Fault-Tolerant Quantum Error Correction

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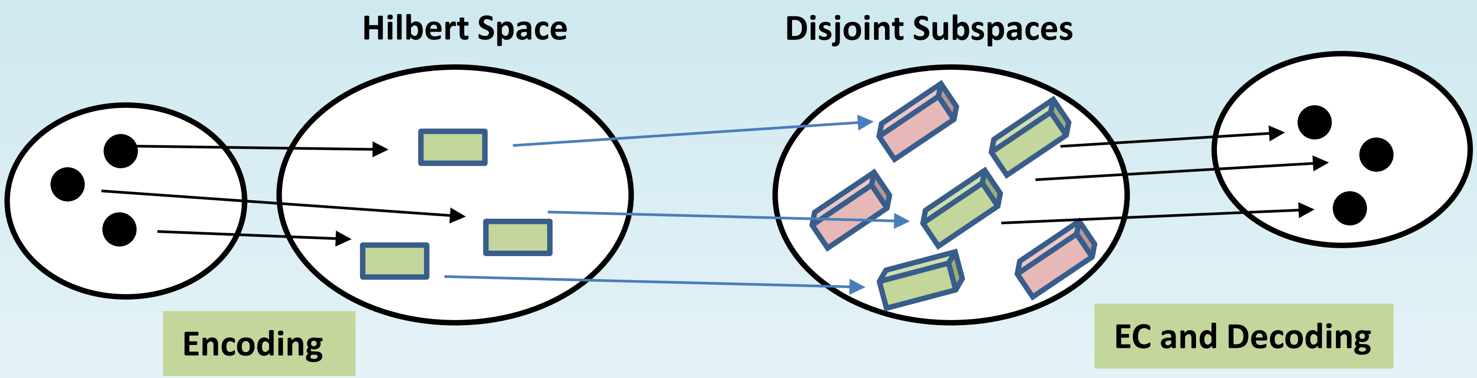
OUTLINE

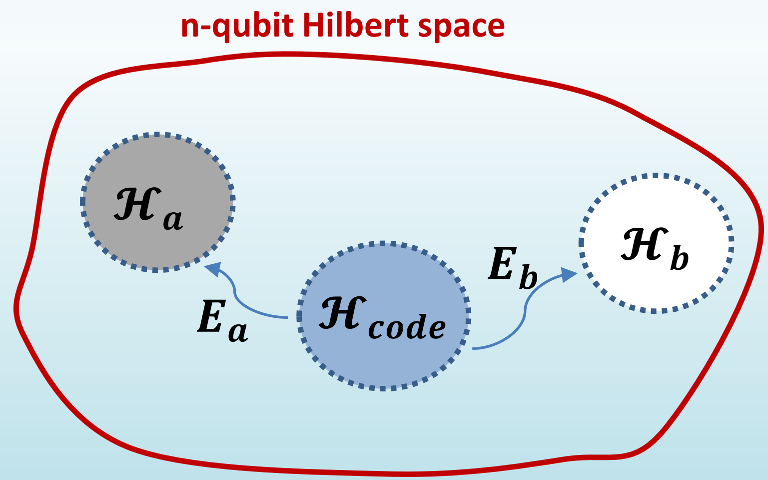
- **Quantum codes**
- **What is fault-tolerance?**
- **Fault-tolerant quantum error correction methods**
- **Universal Fault tolerant quantum computation**

