Heisenberg's Error Disturbance Relation (EDR)

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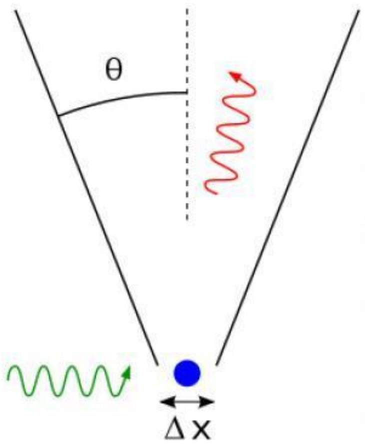
Outline

- Heisenberg-Bohr’s Microscope
- The difference between preparation variances and EDR.
- Werner’s approach to prove the EDR.
- Ozawa’s error disturbance relation and disproving the EDR.
- Experimental test of EDR.

Heisenberg-Bohr's microscope



- **The uncertainty in position is the smallest distance at which two bodies might be distinguished.**

$$\epsilon(x) = \frac{\lambda}{2\sin\theta}$$

- **The disturbance in momentum of the electron is at list equal to the uncertainty in x component of photons momentum.**

$$\eta(p_x) = p\sin\theta$$

What is EDR exactly?

- **It's impossible to measure the position of a mass without disturbing it's momentum,**

$\epsilon(A)$ **Is the error in measurement, and not the variance of distribution.**

$$\epsilon(A)\eta(B) \geq \frac{\left|\langle [A,B] \rangle\right|}{2} = \frac{\hbar}{2}.$$

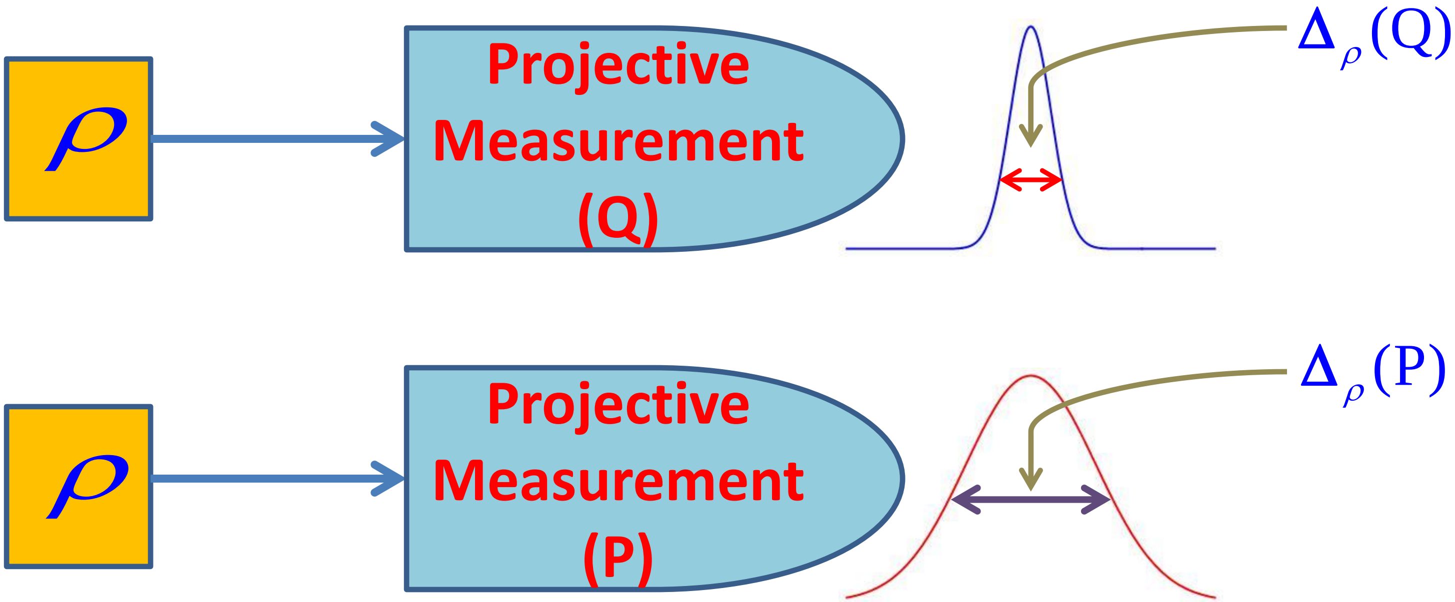
- **It's different from the inequality relation between the variances of A and B,**

