Robustness of Topological Color Codes

Saeed S. Jahromi,

K.N.Toosi University Of Technology, Iran

QI Group. Sharif University of Technology, 1 October 2013

دانشگاه صنعتی خواجه نصیرالدین طوسی

technische universität
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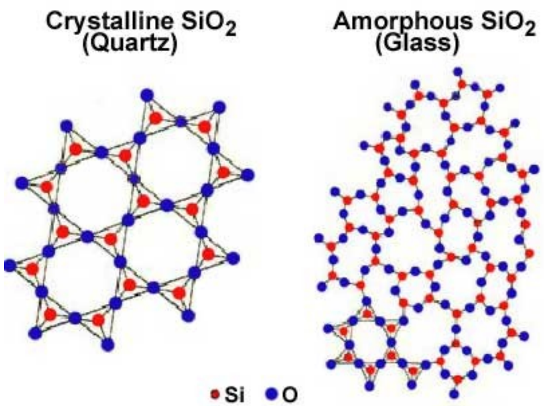
Outline

- Topological Order
- Quantum Computation and Error Correction
- Topological Color Codes
- Color Code-Ising and breakdown of the topological phase
- Mapping the Hamiltonian of TCC
- Perturbative Continuous Unitary Transformation (PCUT)
- Discussion and results

Ordinary Phases of Matter

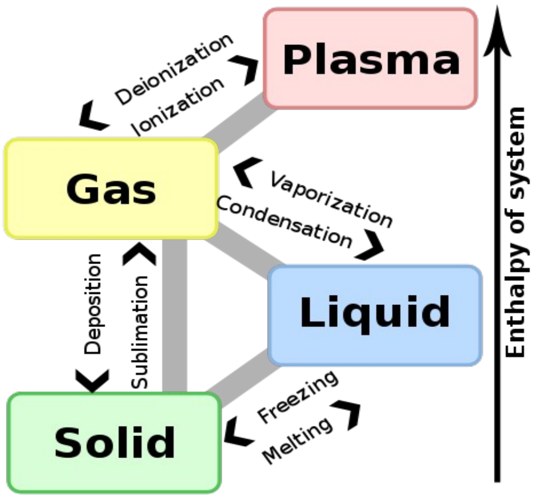
- Solid
- Liquid
- Gas
- Ferromagnetic
- Paramagnetic
- Metal
- Insulator
- ...



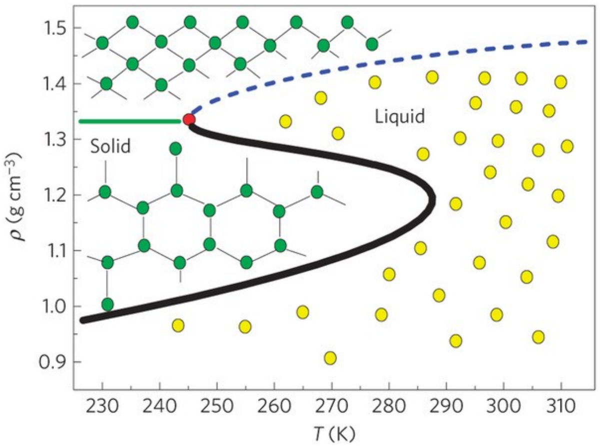
Structure **Correlation** **Symmetry**

Phase Transition

- **Control Parameter (T, P, B, …)**
- **Critical Point (Curi, Neel)**



- Para → Ferro (Anti Ferro)
- Metal → Superconductor
- Metal → Insulator



Classification of Phase Transitions

Ehrenfest Classification

Free Energy

- **First Order**

  Coexistence of Phases

  Sometimes symmetry is broken

  Phase transition point

- **Higher Order (Continuous)**

  Always symmetry is broken

  Order Parameter

  Critical Point

Order Parameter and Landau Theory

Measure of the degree of order in a system

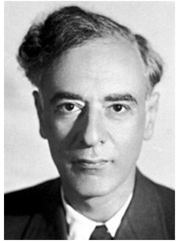
1 → Total order   0 → Total disorder

- **Order Parameter:**

  Vector, Tensor, Scalar, …

50 Years

- **Landau symmetry breaking Theory of Phase transition**



$$\mathcal{L}(m,h,t) = c_1 hm + d_2 tm^2 + c_3 hm^3 + b_4 m^4 \qquad d_2 > 0 \;, \quad b_4 > 0$$

$$t \equiv T - T_c \quad \text{and} \quad h \equiv H - H_c = H$$

Topological Order

Fractional Quantum Hall Effect

2D **Electron Gas** (Low T, High H)

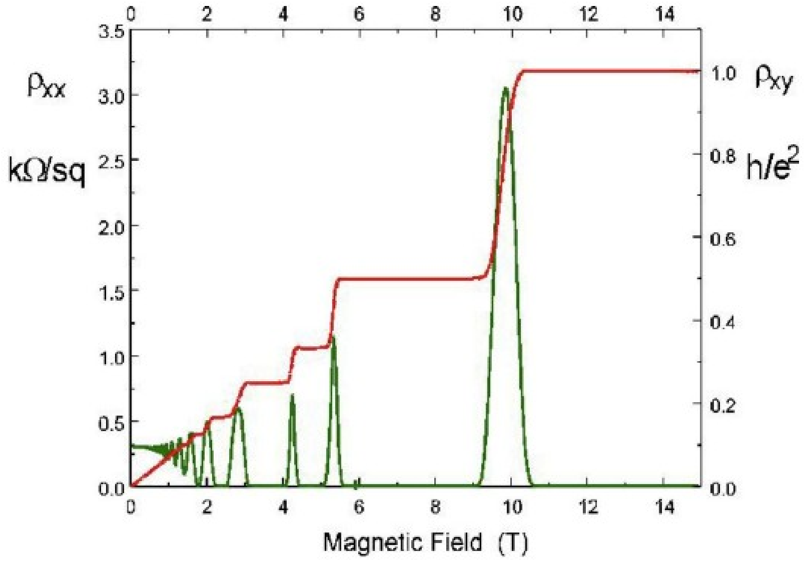
Not a Solid

Uncompressable

Not a liquid

Non localized Electrons

Correlated Motion

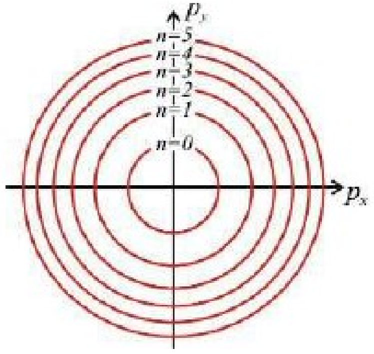


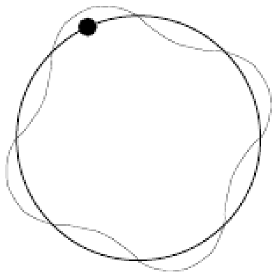
D.C. Tsui, H.L. Stormer, and A.C. Gossard. s.l., Phys. Rev. Lett, 48 1559 (1982).

Topological Order

Quantum Hall Effect

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right), \quad n \geq 0.$$





Correlated Motion

Integer number of steps to dance around the circle in landau levels.