QUANTUM CODES



$$|0\rangle \rightarrow |000\rangle$$
$$|1\rangle \rightarrow |111\rangle$$

$$\mathcal{E} = \{E_a\}$$



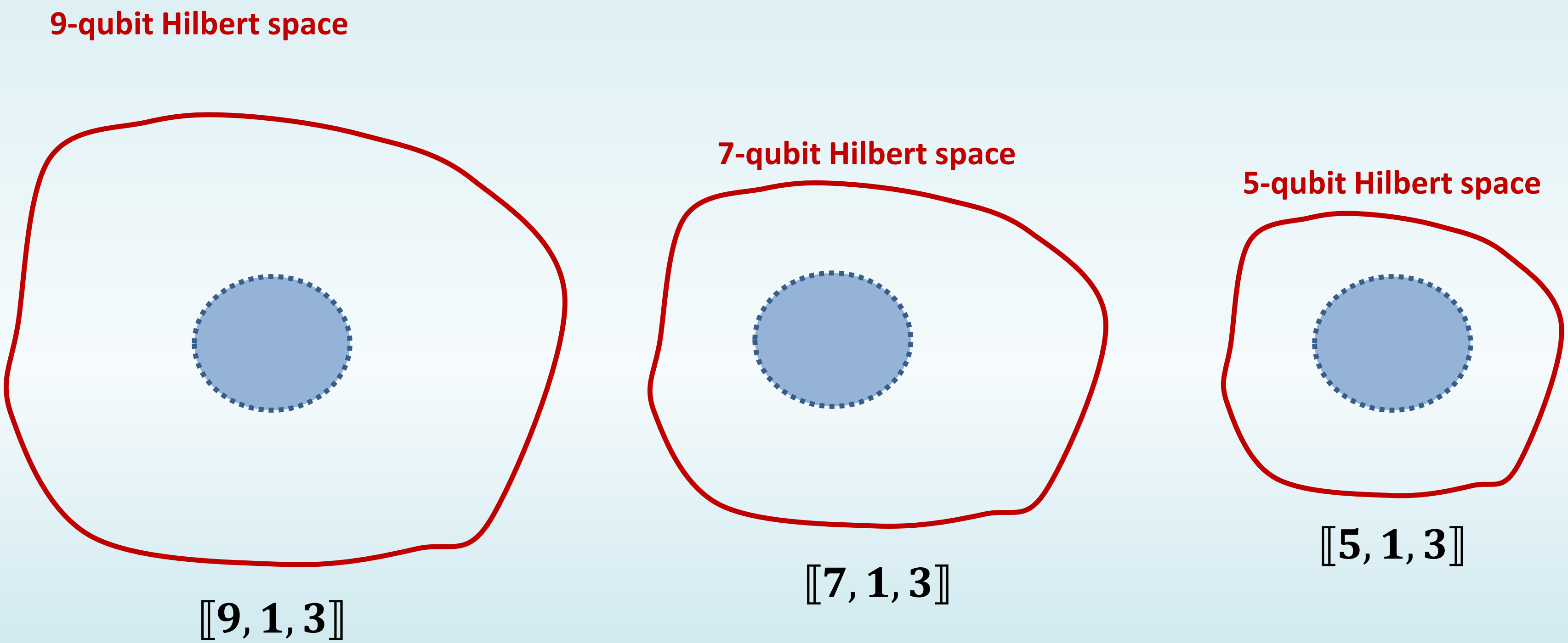
Basis for n − qubit errors :

$$E = O_1 \otimes O_2 \otimes \cdots \otimes O_n \,,\, O_i \in \{\mathbb{I}, X, Y, Z\}$$

QUANTUM CODES

Code parameters : $[\![\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{d}]\!]$ $\quad \boldsymbol{t} = \left[\frac{\boldsymbol{d}-\mathbf{1}}{\mathbf{2}} \right]$

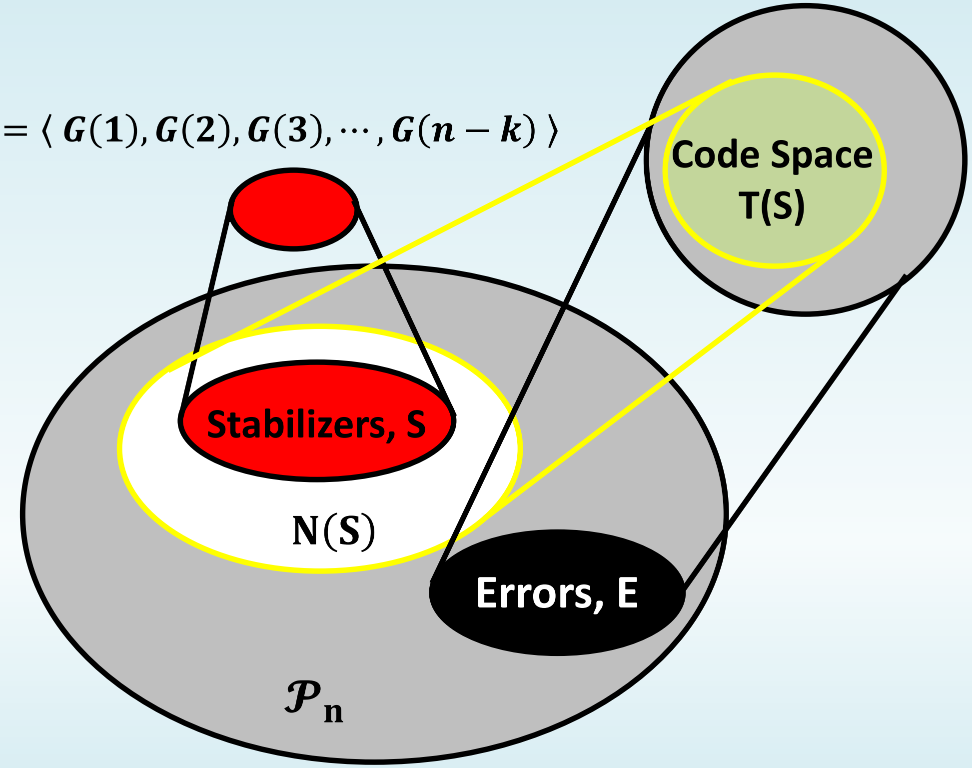
Distance, rate$\left({}^{k}/{}_{n}\right)$, useful for FTQC

