Quantum Dynamical Entropies and Complexity

Fabio Benatti

Department of Phyisics, University of Trieste, Italy

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Outline I

Outline II

Classical Dynamical Entropy

Measure theoretic setting

- $\mathcal{X}$ measure space with a σ-algebra Σ of measurable subsets.
- $\mu : \mathcal{X} \mapsto [0,1]$: probability measure or reference state; $\mu(\mathcal{X}) = 1$.
- $T : \mathcal{X} \mapsto \mathcal{X}$: invertible dynamical map such that $\mu \circ T = \mu$: equilibrium state.
- $\Pi = \{P_i\}_{i=1}^d$: finite, measurable partition,

$$P_i \in \Sigma \ , \ P_i \cap P_j = \delta_{ij} P_i \ , \ \bigcup_{i=1}^d P_i = \mathcal{X} \ .$$

- $\Pi_k = T^{-k}(\Pi) = \{T^{-k}(P_i\}_{i=1}^d$ partition at time $t = k$;
- $\Pi^{(n)} = \{P_{\mathbf{i}^{(n)}}\}$, $\mathbf{i}^{(n)} \in \{1, \ldots, d\}^n$: refined partition up to $t = n - 1$;

$$P_{\mathbf{i}^{(n)}} = P_{i_0} \cap T^{-1}(P_{i_1}) \cap \cdots \cap T^{-n+1}(P_{i_{n-1}}) \ .$$

Dynamical Information Gain

Shannon entropy, information and dynamics: learning from history

- $\Pi_k = \{T^{-k}(P_i)\}_{i=1}^d$: coarse-grained information about $x \in \mathcal{X}$ at $t = k$
- Uncertainty about $x \in \mathcal{X}$ given μ: Shannon entropy,

$$H_\mu(\Pi_k) = H_\mu(\Pi) = -\sum_{i=1}^{d} \mu(P_i) \log \mu(P_i) \ .$$

- $x \in P_{\mathbf{i}^{(n)}}$ implies $T^k x \in P_{i_k}$ for $0 \leq k \leq n-1$
- Uncertainty about the coarse-grained trajectory $\{T^{-k}(P_{i_k})\}_{k=0}^{n-1}$:

$$H_\mu(\Pi^{(n)}) = -\sum_{\mathbf{i}^{(n)}} \mu(P_{\mathbf{i}^{(n)}}) \log \mu(P_{\mathbf{i}^{(n)}}) \ .$$

Maximal Dynamical Information Gain

Maximal coarse-grained time-averaged information

- Average information gain from one step to the next one:

$$h_\mu(T,\Pi) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Big(H_\mu(\Pi^{(k+1)}) - H_\mu(\Pi^{(k)}) \Big)$$
$$= \lim_{n\to\infty} \frac{1}{n} H_\mu(\pi^{(n)}) \ .$$

- Kolmogorov-Sinai dynamical entropy: $h_\mu(T) = \sup_\Pi h_\mu(T,\Pi)$.
- Kolmogorov Theorem: if Π is a generating partition,

$$\bigvee_{n\in\mathbb{Z}} T^n(\Pi) = \Sigma \text{ then } \quad h_\mu(T) = h_\mu(T,\Pi) \ .$$

Baker's map

- Dynamical triple: $\mathcal{X} = \mathbb{T}^2 = \{(x, y) \bmod 1\}$, $\mathrm{d}\mu(x, y) = \mathrm{d}s\mathrm{d}y$

$$B(x,y) = \begin{cases} (2x, y/2) & 0 \leq x < 1/2 \\ 2x - 1, (1+y)/2 & 1/2 \leq x < 1 \end{cases}$$

$$B^{-1}(x,y) = \begin{cases} (x/2, 2y) & 0 \leq y < 1/2 \\ (1+x)/2, 2y - 1 & 1/2 \leq y < 1 \end{cases}$$

- Generating partition: $\Pi = \{P_0, P_1\}$,

$$P_0 = \{(x,y) : 0 \leq x < 1/2\}\ ,\ P_1 = \{(x,y) : 1/2 \leq x < 1\}$$

Baker's map

- Refined partitions: $\Pi^{(n)} = \{P_{\mathbf{i}^{(n)}}\}$

$$P_{\mathbf{i}^{(n)}} = \left\{(x,y) : x_1^{(n)} \leq x \leq x_2^{(n)}, x_2^{(n)} - x_1^{(n)} = 1/2^n\right\}$$

- Entropy rate and KS-entropy:

$$\mu(P_{\mathbf{i}^{(n)}}) = 1/2^n \implies h_\mu(B,\Pi) = \lim_n \frac{1}{n} H_\mu(\Pi^{(n)} = \log 2 = h_\mu(B)$$

Baker's map as a one-dim left-shift

- Write $x = \sum_{k=1}^{\infty} x_k 2^{-k}$, $y = \sum_{j=1}^{\infty} y_j 2^{-j}$ and order (x, y) as a bi-infinite string:

$$(x, y) = \cdots \mathbf{z} = \{z_\ell\}_{\ell=-\infty}^{+\infty} \ , \ z_{-n} = y_n \ , \ z_n = x_{n+1}$$

- B acts as a left-shift:

$$(B\mathbf{z})_n = z_{n+1} \ , \quad B^n \mathbf{z} = 2^n \mathbf{z} \bmod 1$$

- $h_\mu(B) = \log 2$ is the Lyapounov exponent of the Baker map

Dynamical Instability

Lyapounov exponents

- Exponential increase of initial small errors:

$$d(x_1, x_2) = \delta \mapsto d(T^n x_1, T^n x_2) \simeq \mathrm{e}^{\lambda t}\, d(x_1, x_2) \ .$$

- Positive Lyapounov exponents:

$$\lambda(T, x) = \lim_{n \to +\infty} \frac{1}{n} \lim_{\delta \to 0} \log \frac{d(T^n x, T^n(x + \delta))}{d(x, x + \delta)} \ .$$

- $\lambda(T, x) = \lambda(T)$: independence of x for ergodic systems
- Pesin's Theorem: for ergodic systems

$$h_\mu(T) = \sum_i \lambda_i^+(T) \ .$$

Degrees of Statistical Randomness

- ergodicity: time-averaged correlation functions factorize,

$$\lim_{T\to+\infty}\frac{1}{T}\sum_{n=0}^{T-1}\mu(fg\circ T^n)=\mu(f)\mu(g)\ ,\quad \mu(f)=\int_{\mathcal{X}}\mathrm{d}\mu(x)\,f(x)\ .$$

- The only constants of the motion of ergodic are constant on $\mathcal{X}$
- Convex combinations

$$\mu=\sum_i\lambda_i\mu_i\ ,\lambda_i\geq 0\ ,\sum_i\lambda_i=1$$

into T-invariant states, $\mu_i\circ T=\mu_i$, impossible for ergodic states.