$$\sigma(A)\sigma(B) \geq \frac{\left|\langle [A,B] \rangle\right|}{2} = \frac{\hbar}{2}.$$

$$\sigma(A) = \left\langle \left(A - \langle A \rangle\right)^2 \right\rangle^{1/2}$$

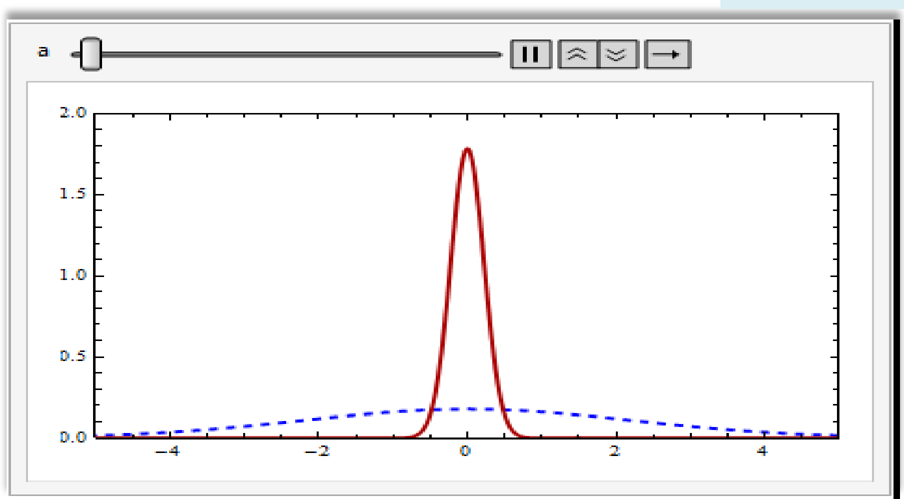
This inequality does not relate to the measurements.

- **It's impossible to prepare a state which has arbitrarily sharp distribution of position and momentum. (There's no simultaneous measurement here.)**