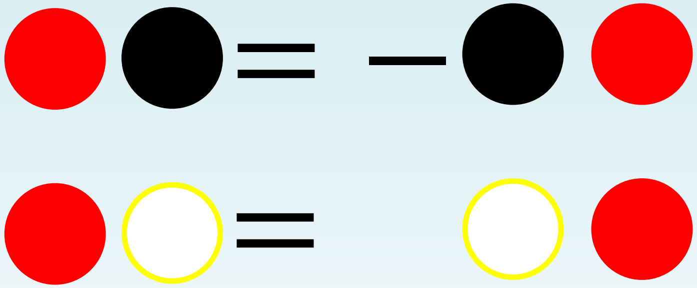


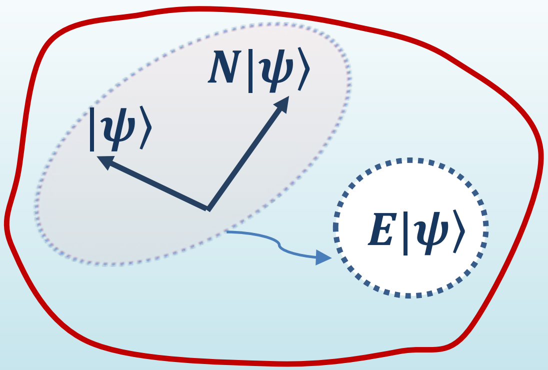
STABILIZER CODES

$$T(S) = \{\, s|\psi\rangle = |\psi\rangle \,\}$$

$$G = \langle\, G(1), G(2), G(3), \cdots, G(n-k) \,\rangle$$



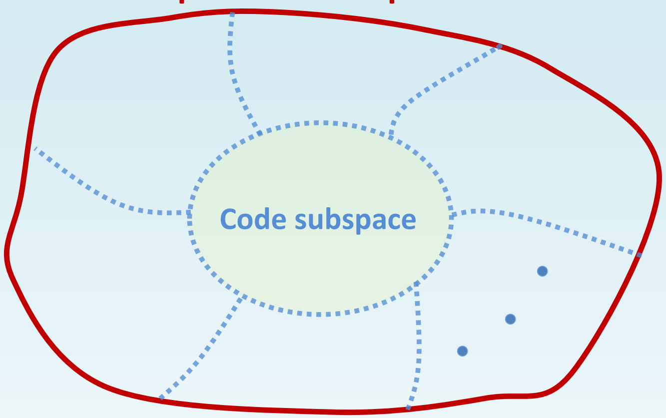




$$\mathcal{P}_\mathrm{n} = c\; O_1 \otimes O_2 \otimes \cdots \otimes O_n$$

STABILIZER CODES

n- qubit stabilizer code: $[\![n, k, d]\!]$



$$G = \langle\, G(1), G(2), G(3), \cdots, G(n-k) \,\rangle$$

$$G^2(i) = \mathbb{1}$$

each subspace $\xrightarrow{\text{LABLE}}$ $e = (e_1, e_2, \cdots, e_{n-k})$

$G(i) \xrightarrow{\text{eigenvalue}} (-1)^{e_i}$

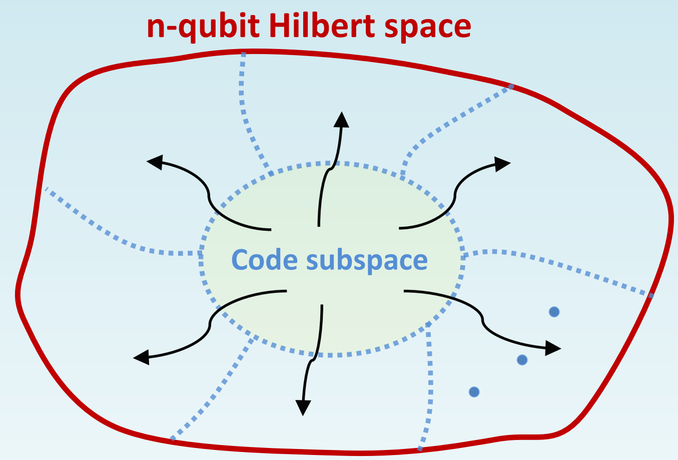
e ≡ subspace syndrome

2^n–D Hilbert space $\xrightarrow{\text{decomposes into } 2^{n-k}}$ disjoint eigenspaces, each of D $=2^k$

HOW TO DETECT ERRORS?

code space syndrome : $\mathbf{e} = (00\cdots 0)$

action of P_a on code space $\xrightarrow{\text{change syndrome}}$ $e = a\Lambda G^T$



$$X\mathbb{I}YZY\mathbb{I} \rightarrow \underbrace{(X\mathbb{I}X\mathbb{I}X\mathbb{I})}_{(101010)}\underbrace{(\mathbb{I}\mathbb{I}ZZZ\mathbb{I})}_{(001110)}$$

$$P_a = \otimes X^{a_i} \otimes Z^{a_{n+i}} \rightarrow \quad a \coloneqq (x_a \vdots z_a)$$

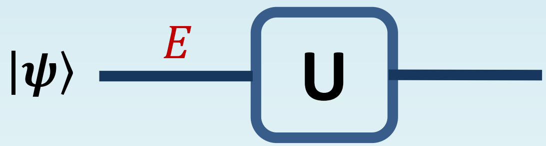
$$x_a \cdot z_b + z_a \cdot x_b = 0$$

$$G(i) = \otimes X^{b_i} \otimes Z^{b_{n+i}} \rightarrow \quad g(i) \coloneqq (x_b \vdots z_b)$$

$$\Lambda = \begin{pmatrix} 0_n & \mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix} \qquad a\Lambda g(i)^T = 0 \qquad G \equiv \begin{pmatrix} g(1) \\ g(2) \\ \vdots \\ g(n-k) \end{pmatrix}$$

WHAT IS FAULT-TOLERANCE?