Mixing and K-mixing

Mixing and Relaxation

- mixing: correlation functions factorize,

$$\lim_{n\to+\infty}\mu(fg\circ T^n)=\mu(f)\mu(g)\ .$$

- Mixing implies ergodicity and tendency to equilibrium: $\nu(g)=\int_{\mathcal{X}}\mathrm{d}\mu(x)\,f_\nu(x)\,g(x)$ perturbation of μ,

$$\begin{aligned}\nu(g\circ T^n) &= \int_{\mathcal{X}}\mathrm{d}\mu(x)\,f_\nu(x)\,g(T^nx)\longrightarrow\mu(f_\nu)\mu(g)\\ &= \nu(\mathcal{X})\mu(g)=\mu(g)\ .\end{aligned}$$

- K-mixing: total memory loss on long times,

$$\lim_n h_\mu(T^n,\Pi)=H_\mu(\Pi)\Longleftrightarrow h_\mu(T,\Pi)>0\ \forall\ \Pi\ .$$

Classical Information Sources

- Classical Information Sources emit finite length binary strings

$$\mathbf{i}^{(n)} \in \Omega^{(n)} = \{\mathbf{i}^{(n)}\} \ , \ \mathbf{i}^{(n)} = i_1 i_2 \ldots i_n \ , \ i_j = 0, 1$$

- with probabilities: $\pi^{(n)} = \{p^{(n)}(\mathbf{i}^{(n)})\}$ such that

$$\sum_{i_n} p^{(n)}(i_1 i_2 \cdots i_n) = p^{(n-1)}(i_1 i_2 \cdots i_{n-1}) \quad \text{compatibility}$$

- Set of binary sequences: $\Omega \ni \mathbf{i} = \{i_j\}_{j=1}^{\infty}$
- Emission of letters, one after the other, left shift on Ω: $\sigma(\mathbf{i})_n = \mathbf{i}_{n+1}$
- Stationary information source:

$$\sum_{i_1} p^{(n)}(i_1 i_2 \cdots i_n) = p^{(n-1)}(i_2 i_3 \cdots i_n) \quad \text{stationarity}$$

Asymptotic Equipartition Theorem

- Kolmogorov-Sinai entropy of a stationary source:

$$h_\pi(\sigma) = \lim_{n\to+\infty} \frac{1}{n} \left(-\sum_{\mathbf{i}^{(n)}} p^{(n)}(\mathbf{i}^{(n)}) \log p^{(n)}(\mathbf{i}^{(n)}) \right)$$

- Typical subset: given $\epsilon > 0$, $\exists N_\epsilon$ such that, for $n \geq N_\epsilon$,

$$A_n^\epsilon = \left\{ \mathbf{i}^{(n)} : 2^{-n(h_\pi(\sigma)+\epsilon)} \leq p(\mathbf{i}^{(n)}) \leq 2^{-n(h_\pi(\sigma)-\epsilon)} \right\}$$

$$\pi(A_n^\epsilon) \geq 1 - \epsilon$$

$$(1-\epsilon)2^{n(h_\pi(\sigma)-\epsilon)} \leq \mathrm{card}(A_n^\epsilon) \leq 2^{n(h_\pi(\sigma)+\epsilon)}$$

Asymptotic Equipartition and compression

- Untypical subset: $\pi\left((A_n^\epsilon)^c\right) \leq \epsilon$,

$$(A_n^\epsilon)^c = \left\{\mathbf{i}^{(n)} : \left|-\frac{1}{n}\log p^{(n)}(\mathbf{i}^{(n)}) - h_\pi(\sigma)\right| > \epsilon\right\}$$

- Encoding A_n^ϵ requires $h_\mu(\pi)$ bits per letter
- sending all $(A_n^\epsilon)^c$ into a same symbol: error with probability $\leq \epsilon$

Algorithmic Complexity

Individual Randomness versus Statistical Randomness

Stationary information sources (of digit strings)

- Source emitting bit-strings $\mathbf{i}^{(n)} = i_1 i_2 \cdots i_n \in \Omega_2^n = \{0,1\}^n$
- with probabilities $\pi^{(n)} = \{p(\mathbf{i}^{(n)})\}_{\mathbf{i}^{(n)} \in \Omega_2^{(n)}}$
- Shannon entropy

$$H(\pi^{(n)}) = -\sum_{\mathbf{i}^{(n)} \in \Omega_2^{(n)}} p(\mathbf{i}^{(n)}) \log p(\mathbf{i}^{(n)})$$

a measure of the average randomness of the emitted strings

Kolmogorov Complexity

Individual randomness: How difficult is to describe a digit string?

- binary program: any string $p \in \Omega_2^{(m)}$
- length of a binary program: $\ell(p) = m$
- Universal, prefix Turing machine $\mathcal{U}$:

$$\Omega_2^{(m)} \ni p \mapsto \mathcal{U}(p) = \mathbf{i}^{(n)} \in \Omega_2^{(n)}$$

- Kolmogorov complexity of $\mathbf{i}^{(n)} \in \Omega_2^{(n)}$: length of the shortest binary program that, run by $\mathcal{U}$, outputs $\mathbf{i}^{(n)}$ and halts:

$$K(\mathbf{i}^{(n)}) = \min\left\{\ell(p) \ : \ \mathcal{U}(p) = \mathbf{i}^{(n)}\right\}$$

Turing Machines

Classical Turing Machines $\mathcal{U}$

- $(0,1)$ on Input and Output Cells: configuration space t
- Read-Write Head Position: position space h
- Internal Control Unit: internal state space q
- $\mathcal{U}$ reversibly operates on input p and halts with $\mathcal{U}(p)$ written on the output tape
- universal TM: capable to act as any other TM
- Prefix UTM: if $\mathcal{U}[p]$ halts, $\mathcal{U}[pq]$ does not halt

Regular bit-strings

Regular bit strings can be shortly described

- $i^{(1000)} = \underbrace{1111111111\cdots 111}_{1000\ bits}$
- short program:

$$\text{Write} \quad 1 \quad 1000 \text{ times} \Longrightarrow K(i^{(1000)}) = \log 1000 \ + \ C$$

Random Bit-Strings

Random bit strings cannot be compressed

- $i^{(1000)} = \underbrace{1001010101000\ldots100}_{1000\ bits}$
- short program:

$$\text{Write} \quad 1,0,0,1,0,1\ldots \Longrightarrow K(i^{(1000)}) = 1000 \ + \ C$$

Counting Argument

- there are 2^n binary strings of length n, but only

$$\sum_{j=0}^{\ell-1} 2^j = 2^\ell - 1 < 2^\ell$$

binary programs p of length $\ell(p) < \ell$

- Thus, there are more than

$$2^n - 2^\ell = 2^n\left(1 - \frac{1}{2^{n-\ell}}\right)$$

Binary strings with complexity larger than ℓ.

