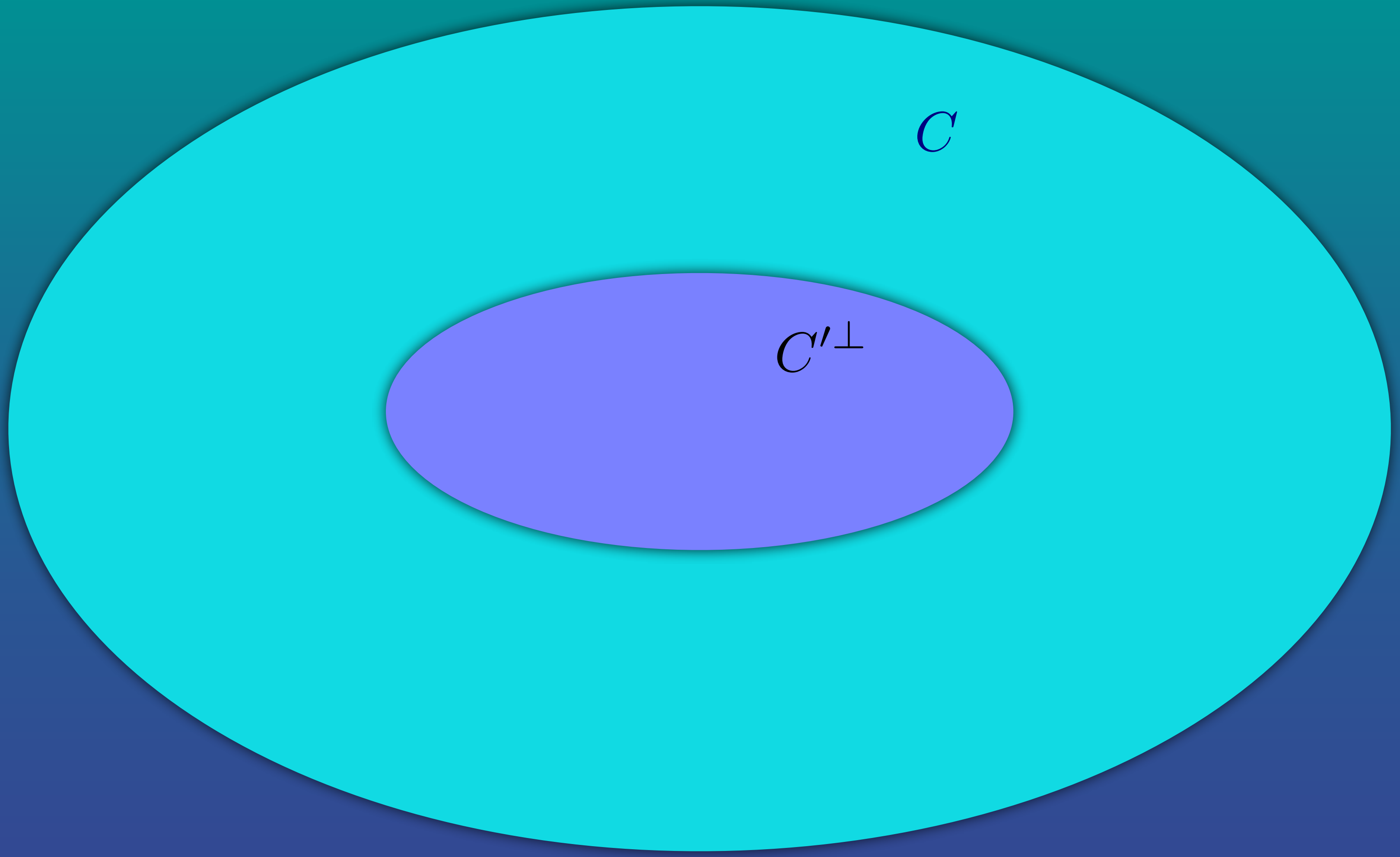
Provided that:

$$H H'^T = 0$$

The meaning of: $H'H^T = 0$

Therefore if: $w \in C'^{\perp} \longrightarrow w = \alpha H' \longrightarrow wH^T = 0$

