# Transition in $Z_d$ Kitaev model

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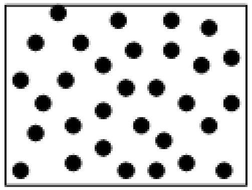
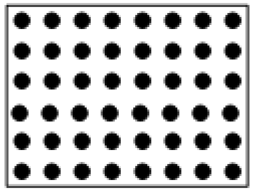
# Phase Transition

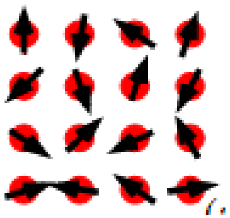
**Landau symmetry-breaking theory(1930-1980)**

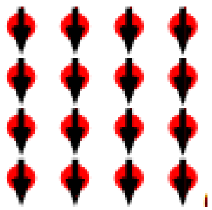
different phases $\equiv$ different symmetry





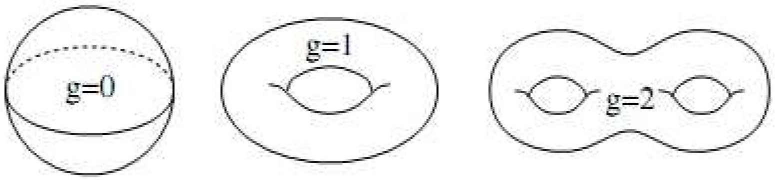
- Liquid → solid
- Paramagnetic → ferromagnetic





# Topological order

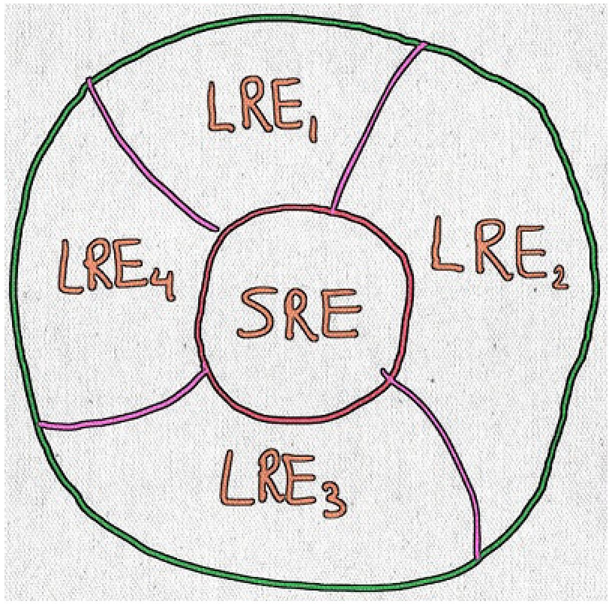
- No symmetry breaking.
- No local order parameter.
- There is a gap.
- Long range entanglement.
- Degeneracy depends on the topology of space.




- Degeneracy cannot be lifted by any local perturbations.

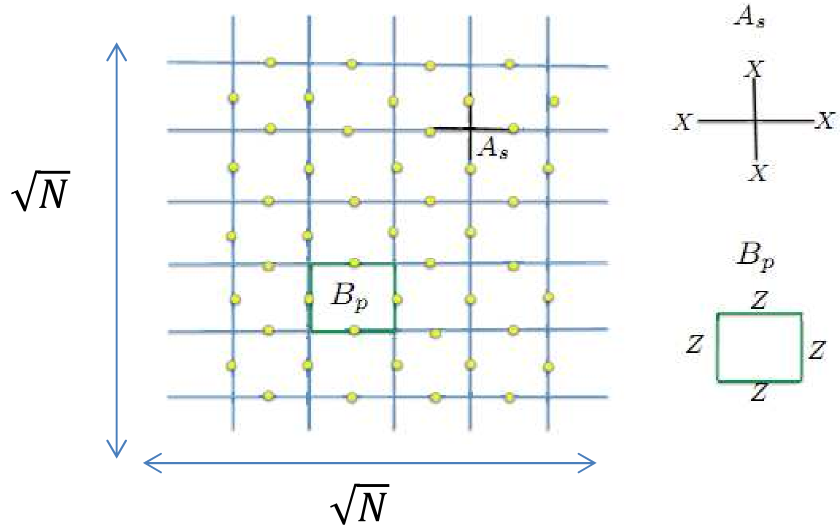
# Local Unitary Transformation and Long Range Entanglement

$$|\phi\rangle \sim |\psi\rangle \quad \longleftrightarrow \quad |\phi\rangle = U_{local}|\psi\rangle$$




Topological Classes

# Toric code




$$\mathcal{L} = \{|\psi\rangle \in \mathcal{H}:\ A_s|\psi\rangle = |\psi\rangle,\ B_p|\psi\rangle = |\psi\rangle \quad for\ all\ s\ and\ p\}$$

$$H = -J\sum_s A_s - K\sum_p B_p$$

# Coding

$$\text{Torus}\begin{cases}\prod_s A_s = 1 \\ \\ \prod_p B_p = 1\end{cases}$$

$$\sqrt{N} \times \sqrt{N} squar\ lattice \begin{cases} Number\ of\ independent\ stabalizers & : 2N - 2 \\ Number\ of\ qubits & : 2N \end{cases}$$