Complexity of a length n binary string with k ones

Trial program p: specify k and the string among $\binom{n}{k}$ strings:

$$\begin{aligned} K(\mathbf{i}^{(n)}) &\leq \ell(p) = C + \log_2 k + \log_2 \binom{n}{k} \\ &\leq C + \log_2 k + n\,H_2(k/n, 1 - k/n) \\ H(k/n) &= -\frac{k}{n}\log_2\frac{k}{n} - (1 - \frac{k}{n})\log_2(1 - \frac{k}{n}) \end{aligned}$$

Indeed,

$$\frac{1}{n+1} 2^{n\,H_2(k/n)} \leq \binom{n}{k} \leq 2^{n\,H_2(k/n)}$$

Kolmogorov complexity and entropy rate

Brudno's theorem

- ergodic sources: $\mathbf{i}^{(n)} \in \Omega_2^{(n)} \longmapsto \mathbf{i} \in \Omega_2 = \{0,1\}^{\infty}$, $\pi^{(n)} \longmapsto \pi$,
- entropy rate

$$h(\pi) = \lim_{n\to\infty} \frac{1}{n} H(\pi^{(n)})$$

- complexity per symbol:

$$k(\mathbf{i}) = \lim_{n\to+\infty} \frac{1}{n} K(\mathbf{i}^{(n)})$$

- for ergodic sources $k(\mathbf{i}) = h(\pi) \qquad \pi - a.e.$

Brudno's theorem at its simplest

- Bernoulli source: $p(i_0 i_1 \cdots i_{n-1}) = \prod_{j=0}^{n-1} p(i_j)$, $p(0) = p$, $p(1) = 1 - p$
- Law of large numbers: $n \geq N_{\epsilon,\delta}$, $N_1(\mathbf{i}^{(n)}) = \frac{1}{n} \sum_{j=0}^{n-1} i_j$

$$\text{Prob}\left\{ |H_2(p) - H_2(N_1(n))| \geq \epsilon \right\} \leq \epsilon$$

- Counting argument:

$$\text{Card}\left\{ \mathbf{i}^{(n)} : \frac{1}{n} K(\mathbf{i}^{(n)}) \leq H_2(p) - \delta \right\} \leq 2^{n(H_2(p)-\delta)}$$

- Typical subset:

$$\text{Card}(A_\epsilon^n) \leq 2^{n(H_2(p)+\epsilon)}$$

Brudno's theorem at its simplest $\mathrm{Prob}\left\{\left|\frac{1}{n}K(\mathbf{i}^{(n)}) - H_2(p)\right| \geq \delta\right\} \leq \delta$

- From law of large numbers and $K(\mathbf{i}^{(n)}) \leq C + \log_2 N_1(\mathbf{i}^{(n)}) + n\, H_2(N_1(\mathbf{i}^{(n)}))$,

$$\mathrm{Prob}\left\{\frac{1}{n}K(\mathbf{i}^{(n)}) \geq H_2(p) + \epsilon\right\} \leq \epsilon$$

- From counting argument and typicality with $\delta > \epsilon$:

$$\mathrm{Prob}\left\{\frac{1}{n}K(\mathbf{i}^{(n)}) \leq H_2(p) - \epsilon\right\} \leq \mathrm{Prob}\left((A_n^\epsilon)^c\right)$$
$$+\mathrm{Prob}\left\{\mathbf{i}^{(n)} \in A_n^\epsilon \ :\ \frac{1}{n}K(\mathbf{i}^{(n)}) \leq H_2(p) - \epsilon\right\}$$
$$\leq \epsilon + 2^{n(H_2(p)+\epsilon)}2^{-n(H_2(p)-\delta)}$$

Algorithmic complexity of point trajectories

- ergodic dynamical system $(\mathcal{X}, \Theta, \mu)$
- coarse-graining of $\mathcal{X}$ by finite partitions $\Pi = \{P_i\}_{i=1}^{p}$
- Coarse-grained trajectory of $x \in \mathcal{X}$: $\mathbf{i}_\Pi^n(x) = i_0(x) i_1(x) \cdots i_{n-1}(x)$:

$$x \in P_{i_0} \ , \ Tx \in P_{i_1} \ , \ \ldots \ T^{n-1} x \in P_{i_{n-1}} \ .$$

- x-dynamical string: $\mathbf{i}_\Pi(x) = \lim_n \mathbf{i}_\Pi^n(x)$

Dynamical Brudno's theorem

- Complexity rate of coarse-grained trajectories:

$$k(\Theta, \mathbf{i}_\Pi(x)) = \lim_{n \to +\infty} \frac{1}{n} K(\mathbf{i}_\Pi^n(x))$$

- maximal complexity rate of coarse-grained trajectories:

$$k_\Theta(x) = \sup_\Pi k_\Pi(\mathbf{i}_\Pi(x))$$

Brudno's theorem

ergodic $(\mathcal{X}, \Theta, \mu)$

$$k_\Theta(\mathbf{i}) = h_\mu(\Theta) \qquad \mu - a.e.$$

Universal probability

- a-priori probability of $\mathbf{i}^{(n)} \in \Omega_2^{(n)}$:

$$\mathbb{P}(\mathbf{i}^{(n)}) := \sum_{p\ :\ \mathcal{U}[p]=\mathbf{i}^{(n)}} 2^{-\ell(p)}\ , \quad \mathcal{U} \text{ prefix UTM}$$

- universality of $\mathbb{P}$: for all semi-computable semi-measures μ:

$$\exists C_\mu > 0\ :\quad C_\mu \mu(\mathbf{i}^{(n)}) \leq \mathbb{P}(\mathbf{i}^{(n)})$$

- universal probability and complexity:

$$K(\mathbf{i}^{(n)}) = -\log \mathbb{P}(\mathbf{i}^{(n)})\ +\ C$$

Why $\sum_{\mathbf{i}^{(n)}} \mathbb{P}(\mathbf{i}^{(n)}) \leq 1$?

- Code for $\mathbf{i}^{(n)} \in \Omega^{(n)}$: $W : \mathbf{i}^{(n)} \mapsto W(\mathbf{i}^{(n)})$
- List the code-words as $\{w_j\}_{j=1}^{2^n}$
- Prefix condition: w_j code words for $\mathbf{i}^{(n)} \in \Omega^{(n)}$ then $w_j w_k$ not code word
- Counting argument: ℓ_j length of w_j,

$$\begin{aligned} N_j &= \text{number of code words with length} \quad \ell_j \\ N_1 &\leq 2^{\ell_1} \,,\; N_2 \leq 2^{\ell_2} \,, \quad N_k \leq 2^{\ell_k} - 2^{\ell_k - \ell_{k-1}} - \cdots - 2^{\ell_k - \ell_1} \end{aligned}$$

- Kraft's inequality:

$$\sum_{j=1}^{M} N_j 2^{-\ell_j} = \sum_{\mathbf{i}^{(n)} \in \Omega^{(n)}} 2^{-\ell(W(\mathbf{i}^{(n)}))} \leq 1$$