Electrons dance around each other

Fermi Statistics, Coulomb Interaction

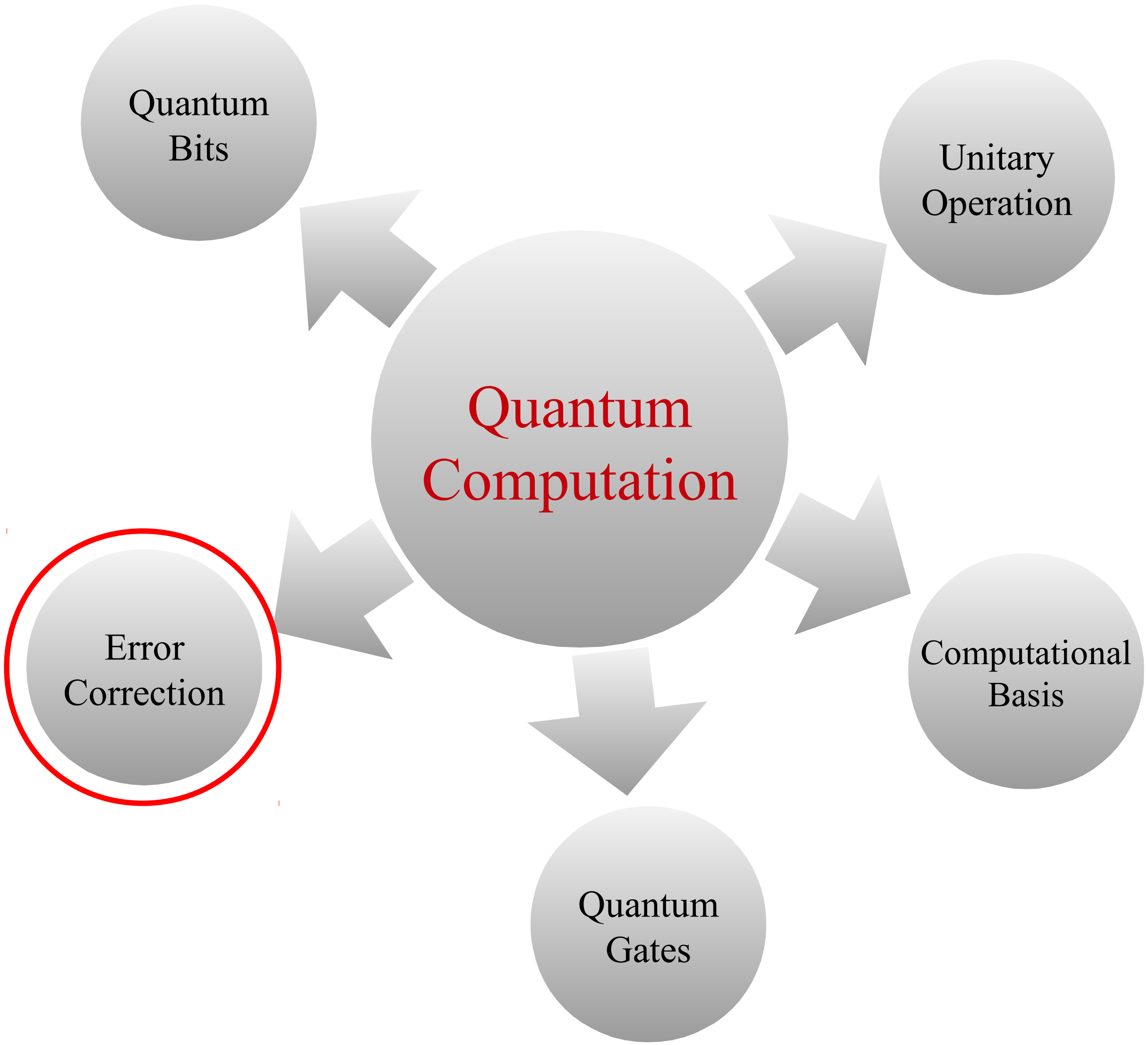
Topological Order

- Ground state degeneracy → Topology of space.
- The ground state degeneracy → not a consequence of symmetry
- Robust against arbitrary perturbations
- Only Change in topological order → change in the ground state degeneracy.

Ground state degeneracy
quantum number → Characterize topological order.

X. G. Wen, PRB 41, 9377 (1990).

Quantum Computation



Errors in Classical Computation

Bit Flip **Phase Flip** **Decoherence**

Errors occur in transmission of Data

Original Data: 101 100 111, Transmitted Data: 101 101 111,

Classical Error Correction: (Repetition code)
Original Data: 101 100 111 101 100 111 101 100 111
Transmitted Data: 101 100 111 101 101 111 101 100 111

Magnetic Storage → Ferromagnetic interaction with neighbors

Fault Tolerant

Errors in Quantum Computation

Information: $|\psi\rangle = a|0\rangle + b|1\rangle$ **a** and **b** are the information.

Information transmission Problems:

1. *No cloning*: It is impossible to create the same quantum state. Therefore, the repetition code cannot be realized:

$$|\psi\rangle \nrightarrow |\psi\rangle|\psi\rangle|\psi\rangle \tag{6.8}$$

2. Measurements destroy a and b: If we measure the state, such as $|\psi\rangle = a|0\rangle + b|1\rangle$, to know what is the error, the state collapse to one of the two states $|0\rangle$ or $|1\rangle$.

3. Linear combination of different types of errors: In case we can detect one error, what if the error is a mixture of different errors? Unfortunately, this is the case.

Quantum Errors Correction

Stabilizer Code

Represent a **quantum state** → By a set of generators **equivalent to observable**.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

ψ is stabilized by σ_{X1} σ_{X2} and σ_{z1} σ_{z2}

$$\sigma_{X1}\sigma_{X2}|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle) = |\psi\rangle, \quad \sigma_{Z1}\sigma_{Z2}|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi\rangle$$

$S = \{\sigma_{X1}\sigma_{X2}, \sigma_{Z1}\sigma_{Z2}\}$ → generators of a vector spac $V_S = \{|00\rangle, |11\rangle\}$

Errors in Quantum Computation

Bit Flip | **Phase Flip** | **Decoherence**

- Search for systemswhich are intrinsically robust against errors.
- States carrying non-local degrees of freedom.
- We can put the quantum information on global degrees of freedom.

↓

Topological Systems

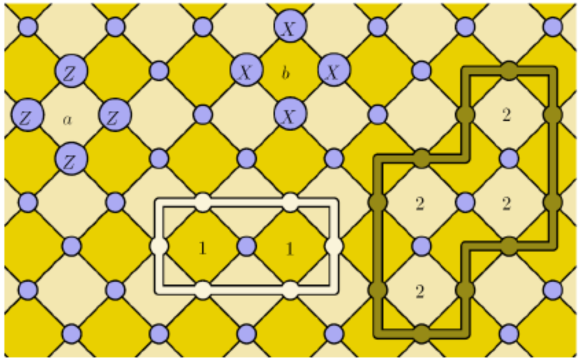
Error Correcting Quantum Codes

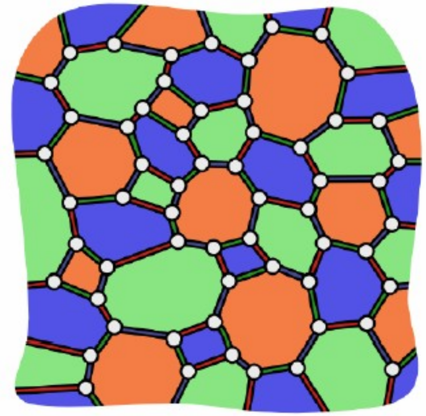
Fault-Tolerant Quantum Computation

Topological Properties protects the system against Errors

Toric Code

Color Code

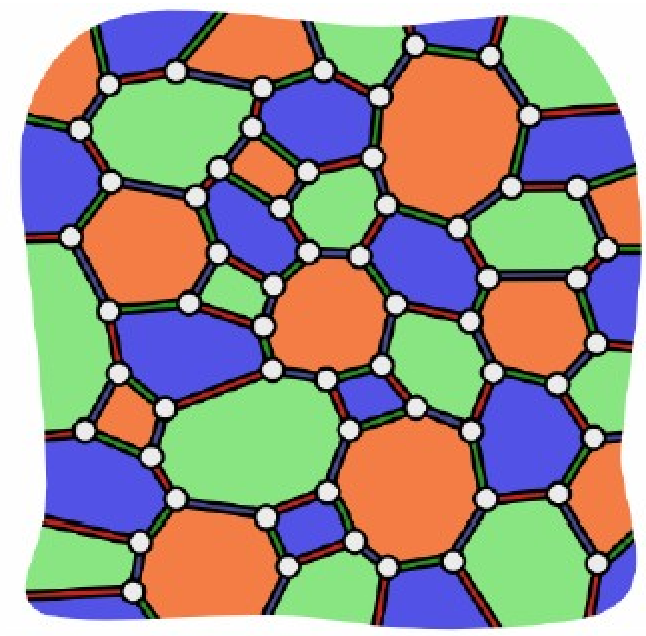


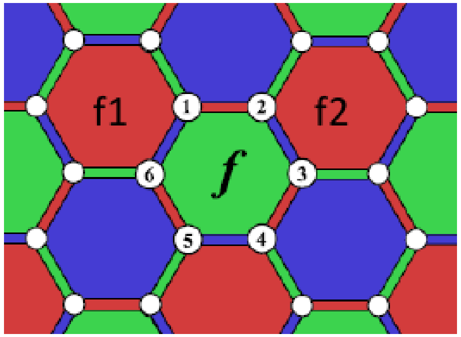


A.Yu. Kitaev, Annals of Physics 303 2-30 (2003).