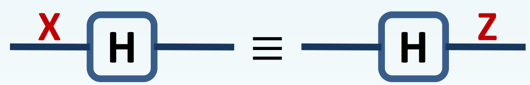
ERROR PROPAGATION :

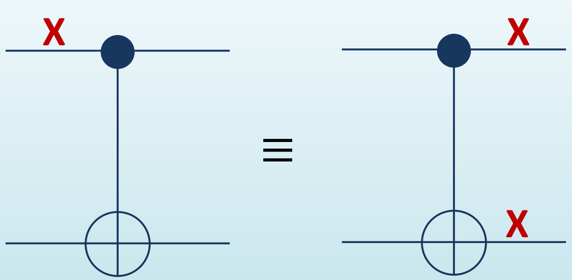


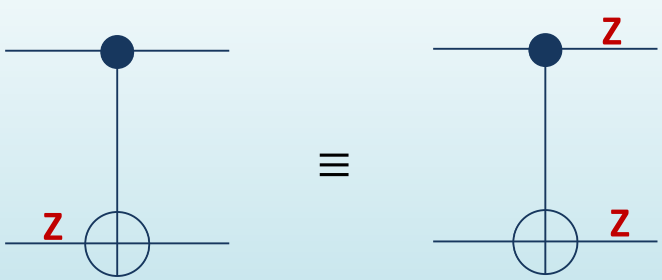
$|\psi'\rangle = U|\psi\rangle$

$$UE|\psi\rangle = UE(U^\dagger U)|\psi\rangle = (UEU^\dagger)|\psi'\rangle$$

$$E \rightarrow UEU^\dagger$$







FAULT-TOLERANCE

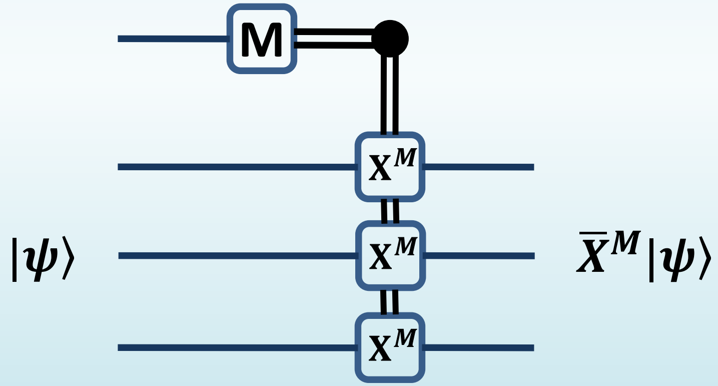
$$\mathbf{X} \otimes \mathbb{I} \rightarrow \mathbf{X} \otimes \mathbf{X}$$

$$\mathbf{Z} \otimes \mathbb{I} \rightarrow \mathbf{Z} \otimes \mathbb{I}$$

$$\mathbb{I} \otimes \boldsymbol{X} \rightarrow \mathbb{I} \otimes \mathbf{X}$$

$$\mathbb{I} \otimes \boldsymbol{Z} \rightarrow \mathbf{Z} \otimes \mathbf{Z}$$

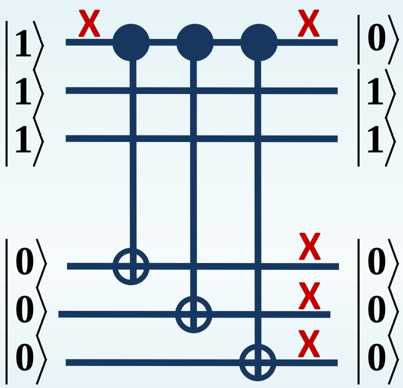
Faulty measurements can cause errors to cascade

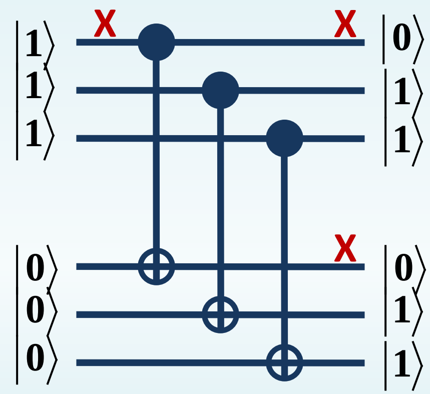


How to avoid cascading?