$$\text{Degeneracy} = \frac{2^{2N}}{2^{2N-2}} = 4$$

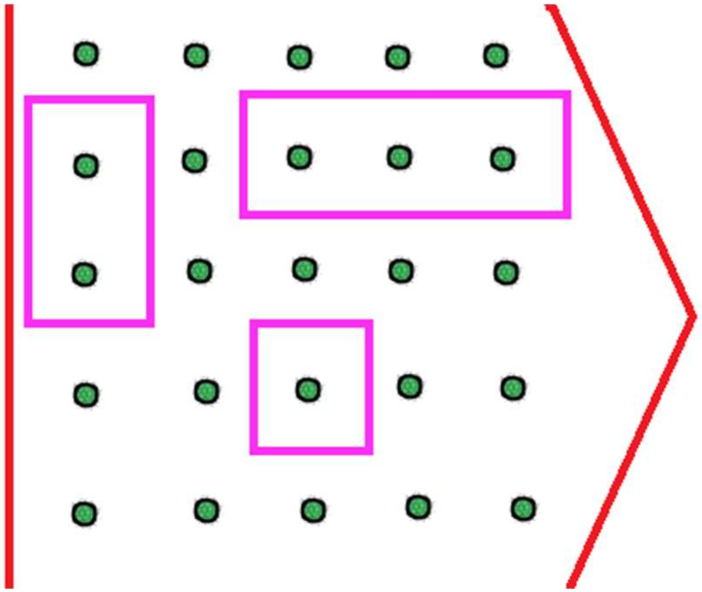
$$|\psi_{00}\rangle = |00\rangle, \qquad |\psi_{01}\rangle = |01\rangle$$

$$|\psi_{10}\rangle = |10\rangle, \qquad |\psi_{11}\rangle = |11\rangle$$

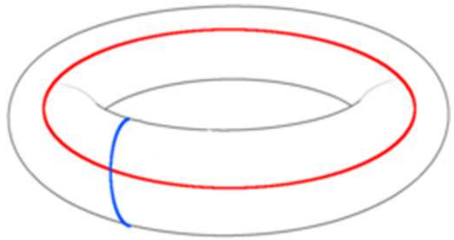
# Ground States

$$|\psi_{00}\rangle = \prod_s (1 + A_s)|0\rangle^{2N}$$

$$= \sum_{c_i}$$

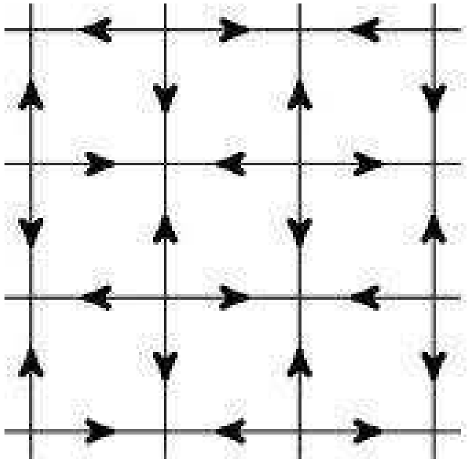


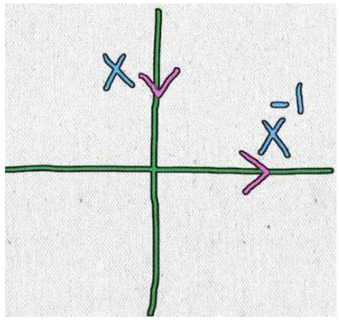
$$|\psi_{ij}\rangle = T_{x,1}{}^{i} T_{x,2}{}^{j} |\psi_{00}\rangle, \qquad i,j = 0,1$$



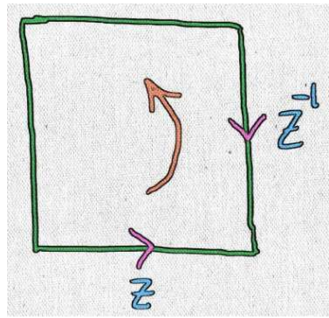
# $Z_d$ Kitaev model,Why?

# What is $Z_d$ Kitaev model?






$A_s$




$B_p$

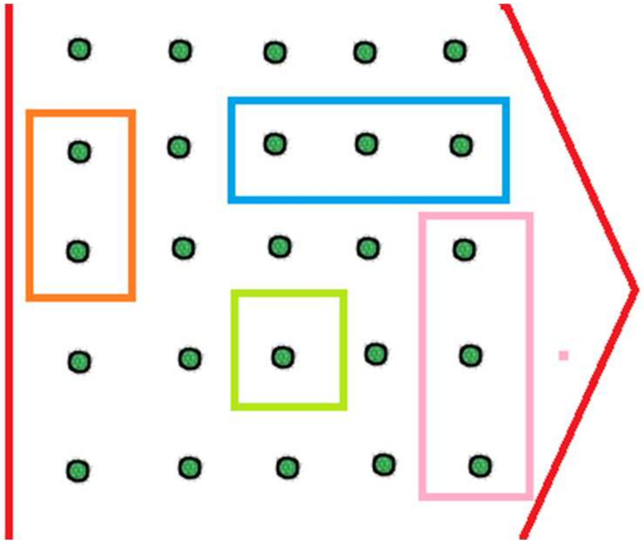
$$H = -J \sum_s (A_s + {A_s}^\dagger) - K \sum_p (B_p + {B_p}^\dagger)$$

Torus $\begin{cases} \prod_s A_s = 1 \\ \prod_p B_p = 1 \end{cases}$ ⇒ Degeneracy: $\frac{d^{2N}}{d^{2N-2}} = d^2$

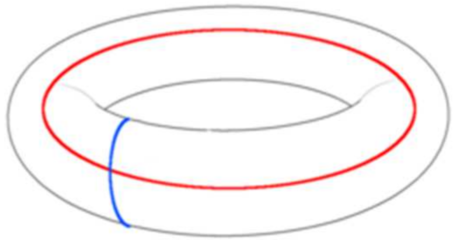
# Ground States

$$|\psi_{00}\rangle = \prod_p (1 + A_s + {A_s}^2 + \cdots + {A_s}^{d-1})|0\rangle^{2N}$$

$$= \sum_{c_i}$$



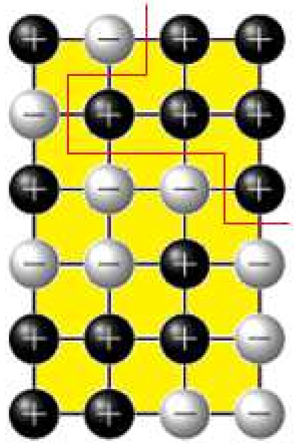
$$|\psi_{ij}\rangle = {T_{x,1}}^i {T_{x,2}}^j |\psi_{00}\rangle \quad i,j = 0, \ldots, d-1$$