H. Bombin, Phys.Rev.Lett. 97 180501 (2006).

Topological Color code





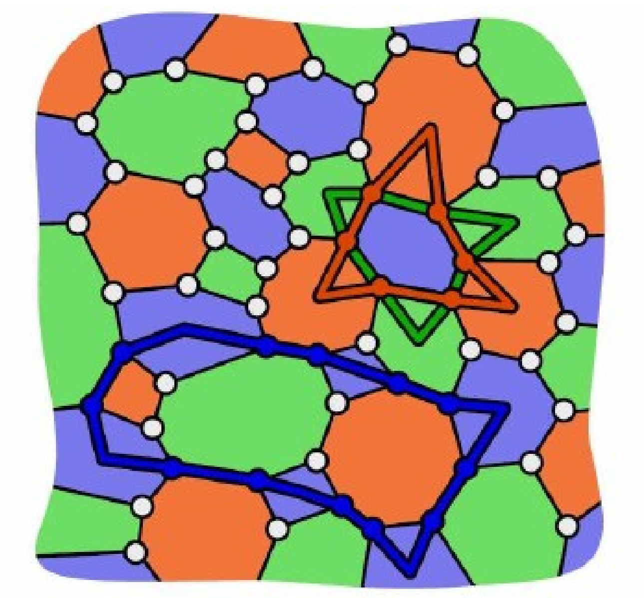
$$H = -\sum_{\mathfrak{p}\in\Lambda}(X_{\mathfrak{p}} + Z_{\mathfrak{p}})$$

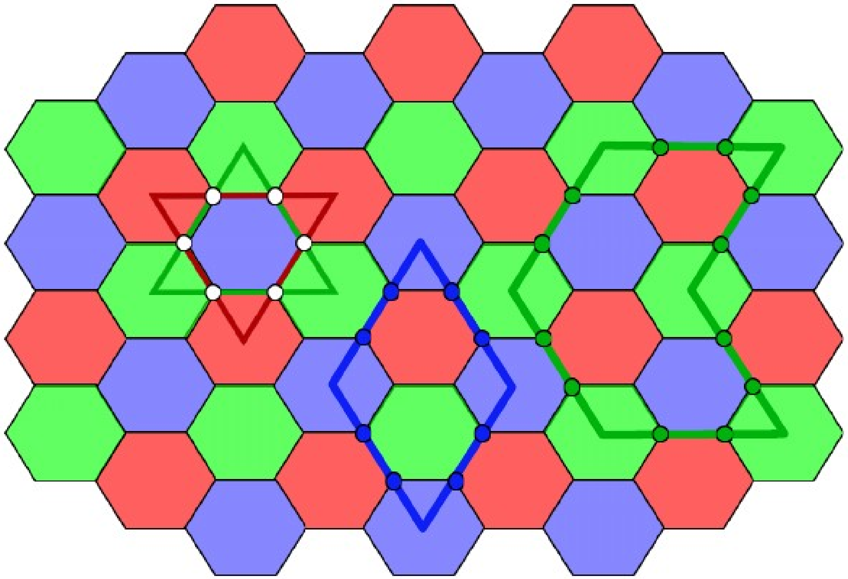
$$X_{\mathfrak{p}} = \bigotimes_{\mathfrak{v}\in\mathfrak{p}} \sigma_{\mathfrak{v}}^{x}$$

$$Z_{\mathfrak{p}} = \bigotimes_{\mathfrak{v}\in\mathfrak{p}} \sigma_{\mathfrak{v}}^{z}$$

H. Bombin, M.A. Martin-Delgado, Phys.Rev.Lett. 97 180501 (2006).
H. Bombin, M.A. Martin-Delgado, Phys. Rev. A 77, 042322 (2008).

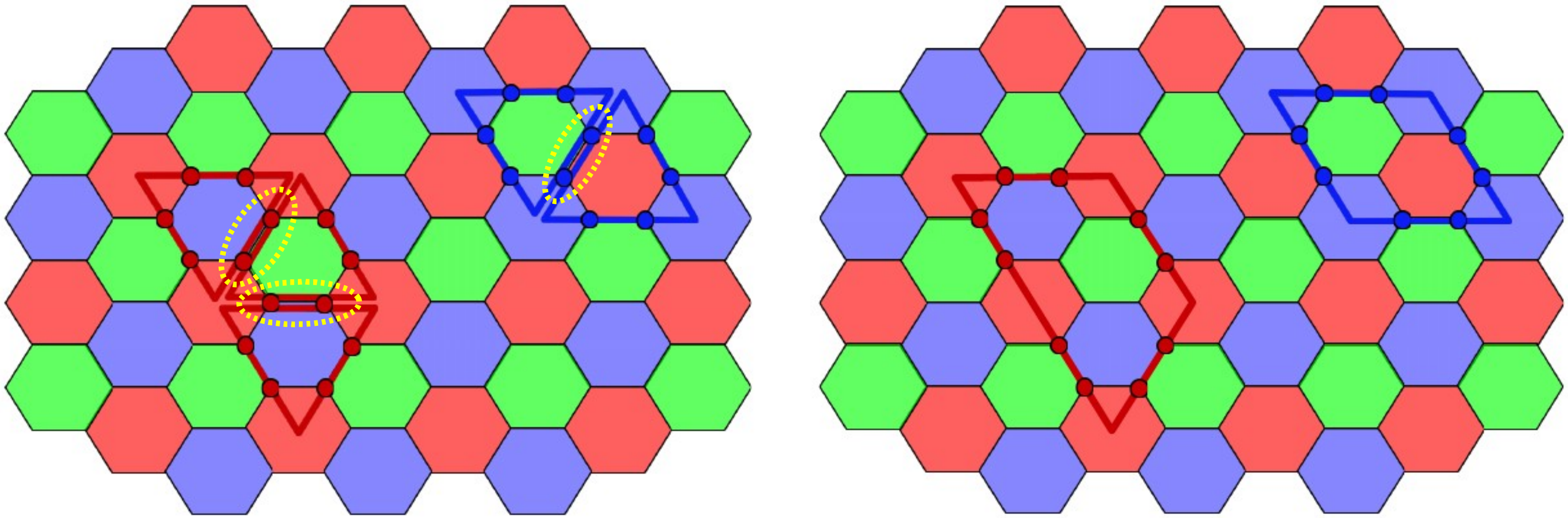
String Operators





$$\mathcal{S}^{CX} = \bigotimes_{\mathfrak{v} \in I} \sigma^x_{\mathfrak{v}}, \quad \mathcal{S}^{CZ} = \bigotimes_{\mathfrak{v} \in I} \sigma^z_{\mathfrak{v}}$$

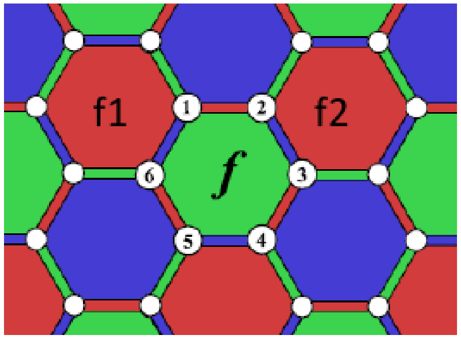
String Operators



Closed strings are extension of plaquette operators

Strings of different shapes are created by product of neighboring plaquette operators

Ground State

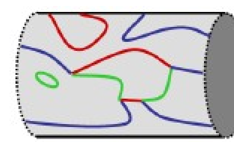


$$H = -\sum_{\mathfrak{p}\in\Lambda}(X_{\mathfrak{p}} + Z_{\mathfrak{p}})$$

$$(X_{\mathfrak{p}})^2 = \mathbb{1} = (Z_{\mathfrak{p}})^2 \qquad X_{\mathfrak{p}} = \pm 1, \quad Z_{\mathfrak{p}} = \pm 1$$

$$[X_{\mathfrak{p}}, Z_{\mathfrak{p}}] = 0, \quad [X_{\mathfrak{p}_1}, X_{\mathfrak{p}_2}] = 0, \quad [Z_{\mathfrak{p}_1}, Z_{\mathfrak{p}_2}] = 0$$

$$|0000\rangle = \prod_{\mathfrak{p}}(1 + X_{\mathfrak{p}})\,|\Uparrow \dots \Uparrow\rangle \Rightarrow |\psi\rangle = \sum_{\gamma}$$