Transversal Gates

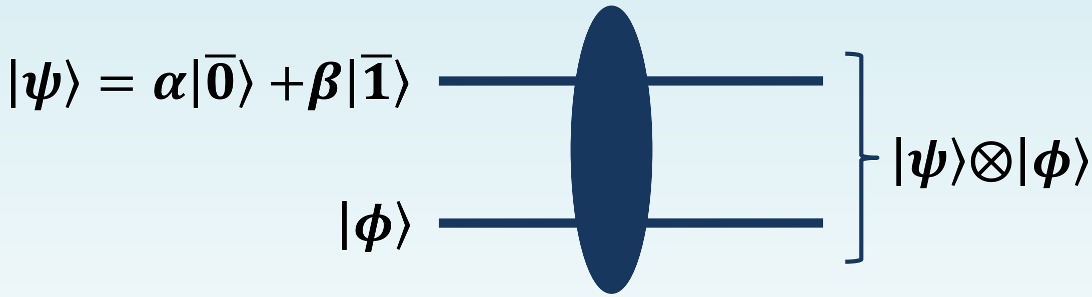
$$|111\rangle|000\rangle \rightarrow |111\rangle|111\rangle$$





FAULT-TOLERANT ERROR CORRECTION METHODS

$$\textbf{Error correction} = \textbf{Syndrome measurement(S.M.)} + \textbf{Recovery step}$$

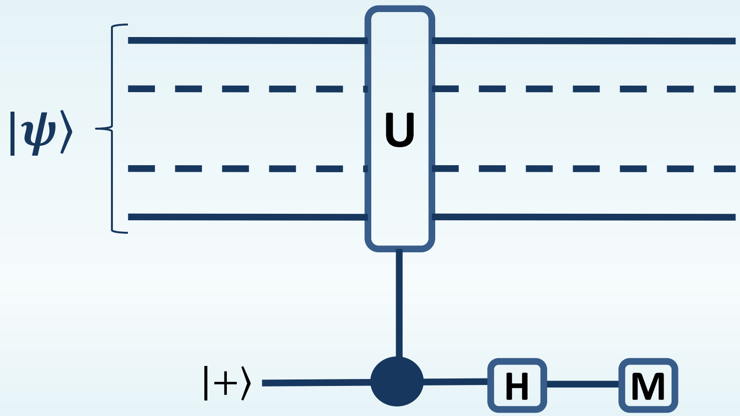


- ✓ **Shor type** **(S.M. using Cat states)**
- ✓ **Steane type** **(S.M. using encoded blocks)**
- ✓ **Knill type** **(S.M via Logical Teleportation)**

SHOR TYPE ERROR CORRECTION

Syndrome measurement using Cat states :

Phase kickback trick :



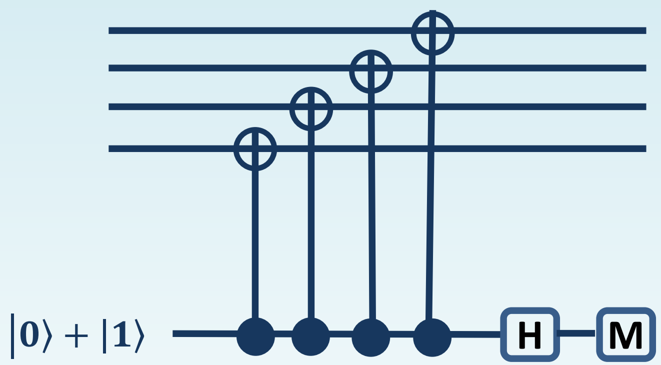
$$|+\rangle|\psi\rangle \xrightarrow{C-U} |0\rangle\otimes|\psi\rangle + |1\rangle\otimes U|\psi\rangle$$

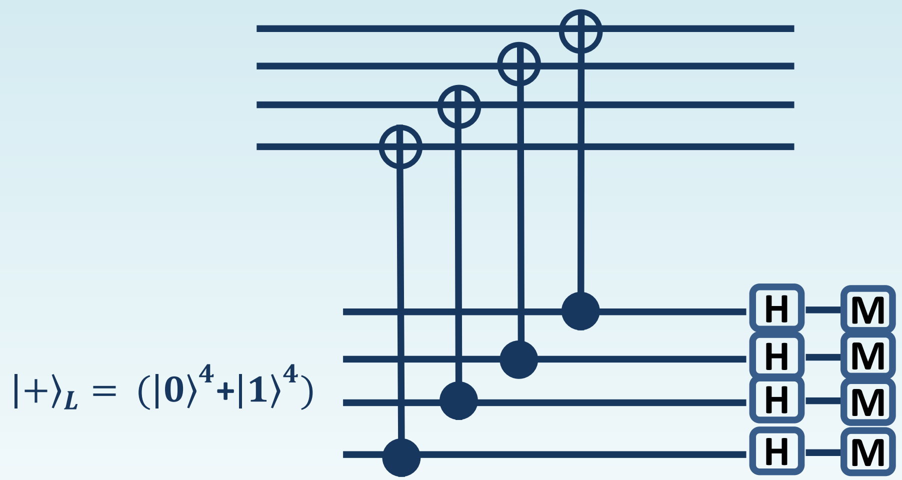
$$= (\,|0\rangle \pm |1\rangle)|\psi\rangle \xrightarrow{H\otimes\mathbb{I}} |0 \; or \; 1\rangle\otimes|\psi\rangle$$

$$G = \langle\, G(1), G(2), G(3), \cdots, G(n-k)\,\rangle$$

SHOR TYPE ERROR CORRECTION

$U = X^{\otimes 4}$





$$H^{\otimes n}(|0\rangle^4 + |1\rangle^4) = \sum_{w(x)=\text{even}} |x\rangle$$

$$H^{\otimes n}(|0\rangle^4 - |1\rangle^4) = \sum_{w(x)=\text{odd}} |x\rangle$$

after the complete syndrome e is measured $\xrightarrow{e = a\Lambda G^T}$ **apply to correct the error**