# Classical Potts model(1952)

$$H_{Ising} = -\frac{1}{2}\sum_{\langle i,j\rangle}(1 + S_i S_j)$$

$$S_i = 1, -1$$

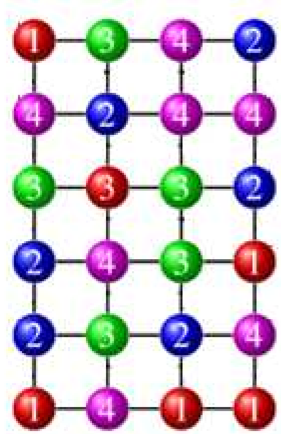


$$= -\sum_{\langle i,j\rangle} \delta_{s_i,s_j}$$

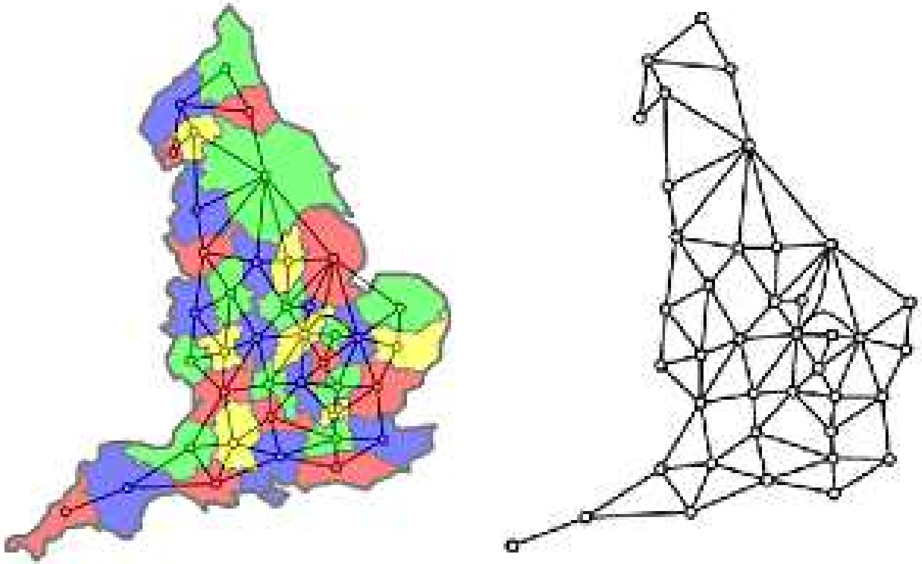
$$H_{Potts} = -\sum_{\langle i,j\rangle} \delta_{s_i,s_j}$$

$$= -\frac{1}{d}\sum_{\langle i,j\rangle}\sum_{r=0}^{d-1}(s_i s_j{}^*)^r$$

$$S_i = 1, \omega, \omega^2, \omega^3, \ldots, \omega^{d-1}$$



# An open problem(1852-1976)



Chromatic polynomial: $P(G, q)$

Chromatic number      :$\chi(G) = \min(q; P(G, q > 0)$

$P_G(4) > 0$      for any planar graph

# Potts model and 4-color problem

$$H_{Potts} = -J \sum_{\langle i,j \rangle} \delta_{s_i,s_j}$$

$$Z(\mathcal{J} \to -\infty) = \sum_{\{\sigma_i\}} e^{\mathcal{J} \sum_{\langle i,j \rangle} \delta_{s_i,s_j}} = P(G,q)$$