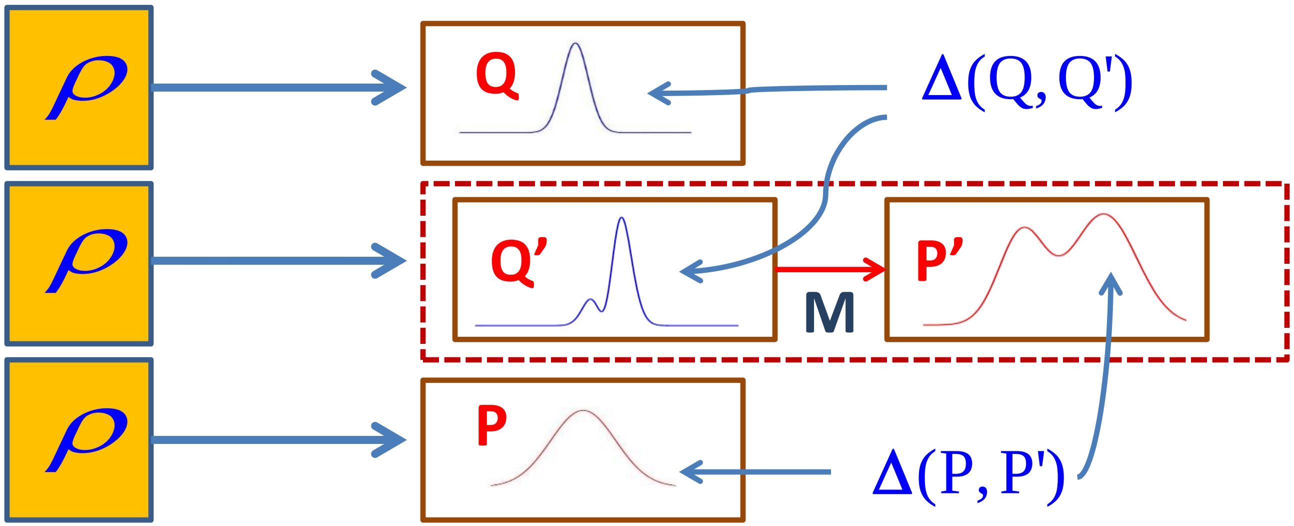


- **No matter what the initial state is, we have:**

$$\Delta_\rho(Q)\Delta_\rho(P) \geq \frac{\hbar}{2}$$



How to define error and disturbance?
Werner's approach

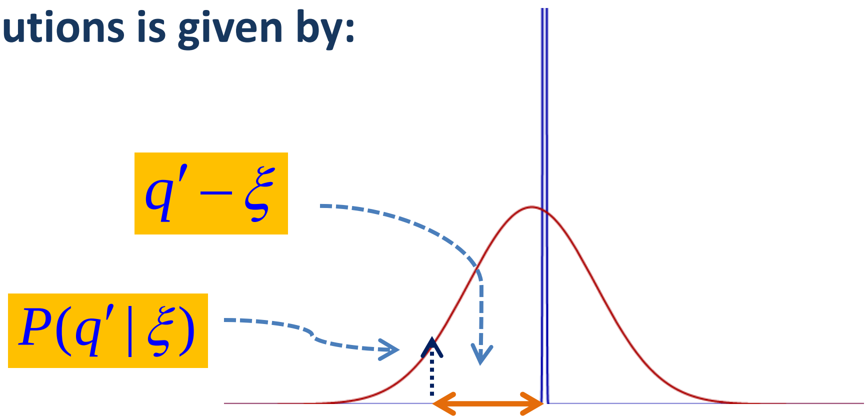


- **Error in position measurement is the difference of the two distributions obtaining from Q' and Q.**
- **Disturbance on momentum caused by position measurement is the difference of two exact momentum measurements.**

Quantifying error and disturbance

Let's consider the distributions which are sharply distributed around a position

Value ξ **, the difference of the two distributions is given by:**



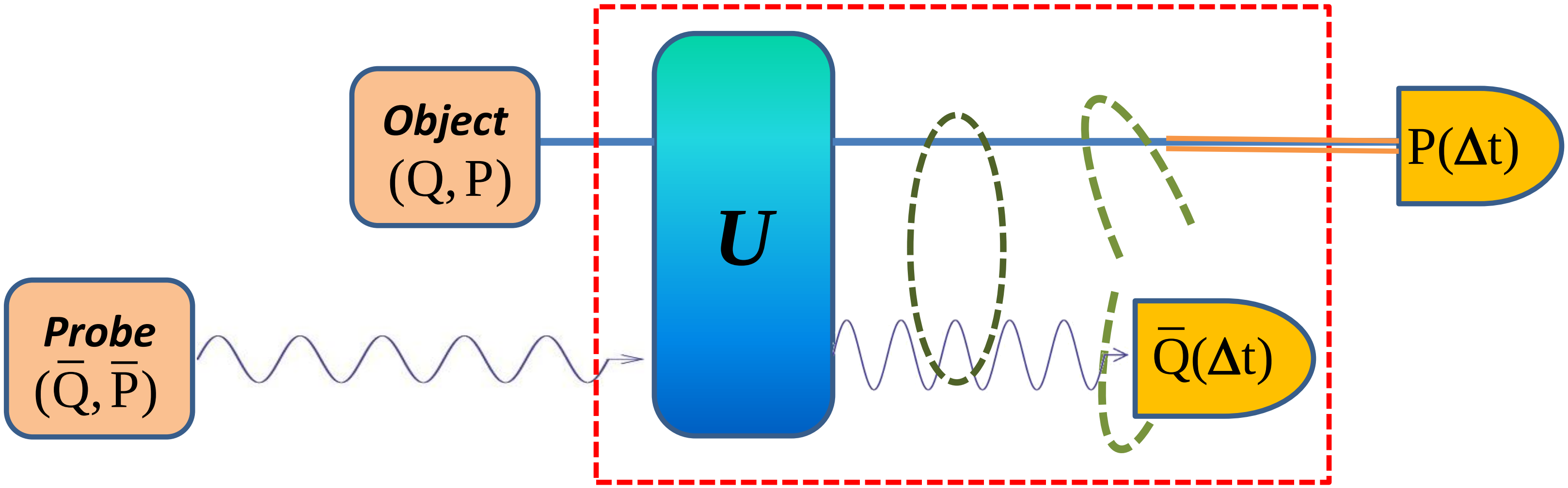
$$D(\rho, \mathrm{Q}'; \xi) = \left\langle (q' - \xi)^2 \right\rangle_{\rho, \mathrm{Q}'}^{1/2}$$