Kolmogorov complexity and entropy

- $\pi^{(n)} = \{p(i^{(n)})\}$ a computable distribution on $\Omega_2^n = \{0,1\}^n$:
- From universality, $2^{-K(\pi)}\pi \leq \mathbb{P}$, and $-\log \mathbb{P}(\mathbf{i}^{(n)}) + C = K(\mathbf{i}^{(n)})$,

$$\langle K \rangle_\pi \leq H(\pi) + C + K(\pi)$$

Shannon entropy=average Kolmogorov complexity

- Convexity: $x(\log x - \log y) \geq x - y$,

$$\begin{aligned} 0 &\leq \sum_{\mathbf{i}^{(n)}} p(\mathbf{i}^{(n)}) \left(\log p(\mathbf{i}^{(n)}) - \log \frac{2^{-K(\mathbf{i}^{(n)})}}{\sum_{\mathbf{i}^{(n)}} 2^{-K(\mathbf{i}^{(n)})}} \right) \\ &= -H(\pi) + \langle K \rangle_\pi + \sum_{\mathbf{i}^{(n)}} p(\mathbf{i}^{(n)}) \log \sum_{\mathbf{i}^{(n)}} 2^{-K(\mathbf{i}^{(n)})} \\ &\leq -H(\pi) + \langle K \rangle_\pi \end{aligned}$$

- Shannon entropy=average Kolmogorov complexity:

$$H(\pi^{(n)}) \simeq \sum_{\mathbf{i}^{(n)} \in \Omega_2^n} p(\mathbf{i}^{(n)}) \, K(\mathbf{i}^{(n)}) := \langle K \rangle_\pi$$

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Quantum Sources

Quantum Spin Chains

- Local algebras: $\mathcal{M}_2^{(n)} = (M_2)^{\otimes n}$
- local states: $M_2^{(n)} \ni \rho^{(n)}$, $\mathrm{Tr}_1 \rho^{(n)} = \mathrm{Tr}_n \rho^{(n)} = \rho^{(n-1)}$
- translation-invariant state on $\mathcal{M} := M_2^{\otimes\infty}$: $\rho^{(n)} \longmapsto \omega$
- von Neumann Entropy Rate

$$
\begin{aligned}
S(\rho^{(n)}) &= -\mathrm{Tr}\, \rho^{(n)} \log \rho^{(n)} \\
s(\omega) &= \lim_{n\to+\infty} \frac{1}{n} S(\rho^{(n)})
\end{aligned}
$$

Quantum-Classical matching

- Classical Partitions$\longmapsto$Quantum Subalgebras:

$$\Pi = \{P_i\}_{i=1}^d \longrightarrow A_\Pi = \{\hat{P}_i\}_{i=1}^d \ , \ \hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i$$

- Shannon entropy of a coarse grained trajectory $\longrightarrow$ von Neumann entropy of the generated algebra?

$$H_\mu(\bigcap_{j=0}^{n-1} \Theta^j(\Pi)) \longrightarrow S(\rho \upharpoonright \bigvee_{j=0}^{n-1} \Theta^j(A_\Pi)) \ ?$$

NO

- Subadditivity does not hold:

$$S(\rho \upharpoonright A \vee \Theta(A)) \nleq S(\rho \upharpoonright A) + S(\rho \upharpoonright \Theta(A))$$

- Already $A_\pi \vee \Theta(A)$ can become infinite dimensional

Relative entropy: $S(\rho_1;\rho_2)=\mathrm{Tr}\Big(\rho_1(\log\rho_1-\log\rho_2)\Big)$

- Positivity: $S(\rho_1;\rho_2)\geq 0$, $S(\rho_1;\rho_2)=0\Longleftrightarrow\rho_1=\rho_2$.
- Joint convexity:

$$S\left(\sum_j\lambda_j\rho_{1j};\sum_j\lambda_j\rho_{2j}\right)\leq\sum_j\lambda_j\,S(\rho_{1j};\rho_{2j})\ .$$

- Monotonicity under completely positive unital maps $\gamma:\mathcal{M}\mapsto\mathcal{M}$:

$$\gamma^T[\rho]=\rho\circ\gamma\ ,\qquad S(\gamma^T[\rho_1];\gamma^T[\rho_2])\leq\ S(\rho_1;\rho_2)\ .$$

Convex Decompositions

Entropy of a subalgebra

- Decompositions of ω:

$$\omega = \sum_j \lambda_j \hat{\omega}_j \ , \ \lambda_j \geq 0 \ , \ \sum_j \lambda_j = 1 \ , \hat{\omega}_j(1) = 1 \ .$$

- Restriction of a state to a subalgebra: $\omega \upharpoonright A \subset \mathcal{M}$
- Entropy of A:

$$\begin{aligned} H_\omega(A) &= \sup_{\omega = \sum \lambda_i \hat{\omega}_i} \sum_i \lambda_i \, S(\hat{\omega}_i \upharpoonright A ; \omega \upharpoonright A) \\ &= S(\omega \upharpoonright A) - \sum_i \lambda_i \, S(\hat{\omega}_i \upharpoonright A) \ . \end{aligned}$$

n-subalgebra Entropy

n subalgebras $A_1, A_2, \ldots, A_n$

- Decompositions of ω, $\mathbf{i}^{(n)} = i_1 i_2 \cdots i_n \in I^n$, I discrete index set:

$$\omega = \sum_{\mathbf{i}^{(n)}} \lambda_{\mathbf{i}^{(n)}} \hat{\omega}_{\mathbf{i}^{(n)}} \ , \ \lambda_{\mathbf{i}^{(n)}} \geq 0 \ , \ \sum_{\mathbf{i}^{(n)}} \lambda_j = 1 \ , \hat{\omega}_{\mathbf{i}^{(n)}}(1) = 1 \ .$$

- sub-decompositions: sums over all indices but one,

$$\begin{aligned} \omega^j_{i_j} &= \sum_{\mathbf{i}^{(n)}, \hat{i}_j} \lambda_{\mathbf{i}^{(n)}} \hat{\omega}_{\mathbf{i}^{(n)}} \ , \ \hat{\omega}^j_{i_j} = \frac{\omega^j_{i_j}}{\omega^j_{i_j}(1)} \\ \lambda^j_{i_j} &= \sum_{\mathbf{i}^{(n)}, \hat{i}_j} \lambda_{\mathbf{i}^{(n)}} \ . \end{aligned}$$

n-subalgebra Entropy: $A = \bigvee_{j=1}^{n} A_j$ algebra generated by $\{A_j\}_{j=1}^{n}$

$$
\begin{aligned}
H_\omega(A_1, A_2, \ldots, A_n) &= \sup_{\omega=\sum \lambda_{\mathbf{i}(n)}\hat{\omega}_{\mathbf{i}(n)}} \Bigg\{ H(\{\lambda_{\mathbf{i}^{(n)}}\}) - \sum_{j=1}^{n} H(\{\lambda^j_{i_j}\}) \\
&+ \sum_{j=1}^{n} \sum_{i_j} \lambda^j_{i_j}\, S(\hat{\omega}^j_{i_j} \upharpoonright A_j ; \omega \upharpoonright A_j) \Bigg\} \\
&= \sup_{\omega=\sum \lambda_{\mathbf{i}(n)}\hat{\omega}_{\mathbf{i}(n)}} \Bigg\{ H(\{\lambda_{\mathbf{i}^{(n)}}\}) - \sum_{j=1}^{n} \Bigg(H(\{\lambda^j_{i_j}\}) \\
&+ \sum_{i_j} \lambda^j_{i_j}\, S(\hat{\omega}^j_{i_j} \upharpoonright A_j) - S(\omega \upharpoonright A_j) \Bigg) \Bigg\} .
\end{aligned}
$$