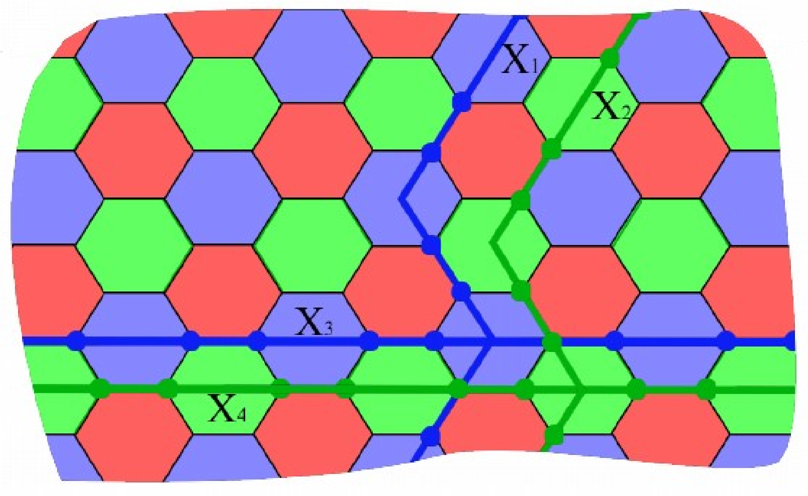


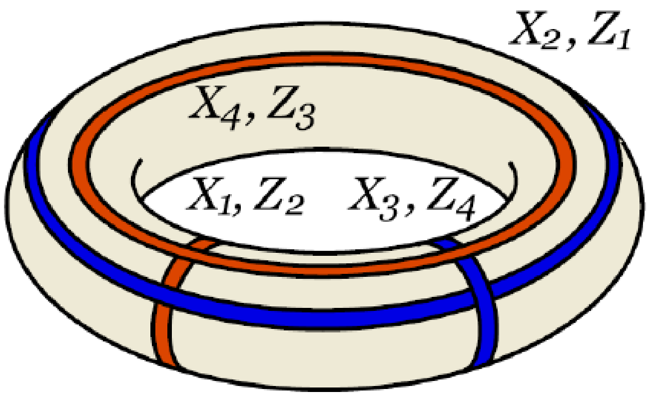
String-net Condensation

X.G. Wen, Quantum Field Theory of Many-body Systems, Oxford Univ. Press, (2004).

Global String Operators

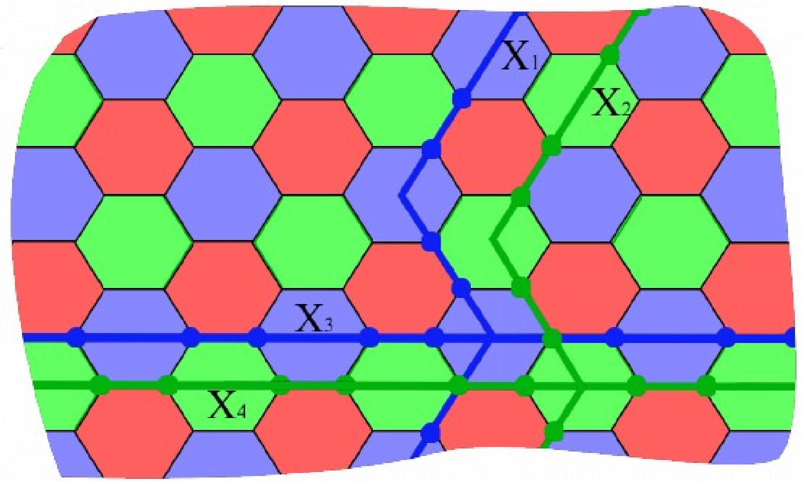
There are another kind of strings which are closed but are not product of pluquette operators

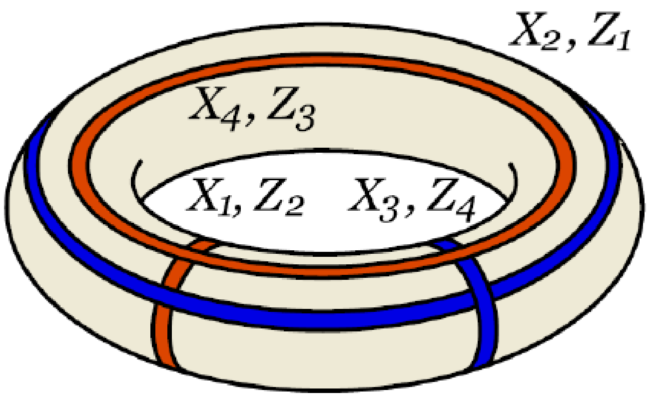




Global Strings (Fundamental non-contractible loops)

Topological Degeneracy



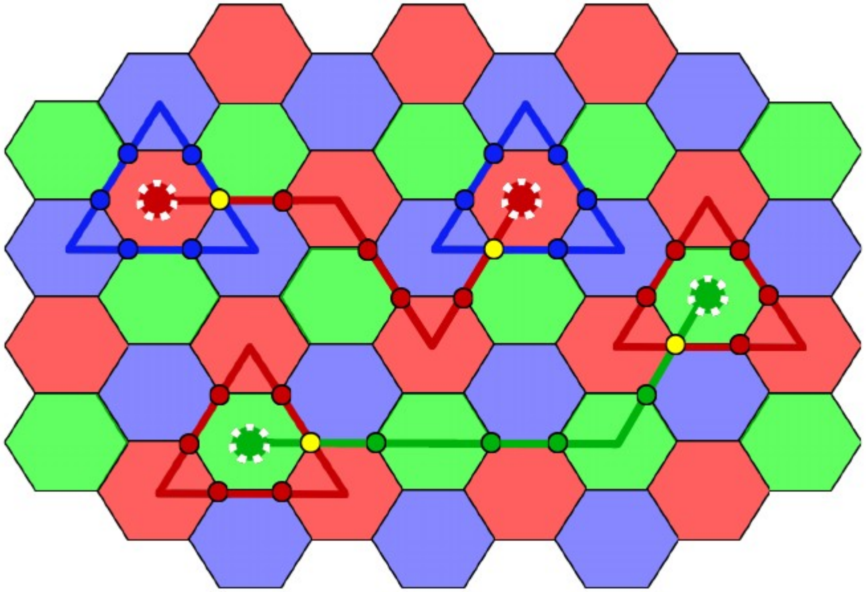


$$X_1 \leftrightarrow \mathcal{S}_2^{BX}, \quad X_2 \leftrightarrow \mathcal{S}_1^{GX}, \quad X_3 \leftrightarrow \mathcal{S}_2^{BX}, \quad X_4 \leftrightarrow \mathcal{S}_1^{GX},$$
$$Z_1 \leftrightarrow \mathcal{S}_1^{GZ}, \quad Z_2 \leftrightarrow \mathcal{S}_2^{BZ}, \quad Z_3 \leftrightarrow \mathcal{S}_1^{GZ}, \quad Z_4 \leftrightarrow \mathcal{S}_2^{BZ}.$$