$$\boldsymbol{\Delta}_c(\mathrm{Q}, \mathrm{Q}') = \lim_{\epsilon \to 0} \left[\sup \left\{ D(\rho, \mathrm{Q}'; \xi) \mid \rho, \xi; D(\rho, \mathrm{Q}; \xi) \leq \epsilon \right\} \right]$$

$$\boldsymbol{\Delta}_c(\mathrm{P}, \mathrm{P}') = \lim_{\epsilon \to 0} \left[\sup \left\{ D(\rho', \mathrm{P}'; \zeta) \mid \rho', \zeta; D(\rho, \mathrm{P}; \zeta) \leq \epsilon \right\} \right]$$

$$\boldsymbol{\Delta}_c(\mathrm{Q}, \mathrm{Q}') \boldsymbol{\Delta}_c(\mathrm{P}, \mathrm{P}') \geq \frac{\hbar}{2}$$

Ozawa's EDR relation



- **To avoid physically inaccessible resources to break Heisenberg's EDR:**

At t=0, for both probe and the system wave functions.

$$\left\| \bar{Q}^m \bar{P}^n \xi \right\| < \infty \;, \quad \forall \{m,n\}$$

- **Gauss introduced the *mean square error and disturbance* as:**

$$\epsilon(Q,\psi) = \epsilon_G\left(Q(0)\,, Q(\Delta t)\right)$$

$$\epsilon_G(\boldsymbol{\Theta}\,, \boldsymbol{\Omega}) = \left(\int\int_{R^2} (\omega-\theta)^2 \mu^{\boldsymbol{\Theta}\,,\boldsymbol{\Omega}}(d\theta, d\omega)\right)^{1/2}$$

$$\eta(P,\psi) = \epsilon_G\left(P(0)\,, P(\Delta t)\right)$$

- $\boldsymbol{\Theta} = \theta$ **is the true value and** $\boldsymbol{\Omega} = \omega$ **is the measured value.**

- **Let's suppose the interaction Hamiltonian to be:**

$$H(\alpha,\beta,\gamma)=\alpha\left(\mathrm{QP}-\overline{\mathrm{QP}}\right)+\beta\overline{\mathrm{Q}}\mathrm{P}+\gamma\mathrm{Q}\overline{\mathrm{P}}$$

- **Solving Heisenberg's equations of motion:**

$$\mathrm{Q}(\Delta t)=a\,\mathrm{Q}(0)+b\,\overline{\mathrm{Q}}(0) \quad , \quad \overline{\mathrm{Q}}(\Delta t)=c\,\mathrm{Q}(0)+d\,\overline{\mathrm{Q}}(0)$$

$$\mathrm{P}(\Delta t)=d\,\mathrm{P}(0)-c\,\overline{\mathrm{P}}(0) \quad , \quad \overline{\mathrm{P}}(\Delta \mathrm{t})=-\mathrm{b}\,\mathrm{P}(0)+a\,\overline{\mathrm{P}}(0)$$