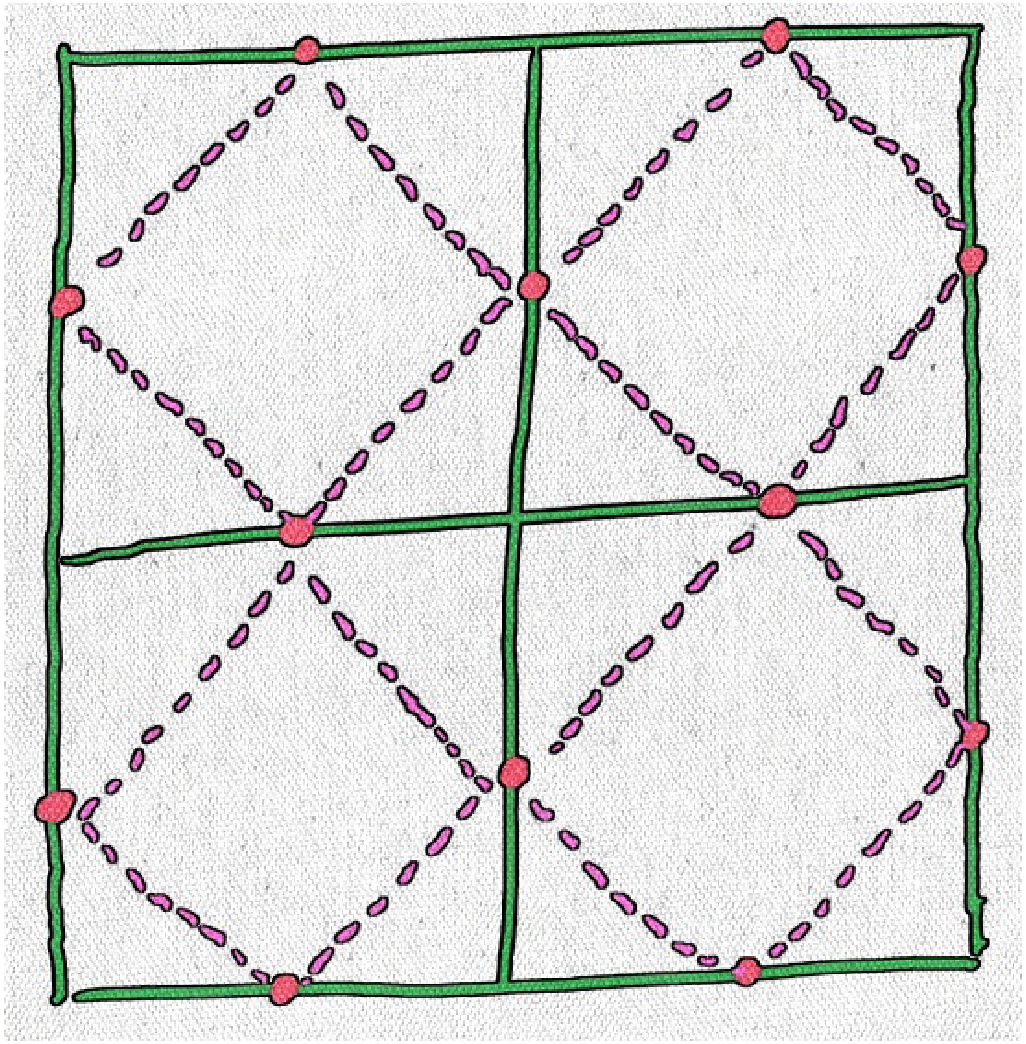
# Quantum Potts model

$$H_P = -\sum_{\langle i,j\rangle}\sum_{r=0}^{d-1}\left((Z_i Z_j{}^{\dagger})^r + (Z_i{}^{\dagger} Z_j)^r\right)$$

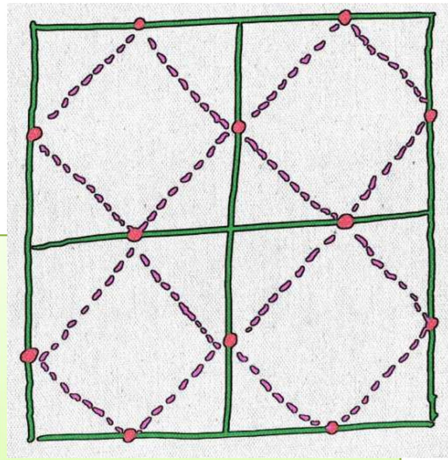
$$|ground\ state\rangle = \begin{cases} |0\rangle^{\otimes N} \\ |1\rangle^{\otimes N} \\ |2\rangle^{\otimes N} \\ \dots \\ |d-1\rangle^{\otimes N} \end{cases}$$

→ degeneracy=d

# Our problem



# Our problem



$$H = H_{Kitaev} + \lambda\, H_{Potts}$$

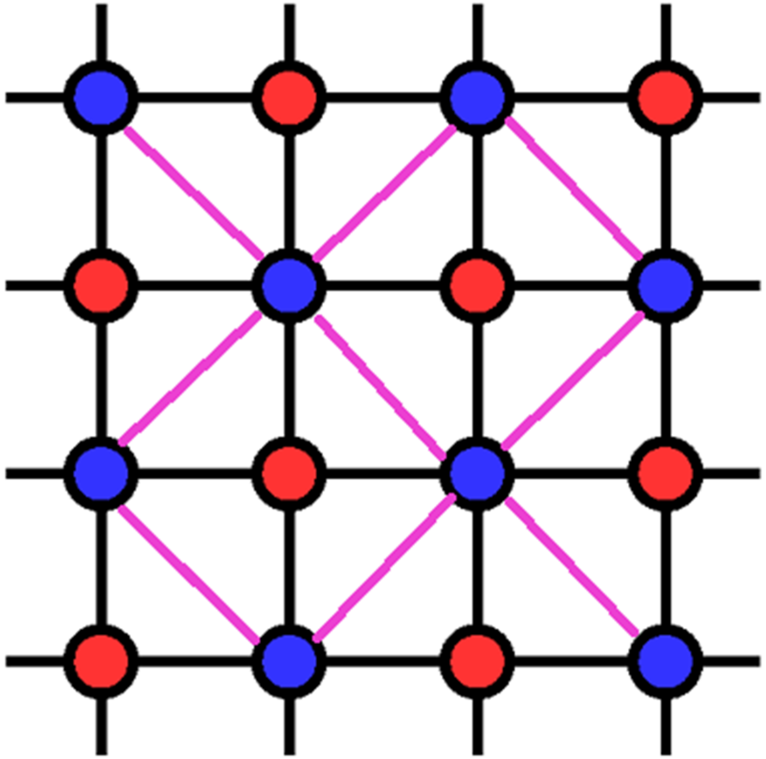
$$H = -J\ \sum_s (A_s + {A_s}^\dagger) - K\ \sum_p\ (B_p + {B_p}^\dagger) - \lambda \sum_{\langle i,j\rangle} \sum_{r=0}^{d-1} ((Z_i {Z_j}^\dagger)^r + ({Z_i}^\dagger Z_j)^r)$$

| $\lambda$ | $0$ | $\infty$ |
|---|---|---|
| Degeneracy | $d^2$ | $d$ |
| Order | topologica | erromagnetic order |

**Phase transition**

# Potts model in magnetic field

$$\widetilde{H_A} = -J\sum_i X_i + X_i{}^\dagger - \frac{\lambda}{d}\sum_{\langle i,j\rangle}\sum_{r=0}^{d-1}((Z_i Z_j{}^\dagger)^r + (Z_i{}^\dagger Z_j)^r) - KN$$