P.W. Shor, FTQC Proc. 35th Ann. Symp. on Foundations of Computer Science (IEEE Press, Los Alamitos, 1996)
D.P. DiVincenzo, P.W. Shor, FTEC with efficient quantum codes, Phys. Rev. Lett. 77 (15) (1996)

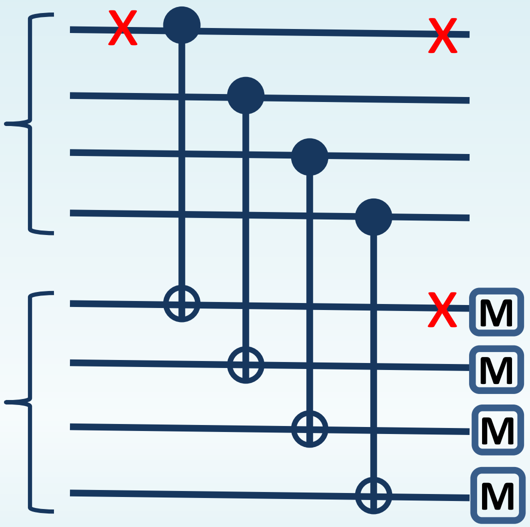
STEANE TYPE ERROR CORRECTION

Ancillas → encoded using the same CSS QECC

To measure Z type generators :

Data block

$|\overline{+}\rangle = |\overline{0}\rangle + |\overline{1}\rangle$



$$Z(b) = \bigotimes_{i=1}^{n} Z^{b_i} \xrightarrow{\text{binary vector}} \boldsymbol{z_b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_n})$$

Measurement result → $\boldsymbol{x_a} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$

Eigenvalue of $\boldsymbol{Z(b) \rightarrow x_a \cdot z_b}$

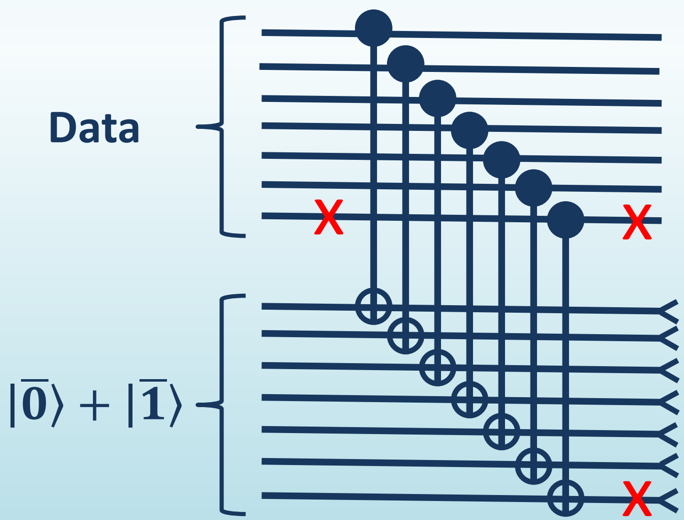
STEANE TYPE ERROR CORRECTION

$7 - \text{qubit code: } [\![7,1,3]\!]$

$$G(1) = \mathbb{I}\mathbb{I}\mathbb{I}XXXX$$
$$G(2) = X\mathbb{I}X\mathbb{I}X\mathbb{I}X$$
$$G(3) = \mathbb{I}XX\mathbb{I}\mathbb{I}XX$$
$$G(4) = \mathbb{I}\mathbb{I}\mathbb{I}ZZZZ$$
$$G(5) = Z\mathbb{I}Z\mathbb{I}Z\mathbb{I}Z$$
$$G(6) = \mathbb{I}ZZ\mathbb{I}\mathbb{I}ZZ$$

$$|\bar{0}\rangle = |0000000\rangle + |0001111\rangle + |0111100\rangle + |1100110\rangle + |0110011\rangle + |1010101\rangle + |1011010\rangle + |1101001\rangle$$

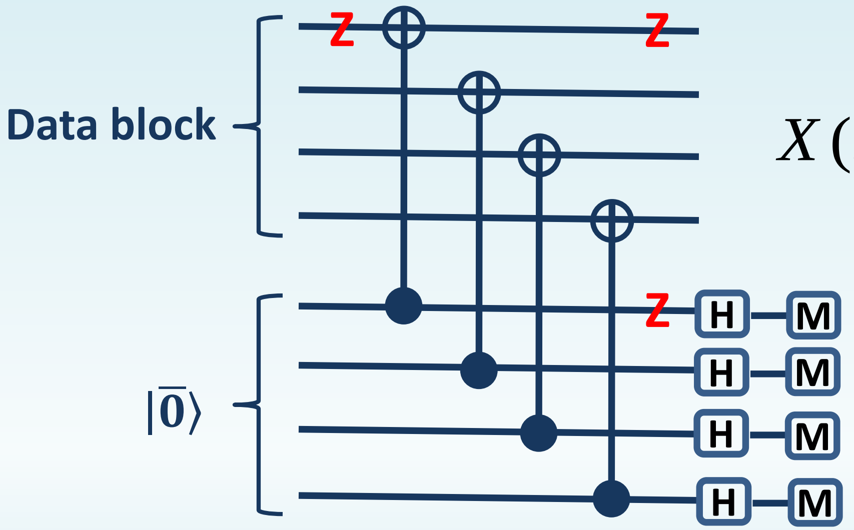
$$|\bar{1}\rangle = |1111111\rangle + |1110000\rangle + |1000011\rangle + |0011001\rangle + |1001100\rangle + |0101010\rangle + |0100101\rangle + |0010110\rangle$$