- **The error at this case would be:**

$$\epsilon(\mathrm{Q},\psi)=\left\|\left[(c-1)\mathrm{Q}(0)+d\,\overline{\mathrm{Q}}(0)\right]\left(\psi\otimes\xi\right)\right\|<\infty$$

- **Root mean square disturbance:**

$$\eta(\mathrm{P},\psi)=\left\|\left[(d-1)\mathrm{P}(0)-c\,\overline{\mathrm{P}}(0)\right]\left(\psi\otimes\xi\right)\right\|<\infty$$

- **Error Disturbance Relation:**

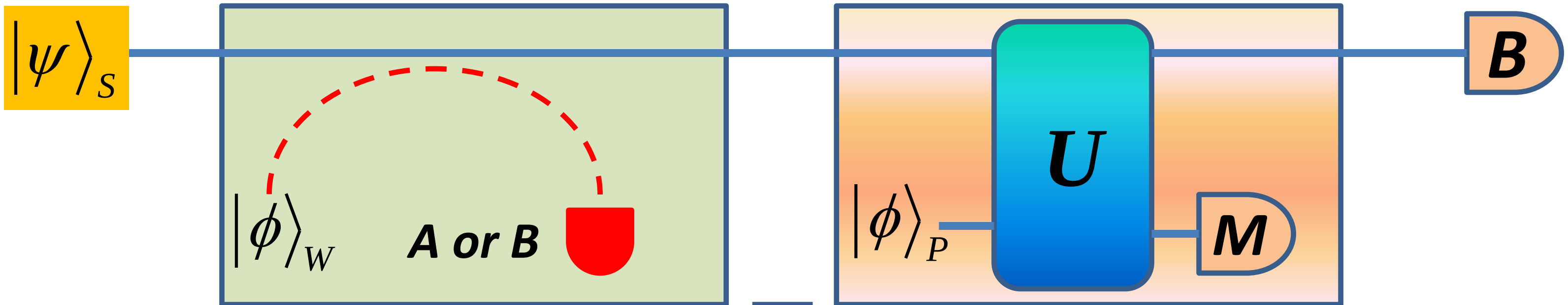
$$\epsilon(\mathrm{Q},\psi)\eta(\mathrm{P},\psi) \geq \frac{|1-c-d|\hbar}{2}$$

- **Error free measurements:**

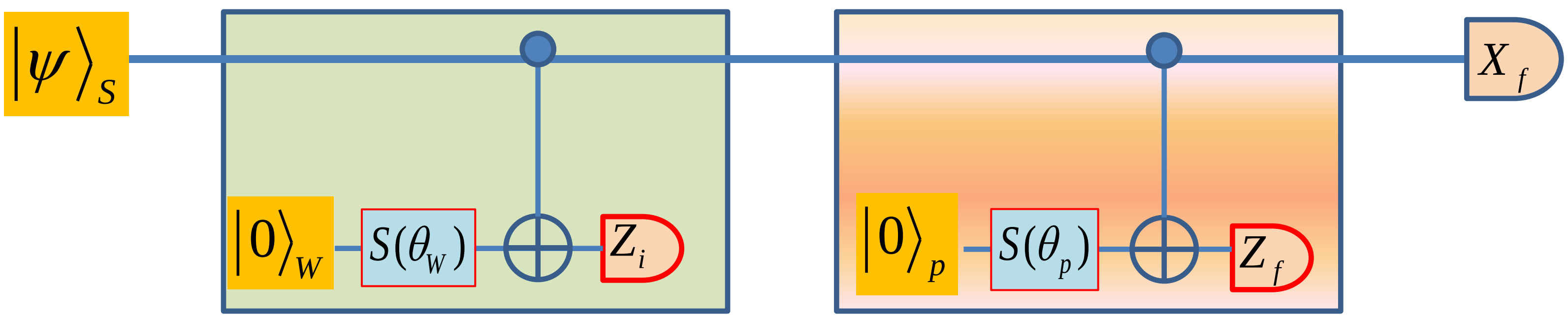
$$b=-1\ ,\ c=1\ ,\ d=0\ ,\ a>-2$$

$$\begin{cases} \epsilon(\mathrm{Q},\psi)=0 \\ \eta(\mathrm{P},\psi)<\infty \end{cases} \Rightarrow \epsilon(\mathrm{Q},\psi)\eta(\mathrm{P},\psi)=0$$