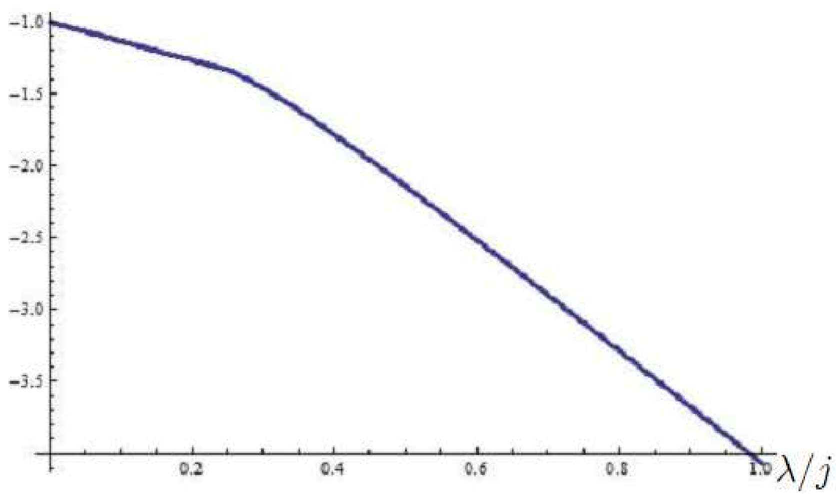
# Mean Field

$$\min_{|\psi\rangle} \langle \psi | H | \psi \rangle = E_{g.s}$$

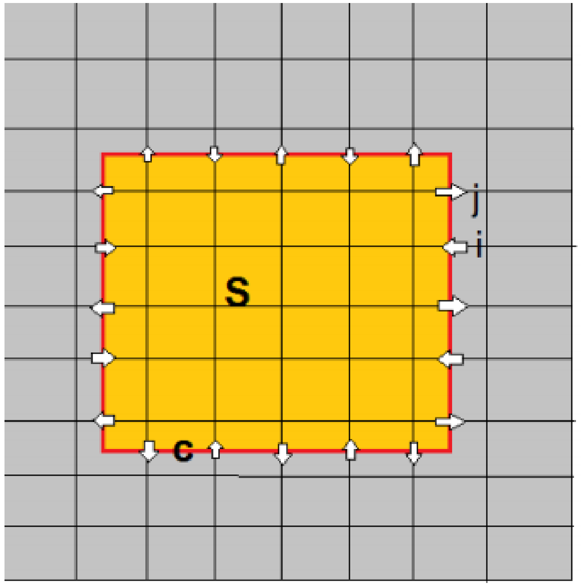
$$\langle \Lambda | H | \Lambda \rangle \geq E_{g.s} \qquad \forall \, |\Lambda\rangle$$

$$|\Lambda\rangle = |\varphi\rangle^{\otimes \frac{N}{2}} \qquad |\varphi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle$$

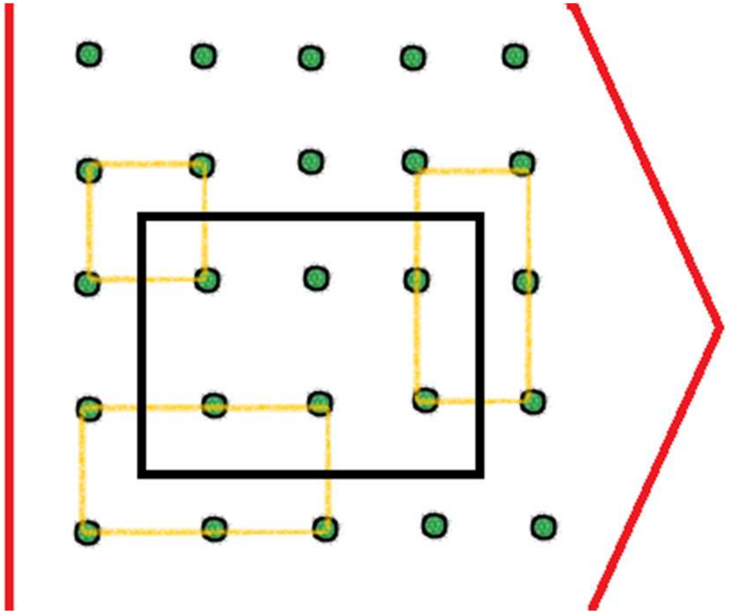
$$E = \langle \Lambda | H_A | \Lambda \rangle = -\mathrm{J}\frac{N}{2} \sum_i (a_i a_{i+1}{}^* + a_{i+1} a_i{}^*) - 4N\lambda \sum_i |a_i|^4$$



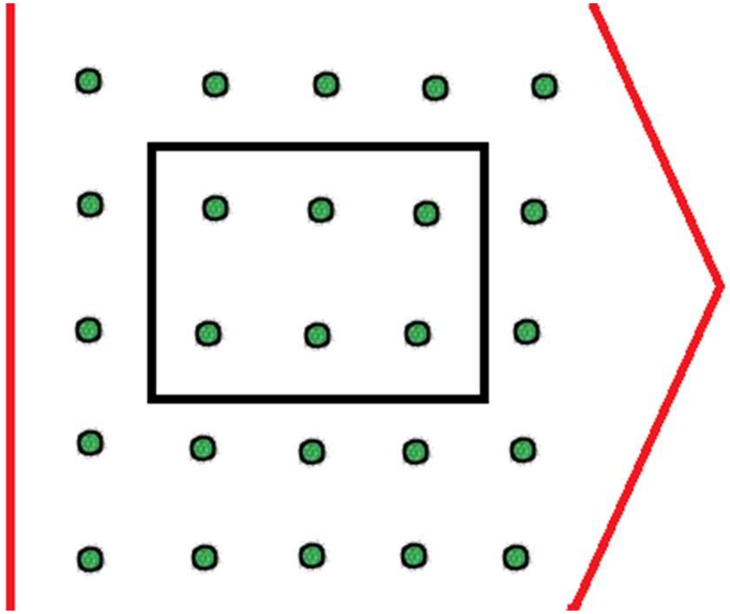
# Wilson loop




$$W_c = \prod_{i \in c} X_i \prod_{j \in c} {X_j}^{\dagger} = \prod_{v \in S} A_v$$