2-subalgebra Entropy

- Quantum upper bound:

$$H_\omega(A_1, A_2) = \sup_{\omega=\sum \lambda_{i_1 i_2}\hat{\omega}_{i_1 i_2}} \Bigg\{ \underbrace{H(\{\lambda_{i_1 i_2}\}) - H(\{\lambda^1_{i_1}\}) - H(\{\lambda^2_{i_2}\})}_{\leq \mathbf{0}}$$

$$+ \sum_{i_1} \lambda^1_{i_1} S(\hat{\omega}^1_{i_1} \restriction A_1; \omega \restriction A_1) + \sum_{i_2} \lambda^2_{i_2} S(\hat{\omega}^2_{i_2} \restriction A_2; \omega \restriction A_2) \Bigg\}$$

$$\leq S(\omega \restriction A_1) + S(\omega \restriction A_2)$$

- Concavity of von Neumann entropy:
  $S(\omega \restriction A_j) \leq H(\{\lambda^j_{i_j}\}) + \sum_{i_j} \lambda^j_{i_j} S(\hat{\omega}^j_{i_j} \restriction A_j)$
- Classical upper bound: $H_\omega(A_1, A_2) \leq \sup_{\omega=\sum \lambda_{i_1 i_2}\hat{\omega}_{i_1 i_2}} H(\{\lambda_{i_1 i_2}\})$

Commutative case: explicit computation

- $A_{1,2}$ subalgebras of A **Abelian**: $H_\omega(A_1, A_2) = S(\omega \upharpoonright A_1 \vee A_2)$
- Minimal projections $\{a^j_{i_j}\}$, $j = 1, 2$, $\sum_{i_j} a^j_{i_j} = 1$, $a^j_{i_j} a^j_{i_k} = \delta_{i_j i_k}$:

$$\omega(a) = \sum_{i_1 i_2} \omega(a^1_{i_1} a^2_{i_2} a) = \sum_{i_1} \omega(a^1_{i_1}) \sum_{i_2} \frac{\omega(a^1_{i_1} a^2_{i_2} a)}{\omega(a^1_{i_1})}$$

$$\hat{\omega}^j_{i_j}(a) = \frac{\omega(a^j_{i_j} a)}{\omega(a^j_{i_j})} , \ \lambda_{i_1 i_2} = \omega(a^1_{i_1} a^2_{i_2} a) , \ \lambda^j_{i_j} = \omega(a^j_{i_j}) .$$

- $\omega^j_{i_j} \upharpoonright A_j = \{\omega^j_{i_j}(a^j_{k_j}\} = \{\delta_{i_j k_j}\}$, $\omega \upharpoonright A_j = \{\omega(a^j_{i_j}) = \lambda^j_{i_j}\}$ imply

$$S(\omega \upharpoonright A_1 \vee A_2) \leq H_\omega(A_1, A_2) \leq H(\{\lambda_{i_1 i_2}\}) = S(\omega \upharpoonright A_1 \vee A_2) .$$

Non Commutative case: factor state on a quantum spin chain

- Decomposition of a density matrix:

$$x_j \geq 0 \ , \ \sum_j x_j = 1 \Longrightarrow \rho = \sum_j \sqrt{\rho} x_j \sqrt{\rho} \ .$$

- $M_{1,2}$ commuting spin algebras at sites $1, 2$, $\omega = \bigotimes_{n\in\mathbb{Z}} (\rho)_n$
- a^j_{ij} eigenprojections of $(\rho)_j$: $\sqrt{(\rho)}_j a^j_{i_j} \sqrt{(\rho)_j} = (\rho)_j a^j i_j$.
- $A_j \subset M_j$: maximally Abelian subslagebra generated by $\{a^j_{i_j}\}$:

$$\begin{aligned} S(\omega \restriction M_1 \otimes M_2) &\geq H_\omega(M_1, M_2) \geq H_\omega(A_1, A_2) = H(\{\omega(a^1_{i_1} a^2_{i_2})\}) \\ &= S(\omega \restriction M_1 \otimes M_2) = 2S(\rho) \ . \end{aligned}$$

- $H_\omega(M_1, M_2) = H_\omega(A_1, A_2) = S(\omega \restriction M_1 \otimes M_2) = 2S(\rho)$

Properties of n-subalgebra entropies

- Positivity and boundedness:

$$0 \leq H_\omega(A_1, A_2, \ldots, A_n) \leq \sum_{j=1}^{n} S(\omega \restriction A_j)$$

- Invariance under repetition:

$$H_\omega(A_1, A_1, A_2 \ldots, A_n) = H_\omega(A_1, A_2, \ldots, A_n) \ .$$

- Invariance under permutations:

$$H_\omega(A_1, \ldots, A_n) = H_\omega(A_{\pi(1)}, \ldots, A_{\pi(n)}) \ .$$

- Invariance under automorphisms $\Theta : \mathcal{M} \mapsto \mathcal{M}$:

$$H_\omega(\Theta(A_1), \Theta(A_2), \ldots, \Theta(A_n)) = H_\omega(A_1, A_2, \ldots, A_n) \ .$$

Properties of n-subalgebra entropies

- Sub-additivity:

$$H_\omega(A_1, \ldots, A_n) \le H_\omega(A_1, \ldots, A_p) + H_\omega(A_{p+1}, \ldots, A_n)$$

- Mononicity under embeddings

$$A_i \subseteq B_i \Longrightarrow H_\omega(A_1, \ldots, A_n) \le H_\omega(B_1, \ldots, B_n)$$

$$A = \bigvee_{j=1}^{n} A_j \Longrightarrow H_\omega(A_1, A_2, \ldots, A_n) \le H_\omega(A) \ .$$

CNT-Entropy

- Quantum triple: $(\mathcal{M}, \Theta, \omega)$, $\omega \circ \Theta = \omega$
- Dynamical information rate about $A \subseteq \mathcal{M}$:

$$h_\omega^{\mathrm{CNT}}(\Theta, A) = \lim_{n \to +\infty} \frac{1}{n} H_\omega(A, \Theta(A), \ldots, \Theta^{n-1}(A))$$

- CNT-Entropy:

$$h_\omega^{\mathrm{CNT}}(\Theta) = \sup_{A \subseteq \mathcal{M}} h_\omega^{\mathrm{CNT}}(\Theta, A) \ .$$

- Nested sequence of finite dim. subalgebras: $\bigvee_n A_n = \mathcal{M}$,

$$h^{\mathrm{CNT}}(\theta) = \lim_{n \to +\infty} h_\omega^{\mathrm{CNT}}(\Theta, A_n) \ .$$

Quantum spin chains: CNT-entropy for $\omega = \bigotimes_{n\in\mathbb{Z}}(\rho)_n$