$$|\psi_{ijkl}\rangle = X_1^i X_2^j X_3^k X_4^l |0000\rangle$$

Excitations

Excitations are created at the end points of open strings

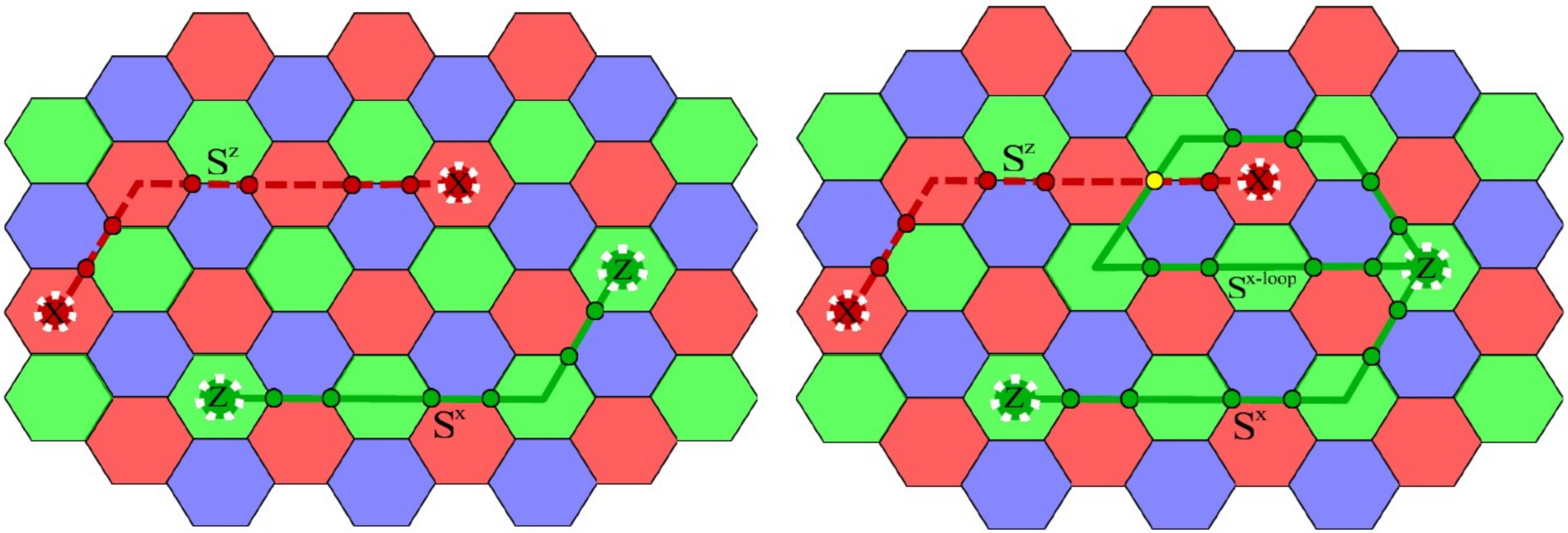


$$X_\gamma := \bigotimes_{\mathfrak{e}\in\gamma} X_{\mathfrak{e}}, \qquad Z_\gamma := \bigotimes_{\mathfrak{e}\in\gamma} Z_{\mathfrak{e}}$$

$$\{X_\gamma, Z_{\mathfrak{p}}\} = 0 \qquad \{Z_\gamma, X_{\mathfrak{p}}\} = 0$$

$$|\Psi_{\mathrm{ex}}\rangle = X_\gamma|\Psi_{\mathrm{gs}}\rangle = -|\Psi_{\mathrm{gs}}\rangle$$

Mutual Statistics and Anyons



$$|i\rangle = S^x S^z |\psi_{gs}\rangle$$

$$|f\rangle = S^{x-loop}|i\rangle = S^{x-loop} S^x S^z |\psi_{gs}\rangle = -|i\rangle$$

"Since interchange of two of these particles can give any phase, I will call them generically **Anyons**."

Frank Wilczek, PRL, **49**, NUMBER 14, (1982).

Perturbed Topological Color Code with magnetic field or Ising interaction

$$H_{\text{TCC}} = -J \sum_{p \in \Lambda} (X_p + Z_p)$$

$$H = H_{\text{TCC}} - \sum_{\alpha} \left(h_\alpha \sum_i \sigma_i^\alpha + j_\alpha \sum_{<ij>} \sigma_i^\alpha \sigma_j^\alpha \right) \qquad \alpha = x, y, z$$

We call $\alpha = x, z$ *parallel* and $\alpha = y$ *transverse*

no longer exactly solvable

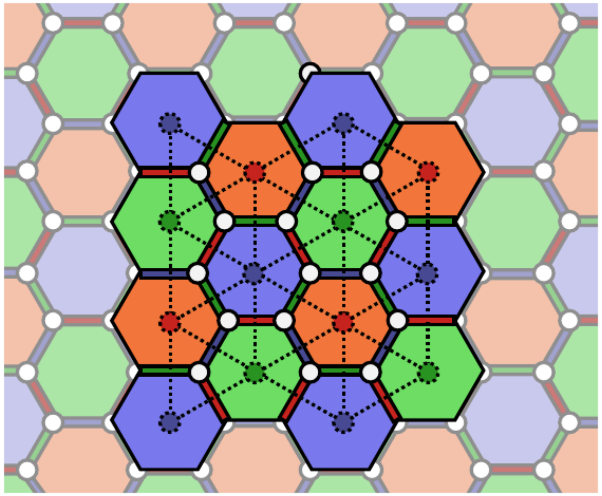
Small Ising Couplings

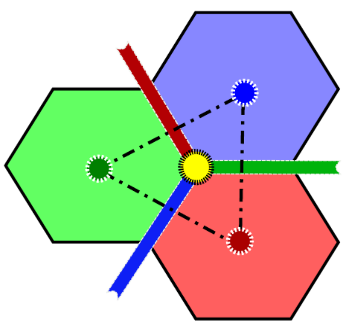
Mapping the Hamiltonian

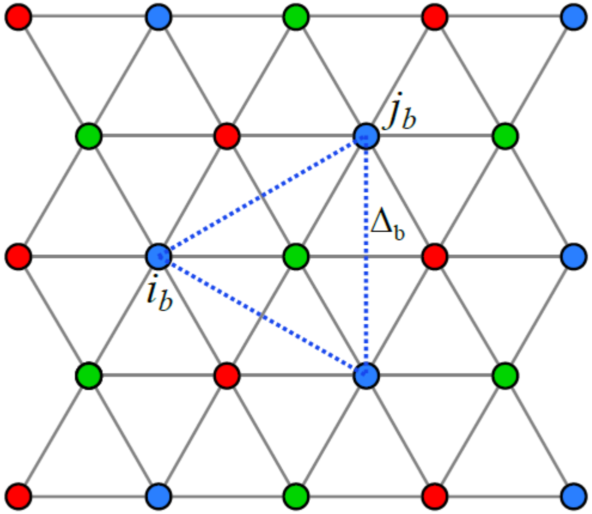
$$H = -J \sum_{p} (X_p + Z_p) - j_x \sum_{<ij>} \sigma_i^x \sigma_j^x$$

$$H = -NJ - J \sum_{p} Z_p - j_x \sum_{<ij>} \sigma_i^x \sigma_j^x$$

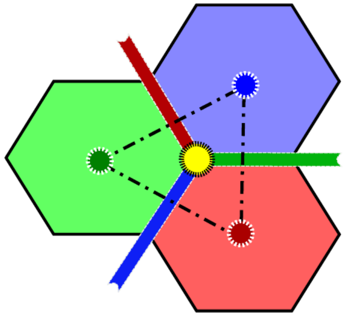
$$H = -NJ - J \sum_{i_c \in \triangle_{r,g,b}} \tau_{i_c}^z - j_x \sum_{<i_c j_c> \in \triangle_{r,g,b}} \tau_{i_c}^x \tau_{j_c}^x$$







Small Field Limit Mapping



$$H = -J \sum_p (X_p + Z_p) - h_x \sum_i \sigma_i^x$$

$$H = -J \sum_p \tau^z + h_x \sum_{\triangle} \tau_r{}^x \tau_g{}^x \tau_b{}^x$$

The perturbation manipulate 3 particle on the lattice

Original Honeycomb lattice → Effective triangular lattice

Continuous Unitary Transformations (CUTs)

$$H = H_0 \longrightarrow H_1 \longrightarrow \ldots \longrightarrow H_m,$$

$$H_i = U_i H_{i-1} U_i^\dagger, \quad i = 1, 2, ...m$$

$$H_{simple} = H_m = U H U^\dagger, \quad U := U_1 U_2 \ldots U_m.$$

$$U = e^\eta$$

Continuous Unitary Transformations (CUTs)

The basic idea is to unitarily transform the initial problem in a continuous fashion.