$|\psi_0\rangle = \prod_s (1 + B_p)|+\rangle^{2N}$



$|\varphi_0\rangle = |0\rangle^{2N}$

$$W_c = \begin{cases} 1, & Kitaev \\ 0, & Potts \end{cases}$$

# Calculation of Wilson loop in $\lambda \ll \lambda_c$

$$\langle W_c \rangle = \frac{\langle \Psi | W_c | \Psi \rangle}{\langle \Psi | \Psi \rangle} \cong e^{\frac{-4ax^2|\partial S|}{d^2}}$$

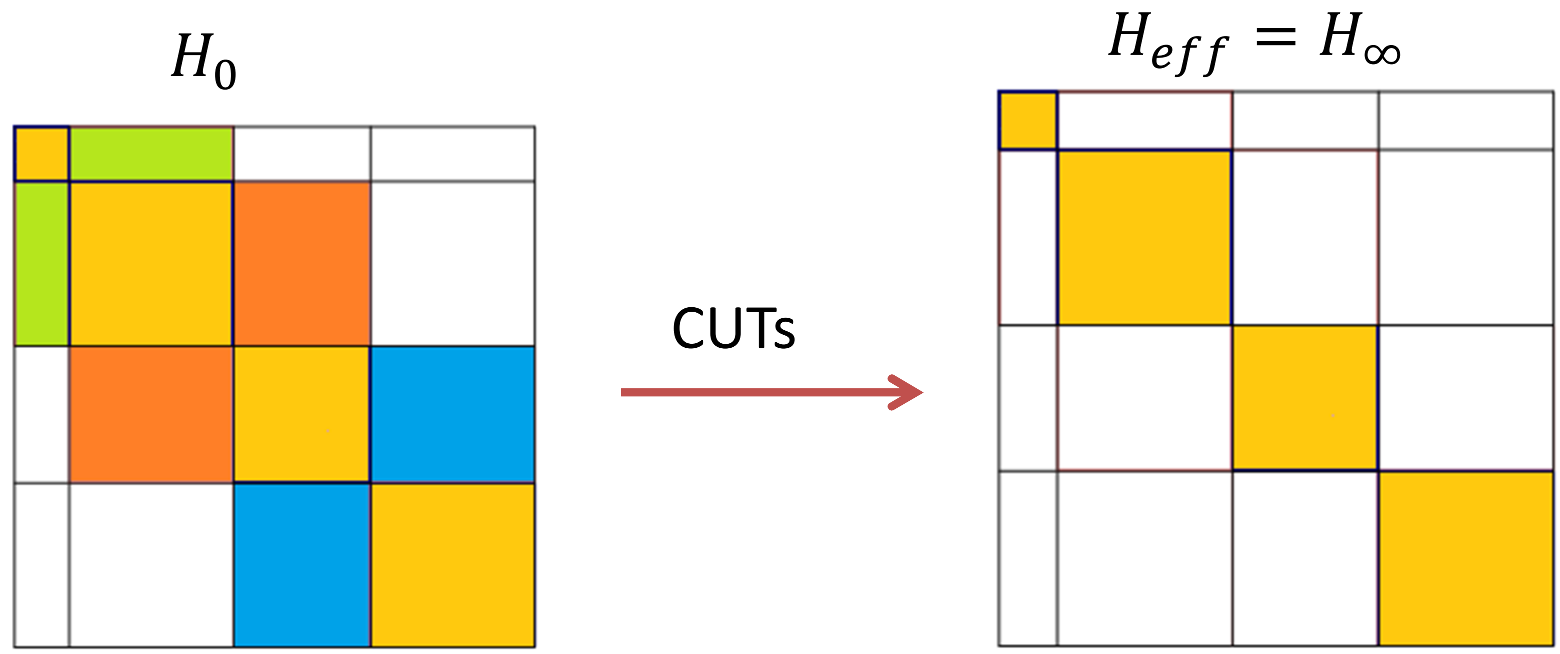
# Calculation of Wilson loop in $\lambda \gg \lambda_c$

$$\langle W_c \rangle = \frac{\langle \Psi | W_c | \Psi \rangle}{\langle \Psi | \Psi \rangle} \cong \frac{1}{2} e^{-|S| \ln \frac{1+\lambda^{-2}}{2\lambda^{-1}}}$$

# Continuous Unitary Transformations(CUTs)

$$D = U\,H\,U^{\dagger}$$

$$H \rightarrow H(l), \qquad H_{eff} = H(l=\infty)$$



# ...CUTs

$$
\begin{cases} H(l) = U(l)\,H(0)\,U^{\dagger}(l) \\ \dfrac{dU(l)}{dl} = \eta(l)U(l) \end{cases}
\Longrightarrow
\boxed{\dfrac{dH(l)}{dl} = [\eta(l), H(l)]}
$$

# Wegner's generator

$$\eta(l) = [H_d(l), H(l)]$$

$$h_{ij}(l) = \langle v_i | H(l) | v_j \rangle \quad \epsilon_i(l) = \langle v_i | H(l) | v_i \rangle$$

$$\sum_{k,i\neq k} \frac{\partial}{\partial l} |h_{ki}|^2 = -2 \sum_{i,k} (\epsilon_i - \epsilon_k)^2 |h_{i,k}|^2$$

$$\frac{dH(l)}{dl} = [[H_d(l), H(l)], H(l)]$$

# Pertubative Continuous Unitary Transformations

$$H(x) = Q + x\,V$$

1) The unperturbed Hamiltonian $Q$ must have an equidistant spectrum bounded from below.By $U_n$ the corresponding subspaces are denoted.