$\longrightarrow C'^{\perp} \subset C$



$$X^a|s\rangle = |s+a\rangle$$

$$S'_j|g_k\rangle = |g_k + e'_j\rangle$$

$$|[g_k]\rangle := \sum_{v \in C'^{\perp}} |g_k + v\rangle$$

$$S'_j|[g_k]\rangle = |[g_k]\rangle$$

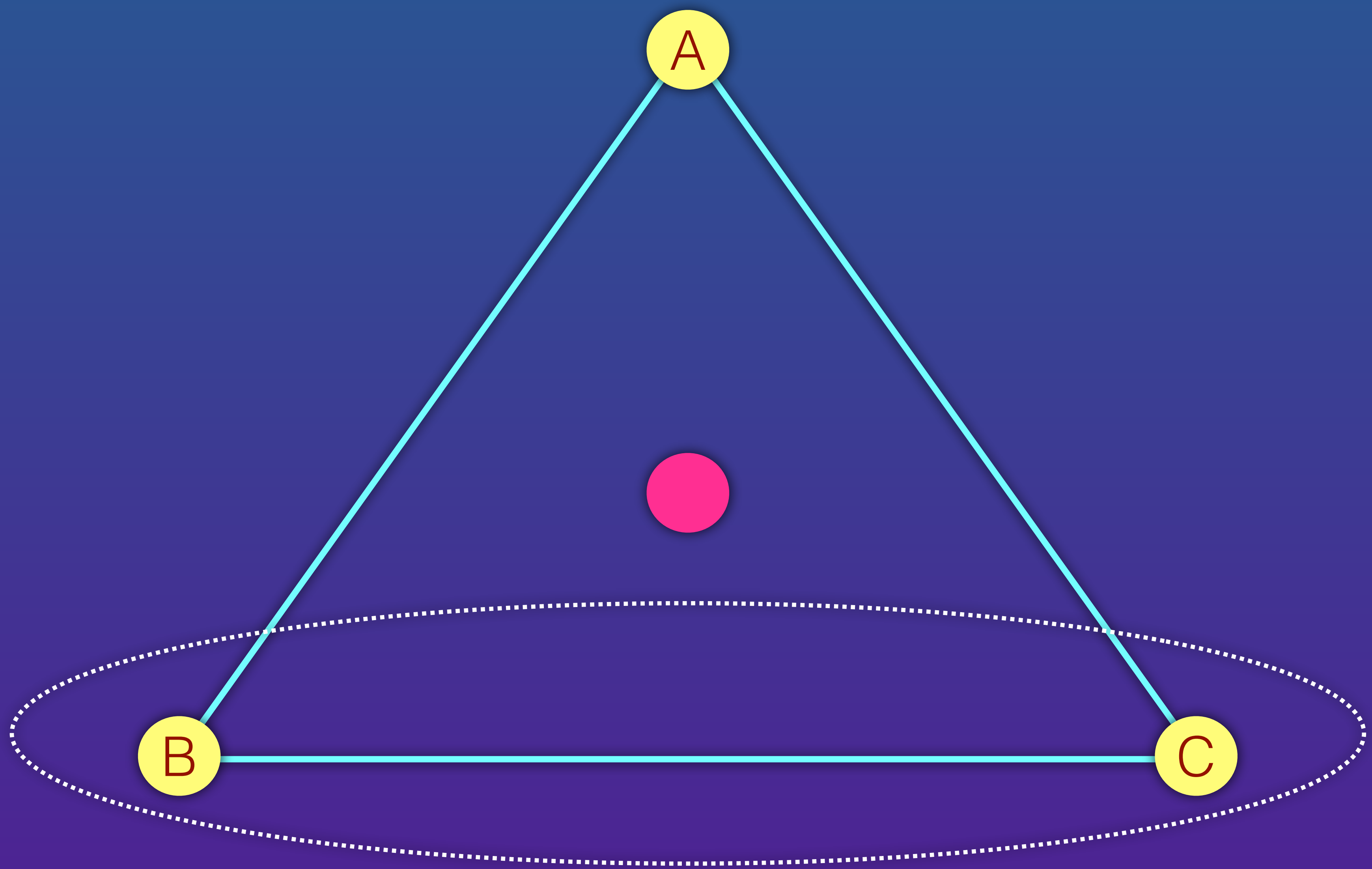
$$S'_j|w\rangle = |w\rangle$$

The final form of code states:

$$|w\rangle = \sum_k \alpha_k |[g_k]\rangle$$

Error Correction and Quantum State Sharing

We want to share a quantum state to A, B and C such that no one can find it, but any two of them can retrieve it if they collaborate.