$$H \longrightarrow H(\ell), \quad \ell \in \mathbf{R}_0^+$$

$$U(\ell) = e^{\eta(\ell)}$$

Hamiltonian

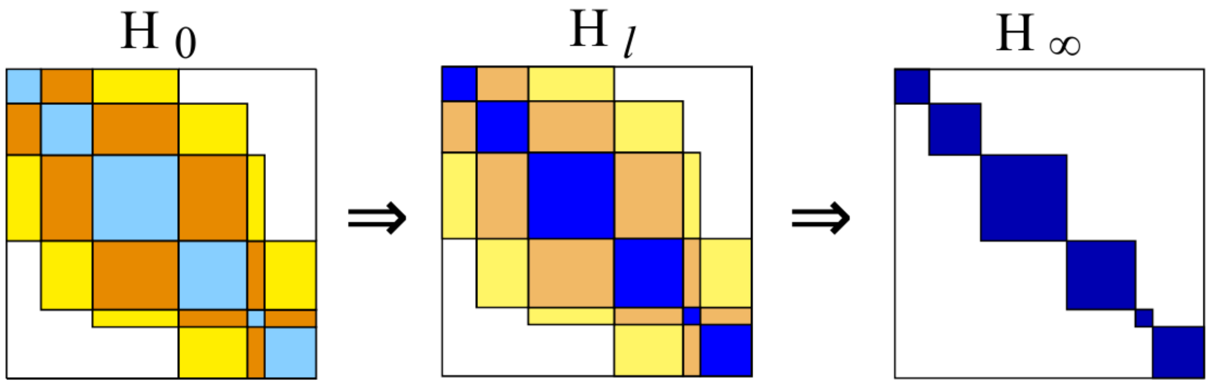
$$\mathcal{H}(l) = U(l)\mathcal{H}U(l)^\dagger$$

$$\Longrightarrow \frac{d\mathcal{H}(l)}{dl} = [\eta(l), \mathcal{H}(l)]$$

$$\mathcal{H}(0) = \mathcal{H} \text{ and } \mathcal{H}(\infty) = \mathcal{H}_{\text{eff}}$$

F.J. Wegner, Ann. Phys. (1994)
S.D. Glazek and K.G. Wilson, Phys. Rev. D (1994)

$$\eta(\ell) = [H_d(\ell), H(\ell)]$$



Perturbative Continuous Unitary Transformations (PCUT)

$$H(x) = U + xV$$

(A) The unperturbed part U has an equidistant spectrum bounded from below. The difference between two successive levels is the energy of a particle, i.e. $Q = U$.

(B) There is a number $\mathbb{N} \ni N > 0$ such that the perturbing part V can be split according to $V = \sum_{n=-N}^{N} T_n$ where T_n increments (or decrements, if $n < 0$) the number of particles by n: $[Q, T_n] = nT_n$.

$$H = Q + T_{-N} + \ldots + T_0 + \ldots + T_{-N}$$

$$H(x;\ell) = Q + xV(\ell) = Q + \sum_{k=1}^{\infty} x^k \sum_{|\underline{m}|=k} F(\ell;\underline{m}) T(\underline{m})$$

A. Mielke, EPJB (1998)
C. Knetter and G. S. Uhrig, EPJB (2000)

Perturbative Continuous Unitary Transformations (PCUT)

$$H(x;\ell) = Q + xV(\ell) = Q + \sum_{k=1}^{\infty} x^k \sum_{|\underline{m}|=k} F(\ell;\underline{m})T(\underline{m})$$

$$\frac{dH(\ell)}{d\ell} = \eta(\ell)H(\ell)$$

$$Q|i\rangle = q_i|i\rangle$$

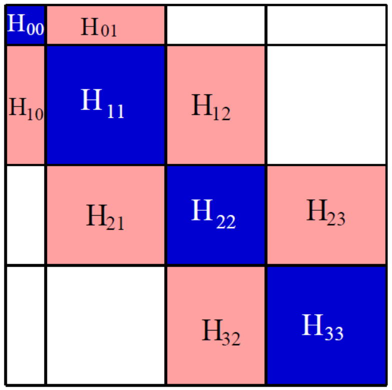
Quasiparticle conserving Generator

$$\eta_{ij}(\ell) = (q_i - q_j)h_{ij}(\ell)$$

A. Mielke, EPJB (1998)
C. Knetter and G. S. Uhrig, EPJB (2000)

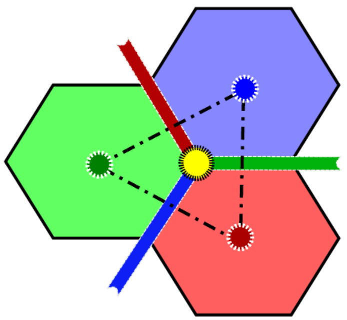
$$H_{\text{eff}} = Q + \sum_{k=1}^{\infty} x^k \sum_{\substack{|\underline{m}|=k \\ M(\underline{m})=0}} C(\underline{m})T(\underline{m})$$

- Effective Hamiltonian conserves number of quasi-particles $[H_{\text{eff}}, Q] = 0$
- Coefficients given as ratio of integer number
- $T(m) = T_{m_1}T_{m_2}T_{m_3}\cdots T_{m_k}$ $\quad m = (m_1, m_2, m_3, \ldots, m_k)$ $\quad m_i \in \{0, \pm 1, \pm 2, \ldots, \pm N\}$



Mapping to a quasi-particle conserving Hamiltonian by PCUT (TCC+Field)

$$\frac{H}{2J} = -\frac{1}{2}\sum_i \tau_i^z + \frac{h_x}{2J}\sum_{\langle ijk\rangle} \tau_i{}^x \tau_j{}^x \tau_k{}^x$$



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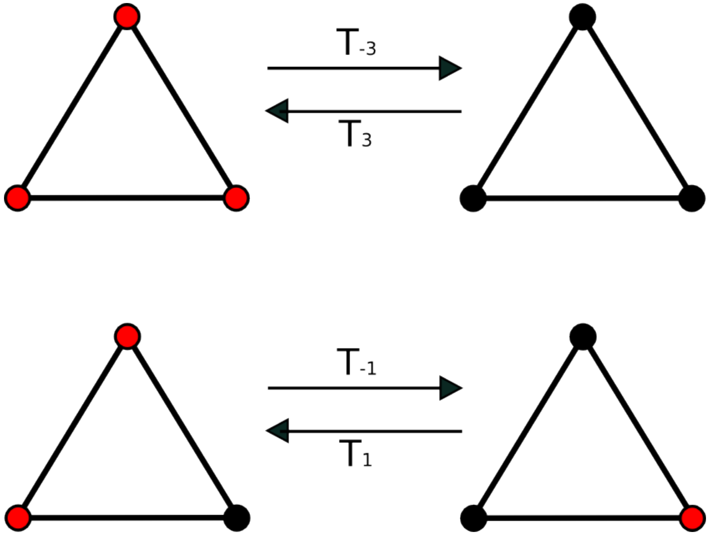
$E_2=-N/2+2$