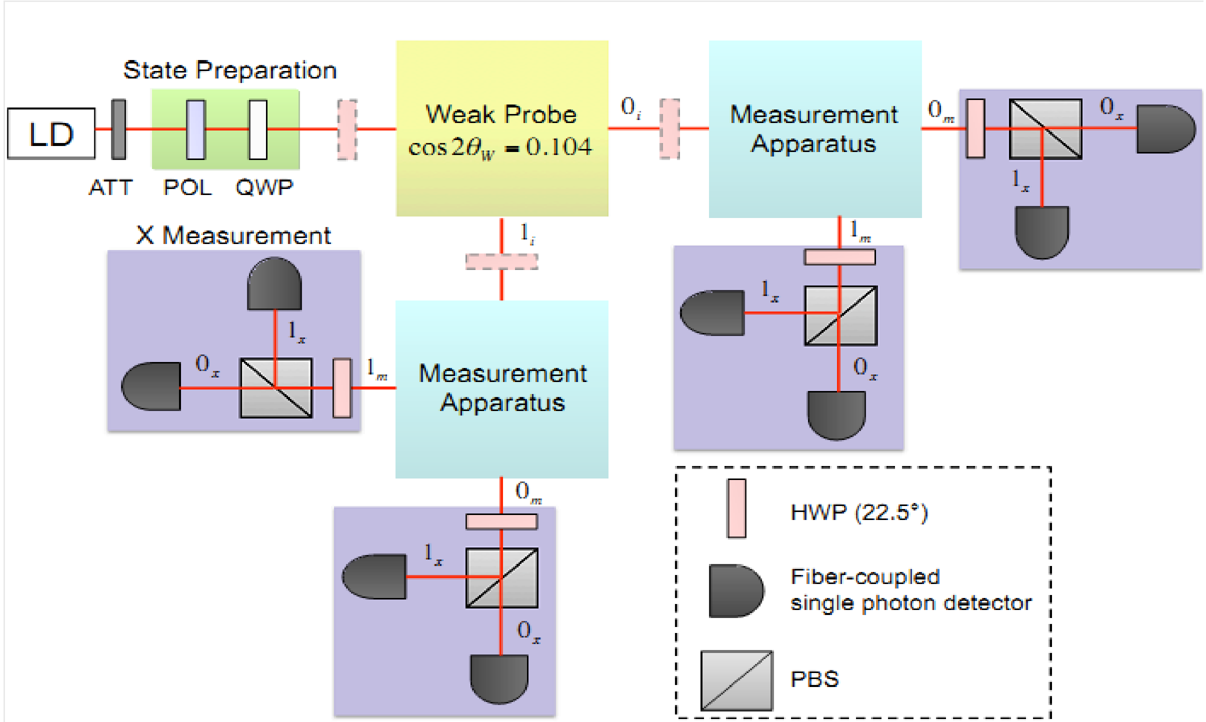
Experimental test of EDR



Spin measurement in directions X and Z:



$$S(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\sin\theta\end{pmatrix}\;,\quad E_{Z_f}=\frac{1}{2}\left(I\pm(\cos 2\theta)Z\right)$$