$$G(4) \xleftrightarrow{\text{Binary vector}} z_b := (0001111)$$

STEANE TYPE ERROR CORRECTION

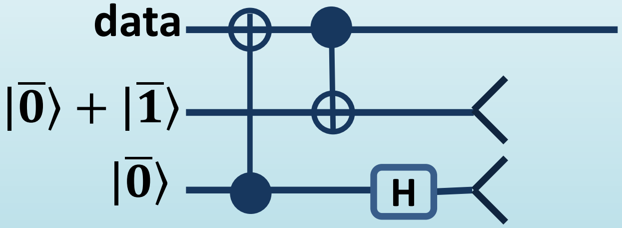
To measure X type generators :



$$X(b) = \bigotimes_{i=1}^{n} X^{b_i} \xrightarrow{\text{binary vector}} \boldsymbol{x_b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \dots, \boldsymbol{b_n})$$

Measurement result $\rightarrow \boldsymbol{z_a} = (\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_n})$

Eigenvalue of $\boldsymbol{X(b)} \rightarrow \boldsymbol{z_a} \cdot \boldsymbol{x_b}$



Universal Fault tolerant quantum computation

U(arbitrary unitary) $\rightarrow$ How does this affect code stabilizers?

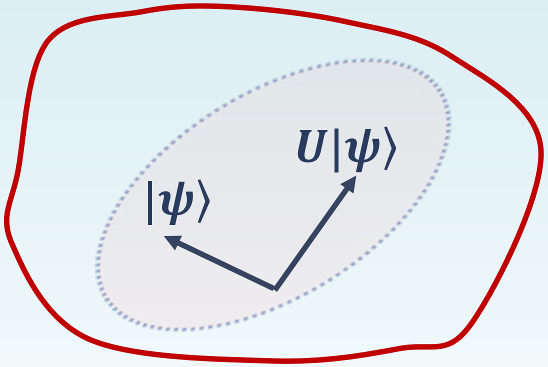
$$S|\psi\rangle = |\psi\rangle$$

$$US|\psi\rangle = U|\psi\rangle$$

$$(USU^{\dagger})U|\psi\rangle = U|\psi\rangle$$

$$USU^{\dagger} \rightarrow S$$

n-qubit Hilbert space



Clifford operators :

$$C_2 = \{\mathrm{U}|\mathrm{UQU}^{\dagger} \in \mathcal{P}_{\mathrm{n}}\} \rightarrow \text{generators : } \{H, P, CNOT\}$$

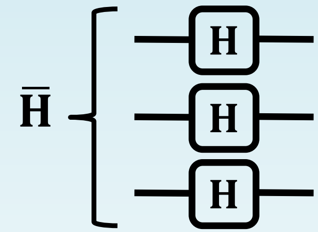
$$H : \begin{cases} \mathrm{X} \rightarrow Z \\ \mathrm{Z} \rightarrow \mathrm{X} \end{cases} \qquad P : \begin{cases} \mathrm{X} \rightarrow \mathrm{iY} \\ \mathrm{Z} \rightarrow \mathrm{Z} \end{cases}$$

Universal gate sets for QC : $\{H, P, CNOT, T_{\pi/8}\}$

F-T IMPLEMENTATION OF CLIFFORD GROUP

Fault-Tolerant Hadamard Gate :

H: switches X and Z $\rightarrow$ Symmetry $\rightarrow$ **self duality**

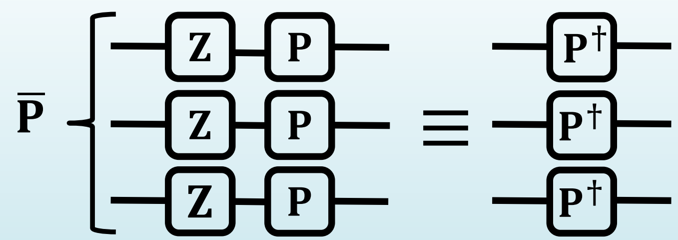


$$G(1) = \mathbb{I}\mathbb{I}\mathbb{I}XXXX \quad G(4) = \mathbb{I}\mathbb{I}\mathbb{I}ZZZZ$$
$$G(2) = X\mathbb{I}X\mathbb{I}X\mathbb{I}X \quad G(5) = Z\mathbb{I}Z\mathbb{I}Z\mathbb{I}Z$$
$$G(3) = \mathbb{I}XX\mathbb{I}\mathbb{I}XX \quad G(6) = \mathbb{I}ZZ\mathbb{I}\mathbb{I}ZZ$$

$$\bar{X} = XXX\mathbb{I}\mathbb{I}\mathbb{I}\mathbb{I}$$
$$\bar{Z} = ZZZ\mathbb{I}\mathbb{I}\mathbb{I}\mathbb{I}$$