Quantum Erasure

$$|\psi\rangle = a|000\rangle + b|111\rangle$$

$$\rho = a^2|00\rangle\langle 00| + b^2|11\rangle\langle 11|$$

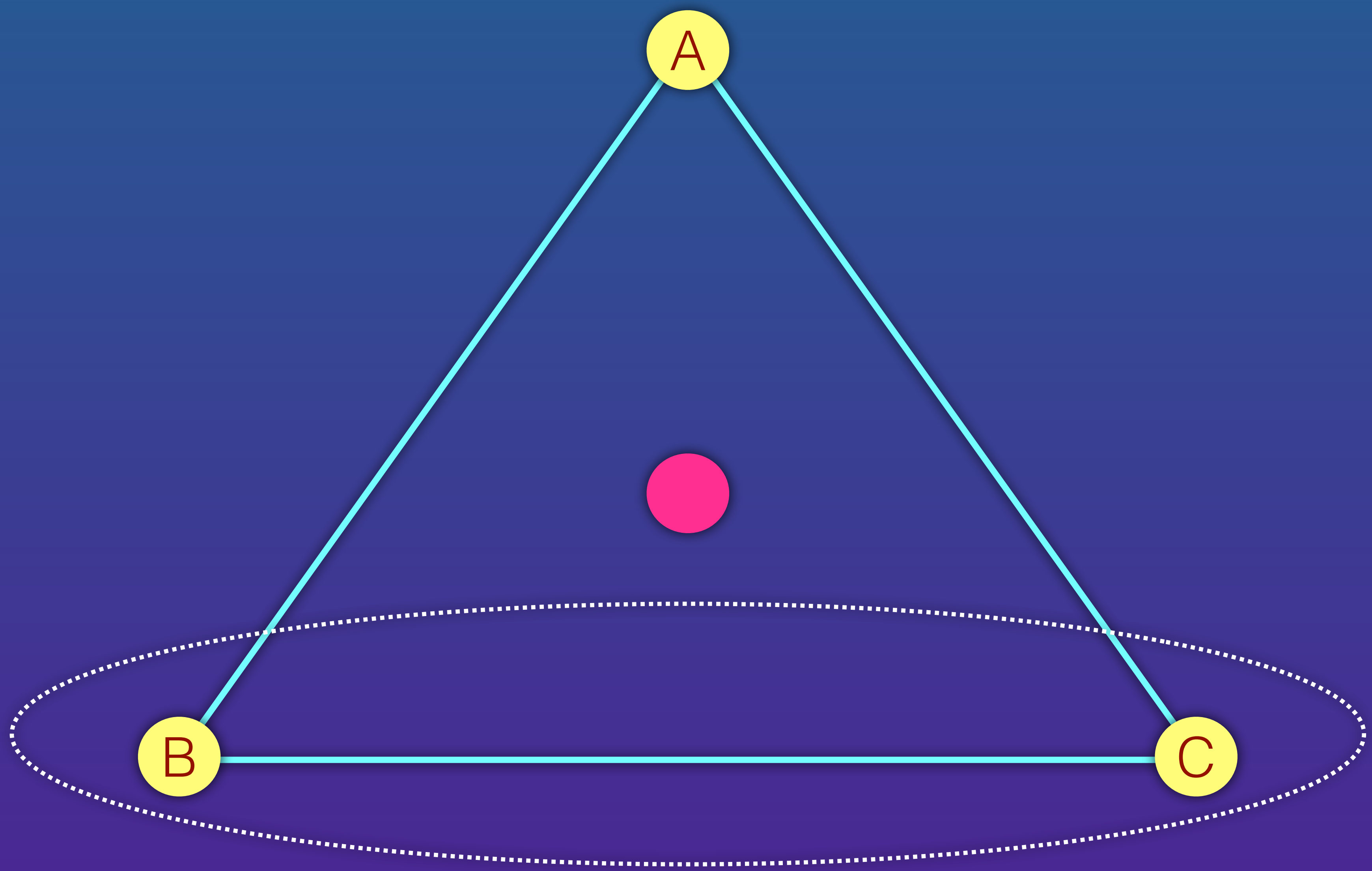
$$|0\rangle \longrightarrow |000\rangle + |111\rangle + |222\rangle \ = |\overline{0}\rangle$$

$$|1\rangle \longrightarrow |012\rangle + |120\rangle + |201\rangle \ = |\overline{1}\rangle$$

$$|2\rangle \longrightarrow |021\rangle + |102\rangle + |210\rangle \ = |\overline{2}\rangle$$

$$a|0\rangle + b|1\rangle + c|2\rangle \longrightarrow a|\overline{0}\rangle + b|\overline{1}\rangle + c|\overline{2}\rangle$$

$$C_{12}C_{21}(a|\overline{0}\rangle + b|\overline{1}\rangle + c|\overline{2}\rangle) \longrightarrow (a|0\rangle + b|1\rangle + c|2\rangle)|\phi\rangle$$