$$Q|n\rangle = n|n\rangle$$

2) The perturbing Hamiltonian $V$ links subspaces $U_i$ and $U_j$only if $|i-j|$ is bounded from above. There is a number $N > 0$ such that $V$ can be written as $V = \sum_{n=-N}^{n=+N} T_n$ where $T_n$ increments(or decrements, if $n < 0$) the number of energy quanta by $n$.

$$[Q, T_n] = nT_n$$

$$H(x) = Q + x\sum_{n=-N}^{n=+N} T_n$$

# Pertubative Continuous Unitary Transformations

$$H(x,l) = Q + x\,V(l)$$

1) The unperturbed Hamiltonian $Q$ must have an equidistant spectrum bounded from below.By $U_n$ the corresponding subspaces are denoted.

$$Q|n\rangle = n|n\rangle$$

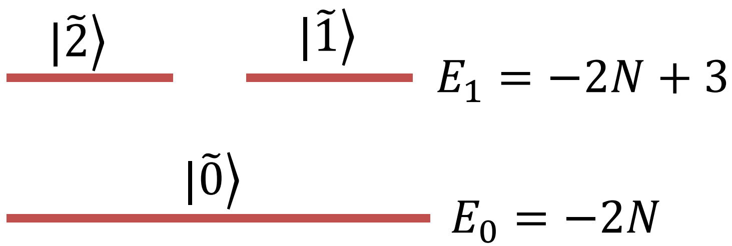
2) The perturbing Hamiltonian $V$ links subspaces $U_i$ and $U_j$only if $|i-j|$ is bounded from above. There is a number $N > 0$ such that $V$ can be written as $V = \sum_{n=-N}^{n=+N} T_n$ where $T_n$ increments(or decrements, if $n < 0$) the number of energy quanta by $n$.

$$[Q, T_n] = nT_n$$

$$H(x,l) = Q + x\sum_{n=-N}^{n=+N} T_n(l)$$

# Kitaev-Potts for d=3

$$\widetilde{H_A} = \underbrace{-\frac{3}{2}\sum_i X_i + {X_i}^\dagger}_{f(Q)} - x \underbrace{\sum_{\langle i,j\rangle}(Z_i {Z_j}^\dagger + {Z_i}^\dagger Z_j)}_{\sum T_n}$$




$$Q_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{-(X_i + {X_i}^\dagger) + 2I}{3} \qquad Q = \sum_i Q_i$$

$$-\frac{3}{2}\sum_i X_i + {X_i}^\dagger = \frac{9}{2} Q$$

# $T_n$ operators

$$Z_i {Z_j}^\dagger |00\rangle = |12\rangle \qquad |n\rangle \rightarrow |n+2\rangle$$

$$|21\rangle = |00\rangle \qquad |n\rangle \rightarrow |n-2\rangle$$

$$|10\rangle = |22\rangle \qquad |n\rangle \rightarrow |n+1\rangle$$

$$|11\rangle = |20\rangle \qquad |n\rangle \rightarrow |n-1\rangle$$

$$|12\rangle = |21\rangle \qquad |n\rangle \rightarrow |n\rangle$$

$$\sum_{\langle i,j \rangle} (Z_i {Z_j}^\dagger + {Z_i}^\dagger Z_j) = T_{-2} + T_{-1} + T_0 + T_{+1} + T_{+2}$$

# PCUT

$$H(l) = \frac{9}{2}Q + \{T_{-2}(l) + T_{-1}(l) + T_0(l) + T_{+1}(l) + T_{+2}(l)\}$$

$$\eta(l) = [Q, H(l)] = -2T_{-2}(l) - T_{-1}(l) + T_{+1}(l) + 2T_{+2}(l)$$

Modified Generator:

$$\eta(l) = [Q, H(l)] = -T_{-2}(l) - T_{-1}(l) + T_{+1}(l) + T_{+2}(l)$$

modified generator preserves the initial block band structure.

# Flow equation

$$\frac{dH(l)}{dl} = [\eta(l), H(l)]$$
$$= [-T_{-2}(l) - T_{-1}(l) + T_{+1}(l) + T_{+2}(l), \mathrm{H}(l)]$$

$$\partial_l T_0(l) = 2[T_{+2}(l), T_{-2}(l)] + 2[T_{+1}(l),\ T_{-1}(l)]$$
$$\partial_l T_{+2}(l) = -9T_{+2}(l) + [T_{+2}(l), T_0(l)]$$
$$\partial_l T_{-2}(l) = -9T_{-2}(l) - [T_{-2}(l), T_0(l)]$$
$$\partial_l T_{+1}(l) = -\frac{9}{2}T_{+1}(l) + 2[T_{+2}(l), T_{-1}(l)] + [T_{+1}(l),\ T_0(l)]$$
$$\partial_l T_{-1}(l) = -\frac{9}{2}T_{-1}(l) + 2[T_{+1}(l), T_{-2}(l)] + [T_0(l),\ T_{-1}(l)]$$