$E_1=-N/2+1$

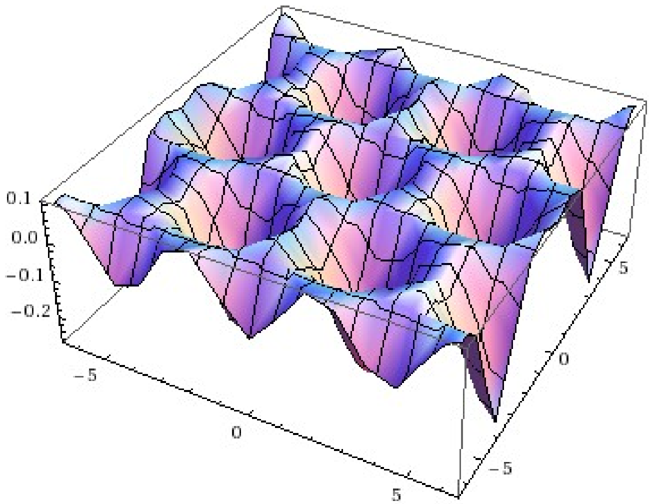
$E_0=-N/2$

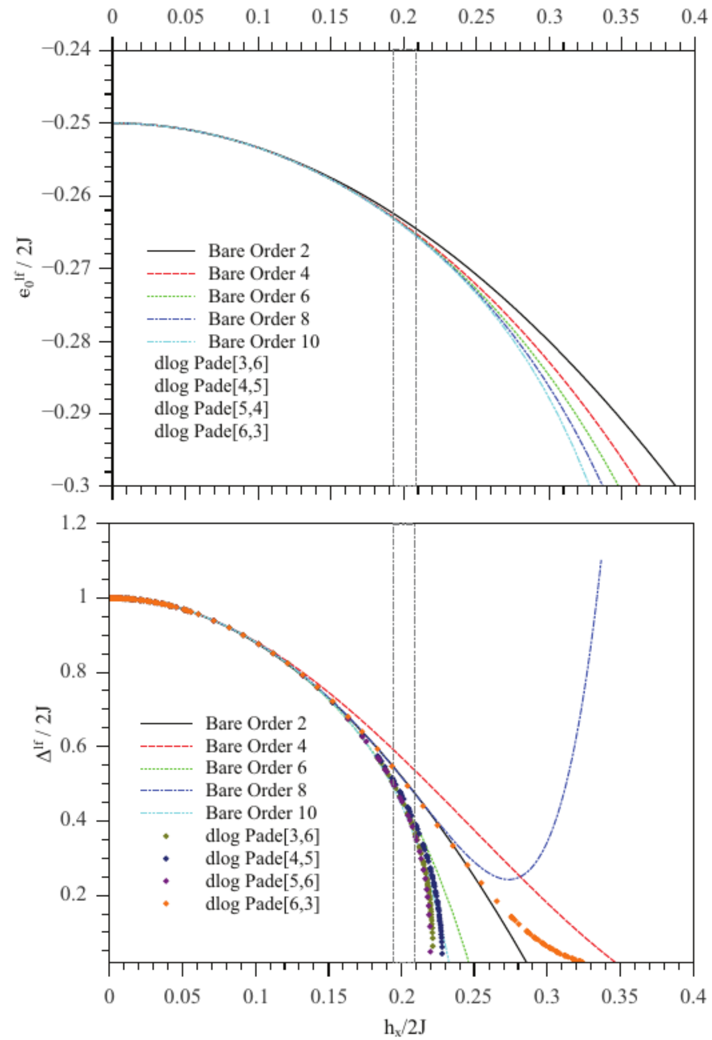


$$\frac{H}{2J} = -\frac{N}{2} + Q + x\sum_{n=-N}^{N} T_n, \quad x = \frac{h_x}{2J}$$

$$\frac{H}{2J} = -\frac{N}{2} + Q + x(T_{-3} + T_{-1} + T_1 + T_3)$$

Small Field Results (TCC+Field)

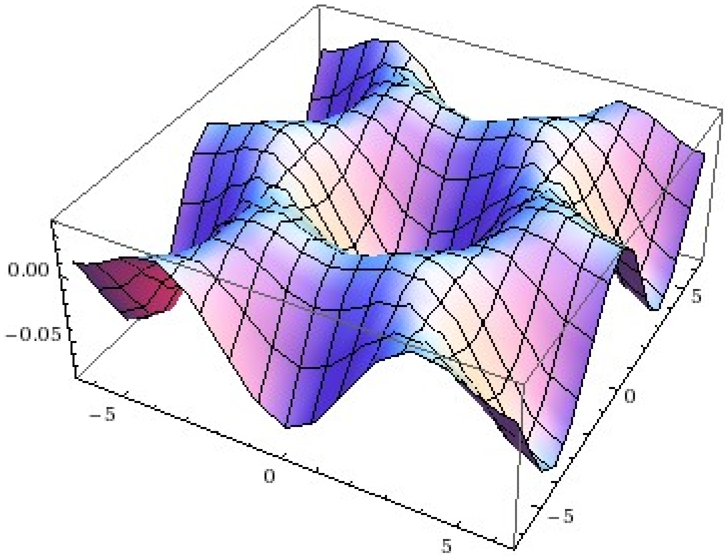




$$\epsilon_0^{\text{lf}} = -\frac{1}{2} - \frac{2}{3}h_x^2 - \frac{19}{27}h_x^4 - \frac{42872}{8505}h_x^6 - \frac{500690327}{10716300}h_x^8 - \frac{148610627638}{281302875}h_x^{10}$$

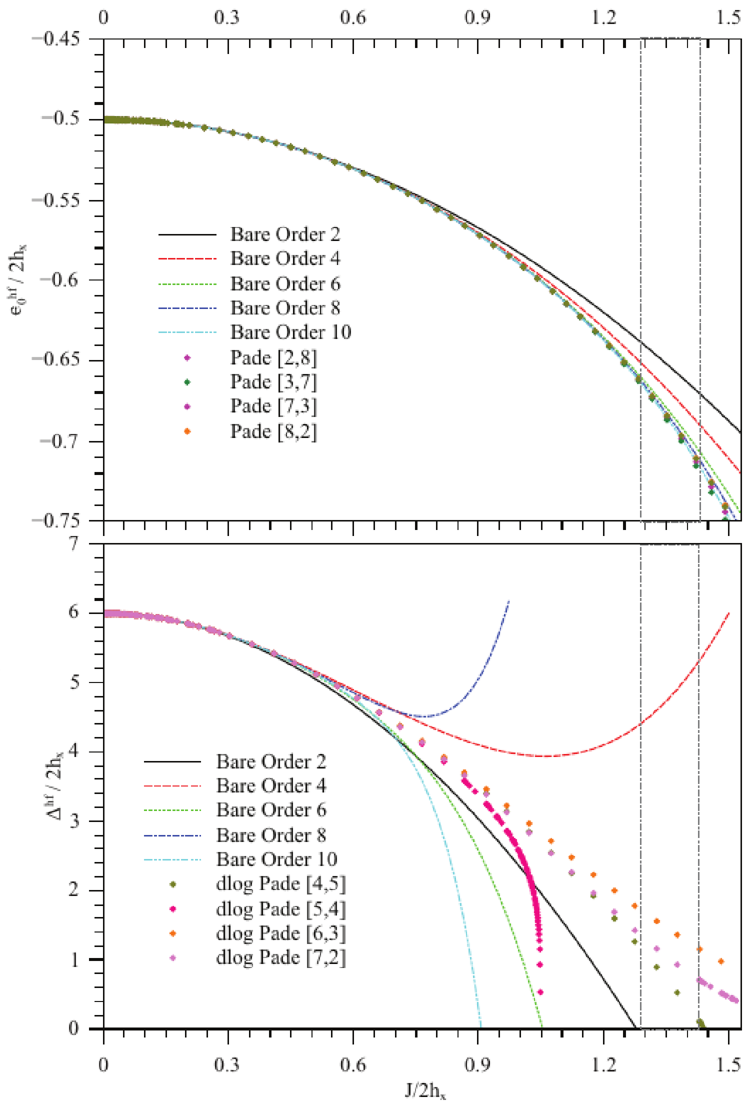
$$\Delta^{\text{lf}} = 1 - 12h_x^2 + 32h_x^4 - \frac{134356}{81}h_x^6 + \frac{18694889252}{893025}h_x^8 - \frac{29786981411535707}{40507614000}h_x^{10}$$

Large Field Results (TCC+Field)