- Quantum spin chain: $(M_d^{\otimes\infty}, \Theta, \omega)$, $\Theta[(M_d)_n] = (M_d)_{n+1}$.
- Nested generating sequence: $A_n = M_{[0,n-1]}$:

$$\begin{aligned} h_\omega^{\mathrm{CNT}}(\Theta) &= \lim_n h_\omega^{\mathrm{CNT}}(\Theta, M_{[0,n-1]}) \\ &\leq \lim_n \lim_k \frac{1}{k} H_\omega(M_{0,n-1]}, M_{[1,n]}, \ldots, M_{[k,n+k-1]}) \\ &\leq \lim_n \lim_k \frac{1}{k} H_\omega(M_{0,n+k-1]}) \\ &\leq \lim_n \lim_k \frac{1}{k} S(\omega(M_{0,n+k-1]}) = s(\omega) = S(\rho) \ . \end{aligned}$$

Quantum spin chains: CNT-entropy for $\omega = \bigotimes_{n\in\mathbb{Z}}(\rho)_n$

- Choose $k = np + q$, $0 \leq q < n$
- $A_{[jn,(j+1)n-1]} \subset M_{[jn,(j+1)n-1]}$ maximally abelian subalgebras generated by the eigenprojections of $\bigotimes_{\ell=jn}^{(j+1)n-1}(\rho)_\ell$:

$$H_\omega(M_{0,n-1]}, M_{[1,n]}, \dots, M_{[k,n+k-1]}) \geq$$
$$\geq H_\omega(M_{0,n-1]}, M_{[n,2n-1]}, \dots, M_{[(p-1)n,pn-1]})$$
$$H_\omega(A_{0,n-1]}, A_{[n,2n-1]}, \dots, A_{[(p-1)n,pn-1]}) \geq$$
$$\geq S(\omega\upharpoonright M_{[0,kn-1]}) = knS(\rho)$$

- $h_\omega^{\mathrm{CNT}}(\Theta, M_{[0,n-1]}) \geq S(\rho) \Longrightarrow h_\omega^{\mathrm{CNT}}(\Theta) = S(\rho)$.

Alicki-Fannes Entropy: Quantum Partitions

- Quantum triple: $(\mathcal{M}, \Theta, \omega)$
- Partition of unity:

$$\mathcal{X} = \{X_i\}_{i=1}^{p} \ , \quad \sum_{i=1}^{p} X_i^\dagger X_i = 1 \ , \quad X_i \in \mathcal{M}_0$$

- Partition of unit time-evolved up to $t = n - 1$:

$$\mathcal{X}^{(n)} = \{X_{\mathbf{i}^{(n)}}\} \ , \quad X_{\mathbf{i}^{(n)}} = \Theta^{(n-1)}[X_{i_{n-1}}] \cdots \Theta[X_{i_1}] X_{i_0}$$

- partition-density matrix:

$$\rho[\mathcal{X}^{(n)}] = [\omega(X^\dagger_{\mathbf{i}^{(n)}} X^\dagger_{\mathbf{j}^{(n)}}]_{p^n \times p^n}$$

Alicki-Fannes entropy

- Entropy of a partition:

$$S[\mathcal{X}^{(n)}] = -\mathrm{Tr}\Big(\rho[\mathcal{X}^{(n)}]\log\rho[\mathcal{X}^{(n)}]\Big)$$

- Partition entropy rate:

$$h_\omega(\Theta,\mathcal{X}) = \limsup_{n\to\infty}\frac{1}{n}S[\mathcal{X}^{(n)}]$$

- Alicki-Fannes Entropy:

$$h_\omega^{\mathrm{AF}}(\Theta) = \sup_{\mathcal{X}} h_\omega(\Theta,\mathcal{X})$$

AF Entropy: another interpretation

- GNS representation: $\rho = \sum_{j=1}^{n} r_j |r_j\rangle\langle r_j|$, $\mathcal{X} = \{X_i\}_{i\in I}$,

$$|\sqrt{\rho}\rangle = \sum_{j=1}^{n} \sqrt{r_j}|r_j\rangle \otimes |r_j\rangle$$

- Reference vector state: $\{|i\rangle\}_{i\in I}$ ONB,

$$|\Psi_{\mathcal{X}}\rangle = \sum_{i\in I}\sum_{j=1}^{n} \sqrt{r_j} X_i |r_j\rangle \otimes |r_j\rangle \otimes |i\rangle$$

- Marginal density matrices: partial traces,

$$\begin{aligned} \mathrm{Tr}_{12}|\Psi_{\mathcal{X}}\rangle\langle\Psi_{\mathcal{X}}| &= \rho[\mathcal{X}] \\ \mathrm{Tr}_{3}|\Psi_{\mathcal{X}}\rangle\langle\Psi_{\mathcal{X}}| &= \mathcal{R}[\mathcal{X}] = \sum_{i\in I} X_i \otimes 1 |\sqrt{\rho}\rangle\langle\sqrt{\rho}| X_i^\dagger \otimes 1 \end{aligned}$$

Successive measurement processes

- Marginal density matrices of pure states: same von Neumann entropy:

$$S(\rho[\mathcal{X}]) = S(\mathcal{R}[\mathcal{X}])$$

- Partition: Measurement processes

$$\mathcal{X} \Longrightarrow \rho \mapsto \mathbb{E}_{\mathcal{X}}[\rho] = \sum_{i\in I} X_i\, \rho\, X_i^\dagger$$

- Repeated measurements at successive times upon the GNS projector:

$$\rho[\mathcal{X}^{(n)}] = \sum_{\mathbf{i}^{(n)}} X_{\mathbf{i}^{(n)}} |\sqrt{\rho}\rangle\langle\sqrt{\rho}| X^\dagger_{\mathbf{i}^{(n)}}$$

Quantum spin chains: AF-entropy

- Quantum spin chain: $(M_d^{\otimes\infty}, \Theta, \omega)$, $\Theta[(M_d)_n] = (M_d)_{n+1}$
- Partition: $\mathcal{X} = \{X_{ij}\}_{i,j=1}^d$

$$X_{ij} = \frac{|i\rangle\langle j|}{\sqrt{d}} \ , \ \{|i\rangle\}_{i=1}^d \ ONB \ in \mathbb{C}^d \ .$$

- Refined partition: $\mathcal{X}^{(n)} = \{X_{\mathbf{i}^{(n)}\mathbf{j}^{(n)}}\}$

$$X_{\mathbf{i}^{(n)}\mathbf{j}^{(n)}} = \frac{1}{\sqrt{d^n}}|i_{n-1}\langle j_{n-1} \otimes \cdots |i_{n-1}\langle j_{n-1} \ .$$

- Partition density matrix: $\rho[\mathcal{X}^{(n)}] = \frac{1}{d^n} \otimes \rho_{[0,n-1]}$,

$$\rho[\mathcal{X}^{(n)}]_{(\mathbf{i}^{(n)}\mathbf{j}^{(n)});(\mathbf{k}^{(n)}\mathbf{p}^{(n)})} = \frac{1}{d^n}\omega\left(\bigotimes_{\ell=0}^{n} |p_\ell\rangle\langle k_\ell|i_\ell\rangle\langle j_\ell|\right) = \prod_{\ell=0}^{n} \frac{\delta_{k_\ell i_\ell}}{d} \rho_{\mathbf{j}^{(n)}\mathbf{p}^{(n)}} \ .$$

Comparison with mean entropy

- Partition entropy:

$$S(\rho[\mathcal{X}^{(n)}]) = S(\rho_{0,n-1}) + n \log d$$

- AF-Entropy:

$$h_\omega^{\mathrm{AF}}(\Theta) = s(\omega) \ + \ \log d = h_\omega^{\mathrm{CNT}}(\Theta) + \log d$$

Has $\log d$ any interpretation in terms of quantum algorithmic complexity?