$C_{12}C_{21}$

$$C_{12}C_{21}|\overline{0}\rangle \longrightarrow |0\rangle|\phi\rangle$$

$$C_{12}C_{21}|\overline{1}\rangle \longrightarrow |1\rangle|\phi\rangle$$

$$C_{12}C_{21}|\overline{2}\rangle \longrightarrow |2\rangle|\phi\rangle$$

$$a|\overline{0}\rangle + b|\overline{1}\rangle + c|\overline{2}\rangle \longrightarrow (a|0\rangle + b|1\rangle + c|2\rangle)|\phi\rangle$$

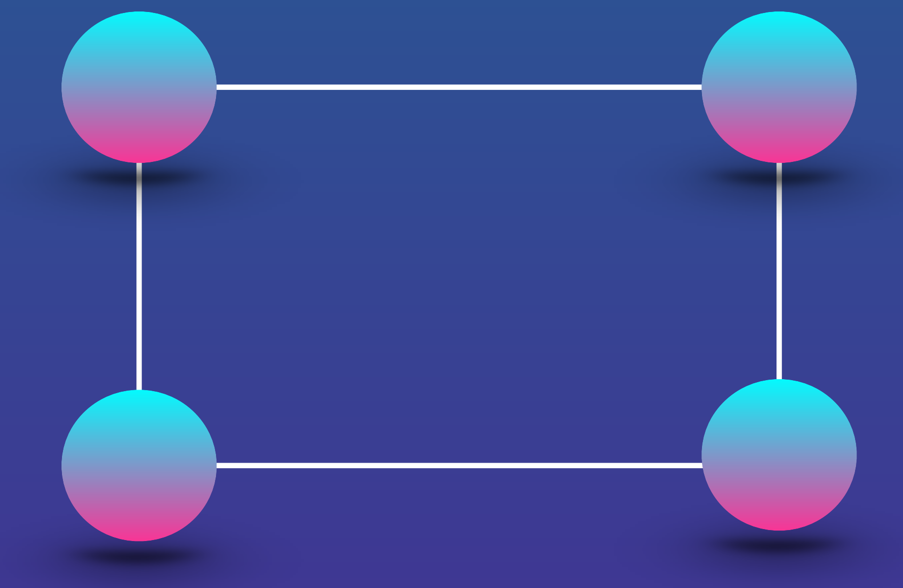
$$|\phi\rangle = |00\rangle + |11\rangle + |22\rangle$$

AME (Absolutely Maximally Entangled) State

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\rho_1 = \rho_2 = \frac{I}{2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$



$$\rho_{12} = \rho_{23} = \rho_{34} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

$$|\psi\rangle = |00\rangle|00\rangle + |01\rangle|11\rangle + |02\rangle|22\rangle + |10\rangle|12\rangle + |11\rangle|20\rangle + |12\rangle|01\rangle + |20\rangle|21\rangle + |21\rangle|02\rangle + |22\rangle|10\rangle$$

$$\rho_{12} = I \qquad \rho_{34} = I \qquad \rho_{23} = I$$

$$|\psi\rangle = |0\rangle|\overline{0}\rangle + |1\rangle|\overline{1}\rangle + |2\rangle|\overline{2}\rangle$$

$$\begin{aligned}|\psi\rangle = |0\rangle(|000\rangle + |111\rangle + |222\rangle) + \\ |1\rangle(|012\rangle + |120\rangle + |201\rangle) + \\ |2\rangle(|021\rangle + |102\rangle + |210\rangle)\end{aligned}$$

$$\begin{aligned}|\psi\rangle = |00\rangle|00\rangle + |01\rangle|11\rangle + |02\rangle|22\rangle + \\ |10\rangle|12\rangle + |11\rangle|20\rangle + |12\rangle|01\rangle + \\ |20\rangle|21\rangle + |21\rangle|02\rangle + |22\rangle|10\rangle\end{aligned}$$

Thank you for your attention