Fault-Tolerant Phase Gate :

$$P : \begin{cases} X \rightarrow iY \\ Z \rightarrow Z \end{cases}$$



take care about overal phase $\rightarrow$ **doubly-even codes**

$\bar{P}$ must be a logical phase gate!!

$$PZXZP^\dagger = -iY$$

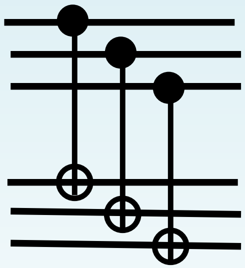
F-T IMPLEMENTATION OF CLIFFORD GROUP

Fault-Tolerant CNOT Gate :

CSS codes $\xrightarrow{\text{are the only codes}}$ for which bitwise CNOT is a valid F–T operation

a generic stabilizer : $S = MN$

$$MN \otimes \mathbb{I} \xrightarrow{CNOT} MN \otimes M$$



$\mathbf{S} \otimes \mathbb{I}$ & $\mathbb{I} \otimes \mathbf{S}$

$\mathbf{S} \otimes \mathbb{I} \rightarrow \mathbf{S} \otimes \mathbf{S}$ if S be X – generator

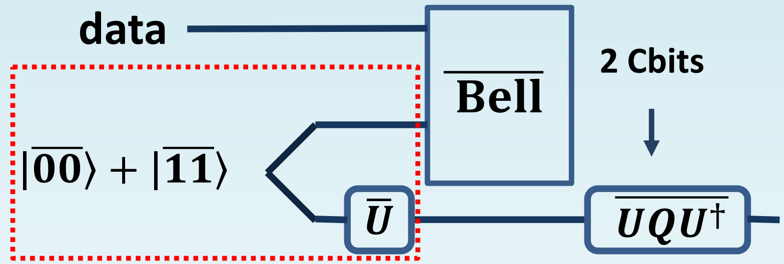
$\mathbb{I} \otimes \mathbf{S} \rightarrow \mathbf{S} \otimes \mathbf{S}$ if S be Z – generator

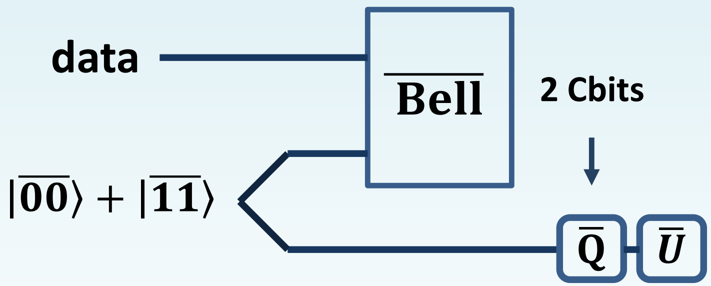
$$\bar{X} \otimes \mathbb{I} \rightarrow \bar{X} \otimes \bar{X}$$
$$\bar{Z} \otimes \mathbb{I} \rightarrow \bar{Z} \otimes \mathbb{I}$$
$$\mathbb{I} \otimes \bar{X} \rightarrow \mathbb{I} \otimes \bar{X}$$
$$\mathbb{I} \otimes \bar{Z} \rightarrow \bar{Z} \otimes \bar{Z}$$

Doubly-even self-dual CSS codes → The best class of codes for FTQC

GATE TELEPORTAATION AND UNIVERSAL FTQC

$$T_{\pi/8}: \begin{cases} X \rightarrow e^{i\pi/4} XP^{\dagger} \\ Z \rightarrow Z \end{cases}$$





$$(\mathbb{I}\otimes\overline{U})(|\overline{00}\rangle + |\overline{11}\rangle)$$

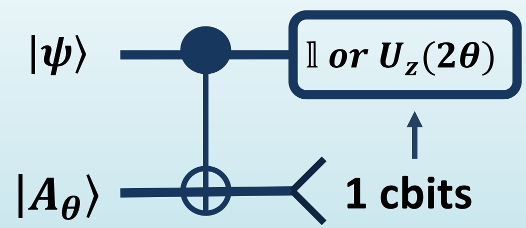
$$(\mathbb{I}\otimes\overline{T_{\pi/8}})(|\overline{00}\rangle + |\overline{11}\rangle)$$

More efficient gate teleportation

$$U_z(\theta) = \exp(-i\frac{\theta}{2}Z)$$

$$|A_\theta\rangle = U_z(\theta)|+\rangle$$

$$T_{\pi/8} = U_z(\pi/4)$$



D. Gottesman & I. Chuang. Viability of universal QC using telep. and 1-qubit operations. Nature, 402:390–393,(199
X. Zhou, D. Leung, and I. Chuang. Methodology for quantum logic gate construction. PRA, 62,052316(2000)

THANKS