# Solving flow equation

$$T_n(l) = \sum_{k=1}^{\infty} {T_n}^{(k)}(l)$$

Example:

$$\partial_l {T_{+2}}^{(k)}(l) = -9{T_{+2}}^{(k)}(l) + \sum_{j=1}^{k-1} [{T_{+2}}^{(j)}(l) \quad , {T_0}^{(k-j)}(l) \quad ]$$

$$\partial_l {T_{+2}}^{(1)}(l) = -9{T_{+2}}^{(1)}(l)$$

$${T_{+2}}^{(1)}(l) = T_{+2}\, e^{-9l}$$

$$H_{eff} = \frac{9}{2}Q + T_0 + \frac{1}{9}([T_{+2}, T_{-2}] + [T_{+1}, T_{-1}]) + \frac{2}{18^2}([T_{+2},[T_0, T_{-2}]] + [[T_{+2}, T_0], T_{-2}]) - \frac{1}{81}([T_{+1},[T_{+1}, T_{-2}]] + [[T_{+2}, T_{-1}], T_{-1}]) + \frac{2}{81}([T_{+1},[T_0, T_{-1}]] + [[T_{+1}, T_0], T_{-1}])$$

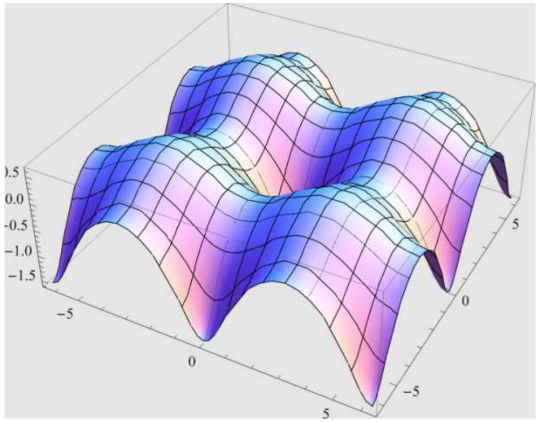
$$H_{eff} = H_0 + H_1 + H_2 + \cdots$$

$$H_0 := E_0 \mathbf{1}$$

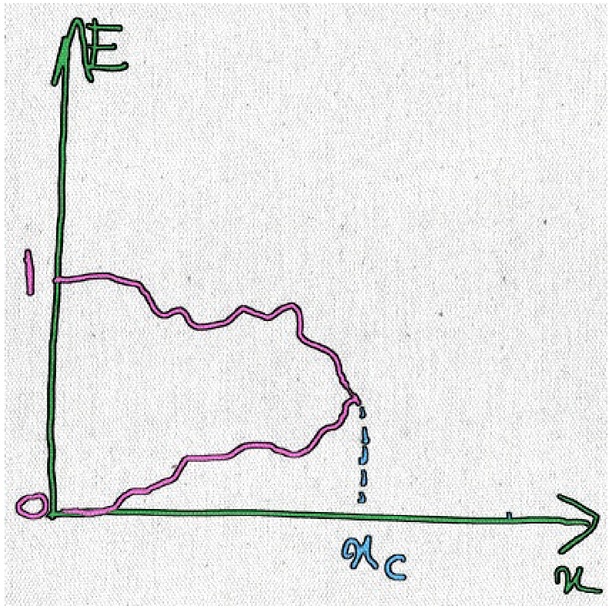
$$H_1 := \sum_{i;j} t_{j;i} e_j^\dagger e_i$$

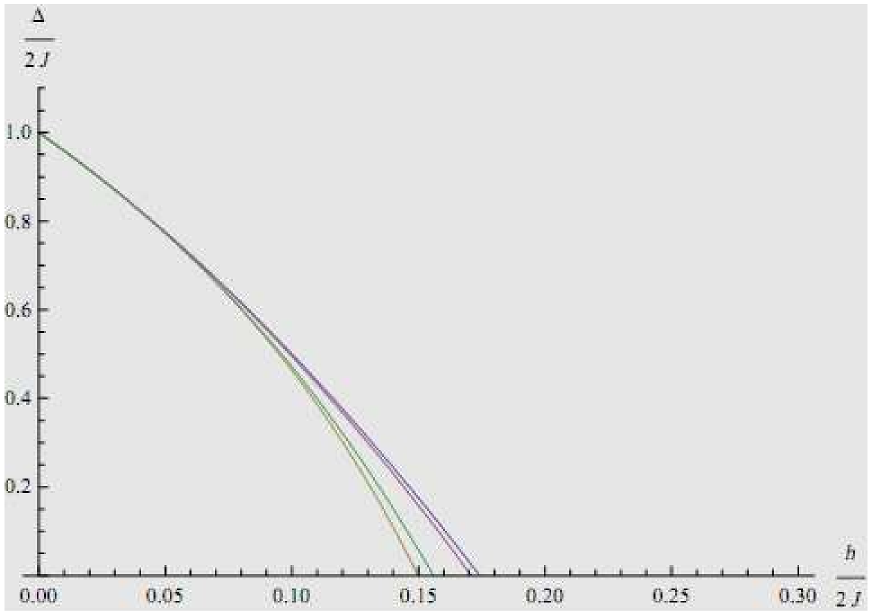
$$H_1 |\mathbf{r}\rangle = \sum_{\mathbf{d}} t_{\mathbf{d}} |\mathbf{r} + \mathbf{d}\rangle \ ,$$

$$
\begin{aligned}
H_1|\mathbf{k}\rangle &= \frac{1}{\sqrt{N}}\sum_{\mathbf{r},\mathbf{d}} e^{i\mathbf{kr}} t_{\mathbf{d}}|\mathbf{r}+\mathbf{d}\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{\mathbf{r},\mathbf{d}} e^{i\mathbf{k}(\mathbf{r}-\mathbf{d})} t_{\mathbf{d}}|\mathbf{r}\rangle \\
&= \underbrace{\sum_{\mathbf{d}} e^{-i\mathbf{kd}} t_{\mathbf{d}}}_{\omega(\mathbf{k};x)} \underbrace{\frac{1}{\sqrt{N}}\sum_{\mathbf{r}} e^{i\mathbf{kr}}|\mathbf{r}\rangle}_{|\mathbf{k}\rangle} \ .
\end{aligned}
$$



# Small coupling results



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