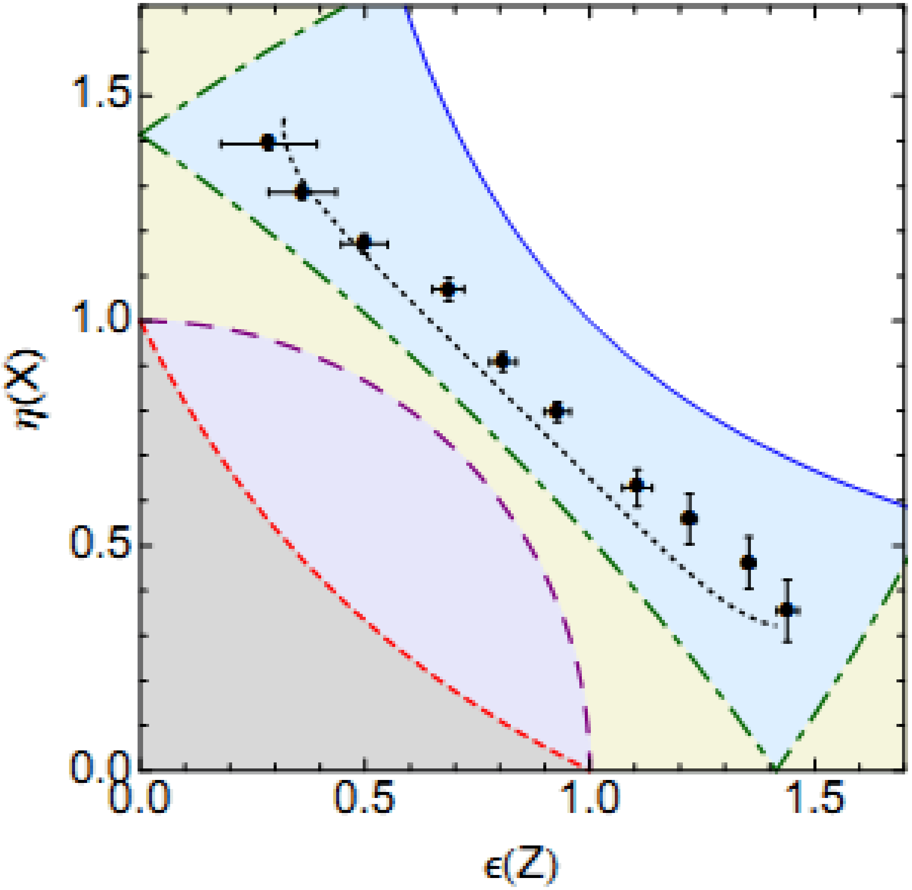


The error in measurement of Z, is given by:

$$\epsilon(Z)^2 = 2\left(1 - \frac{1}{\cos\theta_w}\sum_{i,f} z_f z_i P\left(z_f, z_i\right)\right)$$

$$P\left(z_f = 1, z_i = 1\right) = \sum_k N_{00k} \Big/ \sum_{i,k} N_{i0k}$$

The disturbance of X, is found by substituting x_f and x_i instead of z_f and z_i .



Blue: Heisenberg’s bound---Red (short dashed): Ozawa’s bound---purple (long dashed) and green (dot-dashed): Branciard’s bound---black dot curve: theoretical prediction---Black (filled) circles: experimental data.

Appleby's Formulation for EDR

$$\bar{\epsilon}(Q) = \sup_{\psi} \epsilon(Q,\psi) \ , \ \bar{\eta}(P) = \sup_{\psi} \eta(P,\psi)$$

- **Using This relation Appleby claimed that:**

$$\bar{\eta}(P)\bar{\epsilon}(Q) \geq \frac{\hbar}{2}$$

- **Appleby's formulation does not work for error-free measurements!**

$$\begin{cases} \bar{\epsilon}(Q) = 0 \\ \bar{\eta}(P) \to \infty \end{cases} \Rightarrow \bar{\epsilon}(Q)\bar{\eta}(P) = ?$$

- **Since we can check only a limited set of ψ; the last term is practically zero.**

- **The error and disturbance introduced by Appleby are almost always infinite.**

$$\bar{\epsilon}(\mathrm{Q}) = \begin{cases} d\left\|\bar{\mathrm{Q}}\xi\right\| & , \quad c = 1 \\ \infty & , \quad c \neq 1 \end{cases} \qquad \bar{\eta}(\mathrm{P}) = \begin{cases} d\left\|\bar{\mathrm{P}}\xi\right\| & , \quad d = 1 \\ \infty & , \quad d \neq 1 \end{cases}$$

- **For linear measurements Ozawa showed that Werner's definition of error and disturbance is the same as that of Appleby's.**

Thank you

Red

Blue

Purple

Orange

White