AF-entropy: successive mesurements

- partitions: $\mathcal{X} = \{X_i\}_{i=1}^p$, $\mathcal{X}^{(n)} = \left\{X_{\mathbf{i}^{(n)}} = \Theta^{n-1}[X_{i_{n-1}}] \cdots \Theta[X_{i_1}] X_{i_0}\right\}$
- GNS representation: $(\pi_\omega(\mathcal{M}), U_\omega^\Theta, \Omega_\omega)$
- purification:

$$|\Psi_{\mathcal{X}}^{(n)}\rangle = \sum_{\mathbf{i}^{(n)}} \pi_\omega(X_{\mathbf{i}^{(n)}})|\Omega_\omega\rangle \otimes |\mathbf{i}^{(n)}\rangle$$

- partial traces: $\mathrm{Tr}_I\left(|\Psi_{\mathcal{X}}^{(n)}\rangle\langle\Psi_{\mathcal{X}}^{(n)}|\right) = \rho[\mathcal{X}^{(n)}]$

$$R[\mathcal{X}^{(n)}] = \mathrm{Tr}_{II}\left(|\Psi_{\mathcal{X}}^{(n)}\rangle\langle\Psi_{\mathcal{X}}^{(n)}|\right) = \sum_{\mathbf{i}^{(n)}} \pi_\omega(X_{\mathbf{i}^{(n)}})|\Omega_\omega\rangle\langle\Omega_\omega|\pi_\omega(X_{\mathbf{i}^{(n)}})^\dagger$$

- same entropies: $S(\rho[\mathcal{X}^{(n)}]) = S(R[\mathcal{X}^{(n)}])$

AF-entropy: successive mesurements

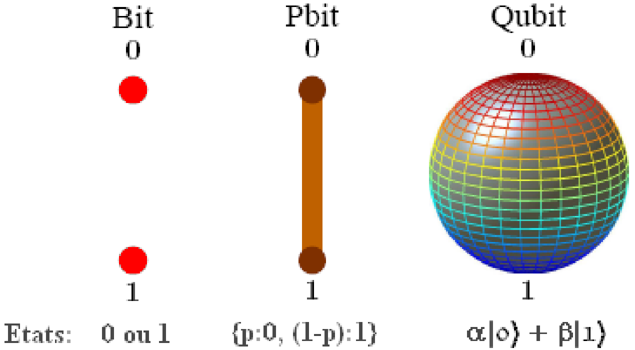
- Density matrix at time $n-1$:

$$R[\mathcal{X}^{(n)}] = \mathrm{Tr}_{II}\left(|\Psi_{\mathcal{X}}^{(n)}\rangle\langle\Psi_{\mathcal{X}}^{(n)}|\right) = \sum_{\mathbf{i}^{(n)}} \pi_\omega(X_{\mathbf{i}^{(n)}})|\Omega_\omega\rangle\langle\Omega_\omega|\pi_\omega(X_{\mathbf{i}^{(n)}})^\dagger$$

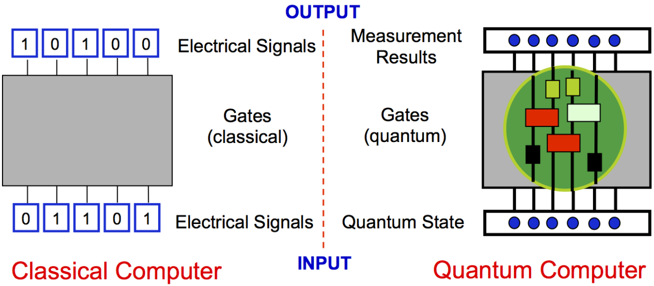
- Subsequent time-evolutions and mesurements:

$$\pi_\omega(X^{(2)}_{i_0 i_1 1_2})|\Omega_\omega\rangle = (U_\omega^\dagger)^2 \underbrace{\pi_\omega(X_{i_2})}_{meas} \underbrace{U_\omega}_{time-ev} \underbrace{\pi_\omega(X_{i_1})}_{meas} \underbrace{U_\omega^\dagger}_{time-ev} \underbrace{\pi_\omega(X_{i_0})}_{meas} |\Omega_\omega\rangle$$

Qubit



Qubit


OUTPUT
1 0 1 0 0
Electrical Signals
Measurement Results
Gates (classical)
Gates (quantum)
0 1 1 0 1
Electrical Signals
Quantum State
INPUT
Classical Computer
Quantum Computer

CTMs vs QTMs

Classical TM $\mathcal{U}$

- $(0,1)$ on Input and Output Cells: configuration space t
- Read-Write Head Position: position space h
- Internal Control Unit: internal state space q
- $\mathcal{U}$ reversibly operates on input p and halts with $\mathcal{U}(p)$ written on the output tape

Quantum TM $\mathbb{U}$

- Tape Hilbert Space: $|t\rangle \in \mathbb{H}_{T_{in}} \otimes \mathbb{H}_{T_{out}}$
- Head Hilbert Space: $|h\rangle \in \mathbb{H}_H$
- Control Unit Hilbert Space: $|q\rangle \in \mathbb{H}_Q$
- QTM Hilbert Space: $\mathbb{H}_{THQ} = \mathbb{H}_{T_{in}} \otimes \mathbb{H}_{T_{out}} \otimes \mathbb{H}_H \otimes \mathbb{H}_Q$

QTM: CP map $\mathbb{R}$

- unitary time evolution U:

$$|\Psi_{in}\rangle \mapsto U|\Psi_{in}\rangle = |\Psi^*\rangle \in \mathbb{H}_{THQ}$$

- QTM halts halts when Control Unit in halting state $|q^*\rangle$
- output obtained by an CP-map:

$$\mathbb{R}[|\Psi_{in}\rangle\langle\Psi_{in}|] = \mathrm{Tr}_{T_{in}HQ}[|\Psi^*\rangle\langle\Psi^*|]$$

Objects to Describe and Descriptions

UTM: $\mathcal{U}$

- Objects to describe : bit-strings $\mathbf{i}^{(n)} \in \Omega_2^{(n)}$
- Described by binary programs: $p \in \Omega_2^{(m)} \; : \; \mathcal{U}(p) = \mathbf{i}^{(n)}$

UQTM: irreversible completely positive map $\mathbb{R}$

- Object to describe: n qubit vectors $|\Psi\rangle \in \mathbb{C}^{2^n}$
- Described by:
  1. quantum states: Quantum Qubit Complexity $\mathrm{QC}_q(\Psi)$

     $$\sigma \in M_{2^m} \Longrightarrow \mathbb{R}[\sigma] \simeq |\Psi\rangle\langle\Psi|$$

  2. bit-strings: Quantum Bit Complexity (Vitanyi 2001) $\mathrm{QC}_c(\Psi)$

     $$p \in \Omega_2^m \Longrightarrow \mathbb{R}[p] \simeq |\Psi\rangle\langle\Psi|$$

Approximate Descriptions

bit-strings vs qubit-strings

- there are 2^n bit-strings of length n
- the n-qubit state-vectors $|\Psi\rangle \in \mathbb{C}^{2^n}$ cover the surface of the unit sphere in $\mathbb{R}^{2^{n+1}}$
- trace distance:

$$D\Big(\mathbb{R}[\sigma], |\Psi\rangle\langle\Psi|\Big) := \mathrm{Tr}\Big|\mathbb{R}[\sigma] - |\Psi\rangle\langle\Psi|\Big|$$

Quantum Qubit Complexity (Berthiaumme et al. 2001)

Quantum Qubit Complexity: $\mathrm{QC}_q(\Psi)$

- Length of a qubit string:

$$\ell(\rho) := \min\left\{n \in \mathbb{N} \,:\, \rho \in M_{2^n}(\mathbb{C})\right\}$$

- Quantum Qubit Complexity:

$$\mathrm{QC}_q(\Psi) = \min\left\{\ell(\sigma) \,:\, D\left(|\Psi\rangle\langle\Psi|, \mathbb{R}[k,\sigma]\right) \leq \frac{1}{k}\ \forall\, k \in \mathbb{N}\right\}$$

Quantum Qubit Complexity: upper bound

$\mathrm{QC}_q(\Psi)$: upper bound

- Simple copying:

$$\mathrm{QC}_q(\Psi) \leq n + C$$

- quantum counting argument: $P = |\Psi\rangle\langle\Psi|$

$$N = \mathrm{card}\{P_i \perp P_j : D(\mathbb{R}[\sigma_i], P_i) \leq \delta\ ,\ \ell(\sigma_i) \leq n\, s(\omega)\}$$

$$N \leq 2^{n\, s(\omega) - \eta(\delta)}$$

Quantum Qubit Complexity: Brudno's relation (F.B. et al. 2006)

Classical Brudno: entropy per symbol = complexity per symbol

ergodic sources $\qquad k(i) = h(\pi) \qquad \pi - a.e.$

$\mathrm{QC}_q(\Psi)$: Brudno's relation

- Ergodic quantum sources with entropy rate $s(\omega)$
- for every $\delta > 0$, there exist typical projections $Q_n \in M_{2^n}$ such that, for n large enough, $\mathrm{Tr}(\rho^n Q_n) \simeq 1$ and

$$s(\omega) - \delta \leq \frac{1}{n}\mathrm{QC}_q(\Psi) \leq s(\omega) + \delta \qquad \forall |\Psi\rangle\langle\Psi| \leq Q_n$$

Universal Density Matrix (Gacs 2001)

Universal probability

:

$$\mathbb{P}(\mathbf{i}^{(n)}) := \sum_{p \,:\, \mathcal{U}[p]=\mathbf{i}^{(n)}} 2^{-\ell(p)}\ , \quad \mathcal{U} \text{ prefix UTM}$$

Universal density matrix

- $\mathbf{i} \in \Omega$: set of all binary strings
- $\boldsymbol{\Omega}$: standard basis of vectors $|\mathbf{i}\rangle$ spanning the Hilbert space $L^2(\boldsymbol{\Omega})$
- Elementary vectors $|\Psi_{el}\rangle = \sum_{\mathbf{i}\in\boldsymbol{\Omega}} C_{\mathbf{i}}|\mathbf{i}\rangle \in L^2(\boldsymbol{\Omega})$: only finitely many algebraic numbers $C_{\mathbf{i}} \neq 0$: $\Psi_{el} = \mathbf{i}_{\Psi_{el}} \in \Omega$
- a-priori semi-density matrix:

$$\mathbb{D} = \sum_{|\Psi_{el}\rangle} \mathbb{P}(\mathbf{i}_{\Psi_{el}})\, |\Psi_{el}\rangle\langle\Psi_{el}|$$

Universality of $\mathbb{D}$ and Operator Complexity

- $\mathbb{P}$ computed with a prefix UTM implies $\mathrm{Tr}(\mathbb{D}) \leq 1$
- Universality of $\mathbb{D}$:
  for all semi-computable, semi-density matrices, $\mathrm{Tr}\rho \leq 1$,

$$\exists\ C_\rho > 0 \quad : \quad C_\rho\, \rho \leq \mathbb{D}\ .$$

- Operator Complexity:

$$\kappa_q = -\log \mathbb{D}$$

- Gacs state complexity:

$$\overline{H}(\rho) = \mathrm{Tr}(\rho\, \kappa_q)$$

$S(\rho) \simeq \overline{H}(\rho)$

- Universality, $\rho \leq C_\rho^{-1}\mathbb{D}$ implies

$$-\log\rho \geq -\log\mathbb{D} + \log C_\rho \Longrightarrow S(\rho) \geq \overline{H}(\rho) + \log C_\rho$$

- Positivity of relative entropy,

$$\mathrm{Tr}\left(\rho\left(\log\rho - \log\frac{\mathbb{D}}{\mathrm{Tr}\mathbb{D}}\right)\right) \geq 0$$

and $\mathrm{Tr}\mathbb{D} \leq 1$ imply

$$S(\rho) \leq \mathrm{Tr}(\rho\,\kappa_q) + \log\mathrm{Tr}(\mathbb{D}) \leq \overline{H}(\rho)\ .$$

Gacs complexity and Alicki-Fannes entropy

Gacs operator complexity and entropy density (CNT-entropy)

- local partitions: $X_i \in M_{[-p,p]}$, $X_{i^{(n)}} \in M_{[-p,p+n-1]}$
- computable X_i and $\rho^{(n)}$ states on $M_d^{\otimes n}$:
- Maximal Gacs Complexity rate:

$$\sup_{\mathcal{X}} \lim_{n\to+\infty} \frac{1}{n} \mathrm{Tr}\left(\rho^{(n)}\,\kappa_q^{(n)}\right) = s(\omega)$$

Gacs complexity and AF entropy

- GNS representation: $(\pi_\omega(\mathcal{M}), U^\sigma_\omega, \Omega_\omega)$;
- local partitions: $X_i \in M_{[-p,p]}$, $X_{i^{(n)}} \in M_{[-p,p+n-1]}$
- computable X_i and $R[\mathcal{X}^{(n)}]$ states on $M_d^{\otimes n} \otimes M_d^{\otimes n}$:

$$\mathrm{Tr}\left(R[\mathcal{X}^{(n)}]\,\kappa_q^{(n)}\right) \simeq S(R[\mathcal{X}^{(n)}]) = S(\rho[\mathcal{X}^{(n)}])$$

- Maximal Gacs Complexity rate:

$$\sup_{\mathcal{X}} \lim_{n\to+\infty} \frac{1}{n}\mathrm{Tr}\left(R[\mathcal{X}^{(n)}]\,\kappa_q^{(n)}\right) = s(\omega) + \log d$$

Quantum Dynamical Systems

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