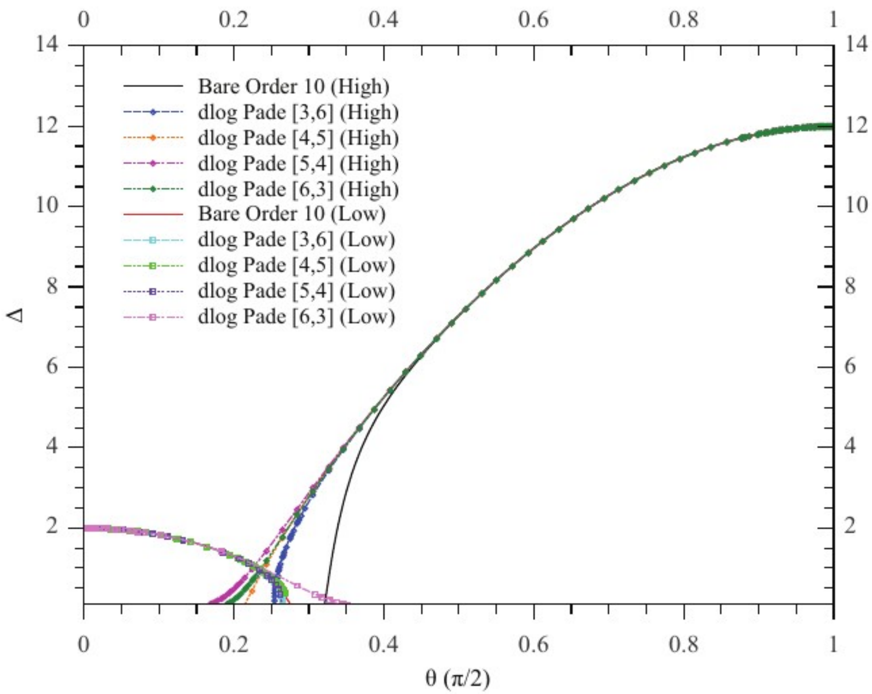


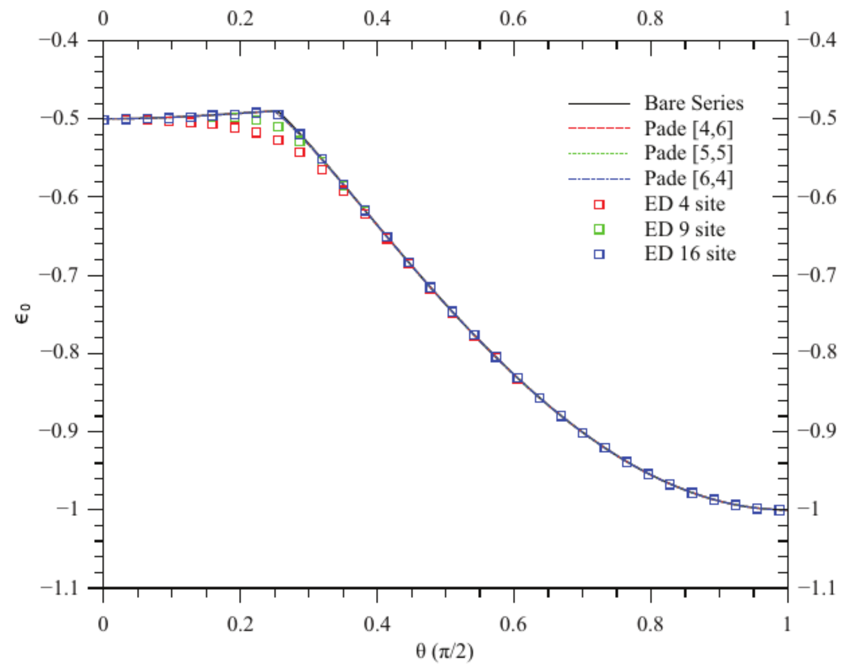
$$\epsilon_0^{\rm hf} = -1 - \frac{1}{6}J^2 - \frac{1}{108}J^4 - \frac{19}{9720}J^6 - \frac{1133}{1866240}J^8 - \frac{12026279}{52907904000}J^{10}$$

$$\Delta^{\rm hf} = 6 - \frac{11}{3}J^2 + \frac{44}{27}J^4 - \frac{413}{144}J^6 + \frac{20157041}{3499200}J^8 - \frac{1446718370831}{105815808000}J^{10}$$



Quantum Phase Transition (TCC+Field)





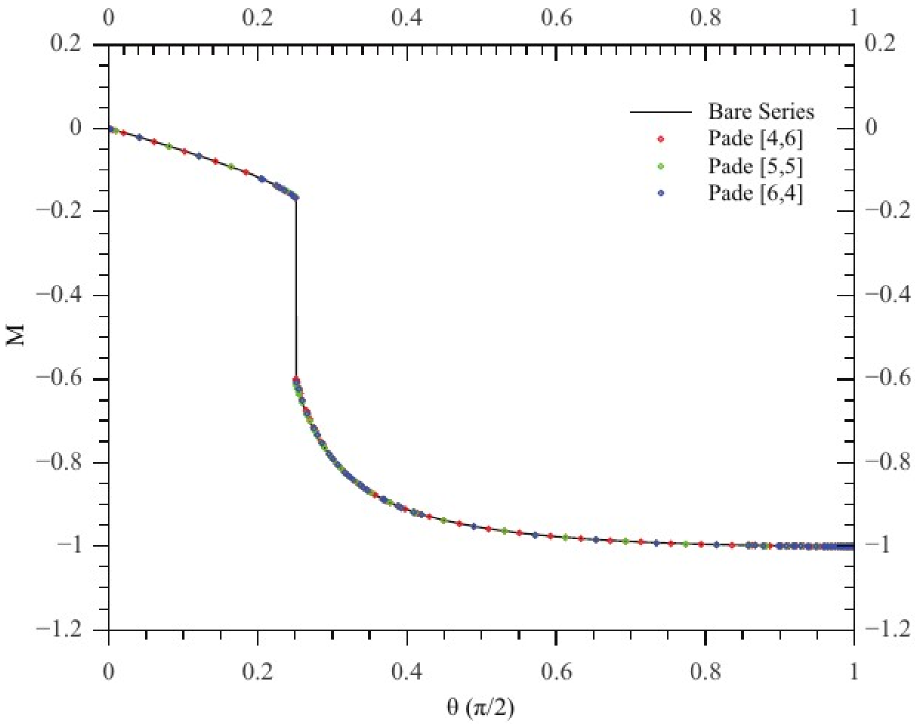
$$h_x = \sin\theta, \quad J = \cos\theta$$

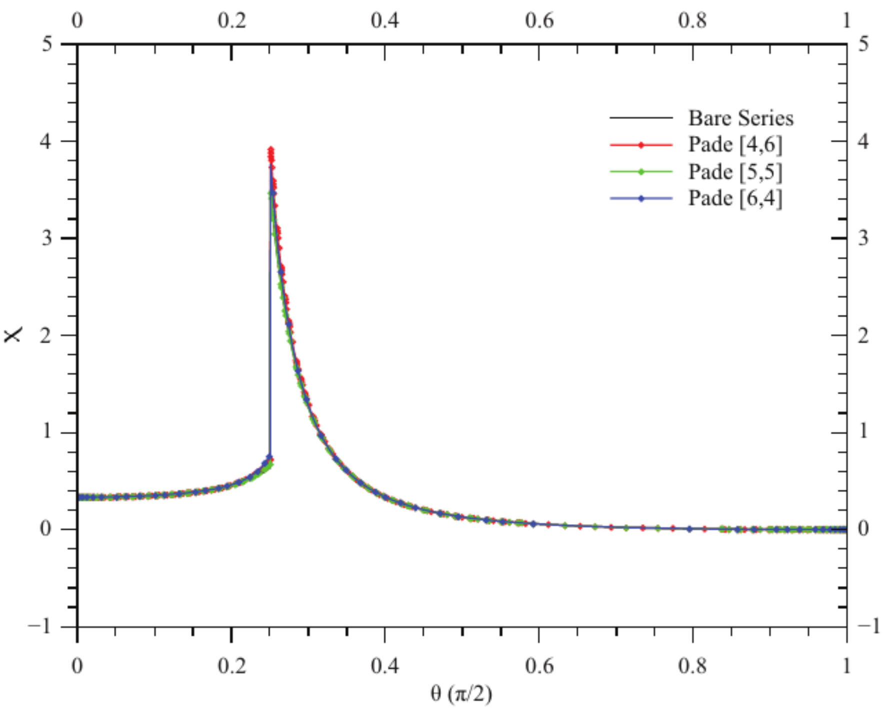
TABLE II: First-order critical point of the small and large field gap intersection and the ground state energy per site.

Gap	θ_c	Ground State	θ_c
dlog Padé [3, 4]	0.382	Padé [1, 7]	0.391
dlog Padé [5, 2]	0.371	Padé [2, 6]	0.396
dlog Padé [6, 1]	0.411	Padé [3, 5]	0.395
dlog Padé [4, 3]	0.377	Padé [4, 4]	0.394
dlog Padé [2, 5]	0.363	Padé [6, 2]	0.394
dlog Padé [1, 6]	0.385	Padé [7, 1]	0.396

S. S. Jahromi, M. Kargarian, S. F. Masoudi, K. P. Schmidt, Phys. Rev. B. 87, 094413 (2013)

Physical Measurement (TCC+Field)



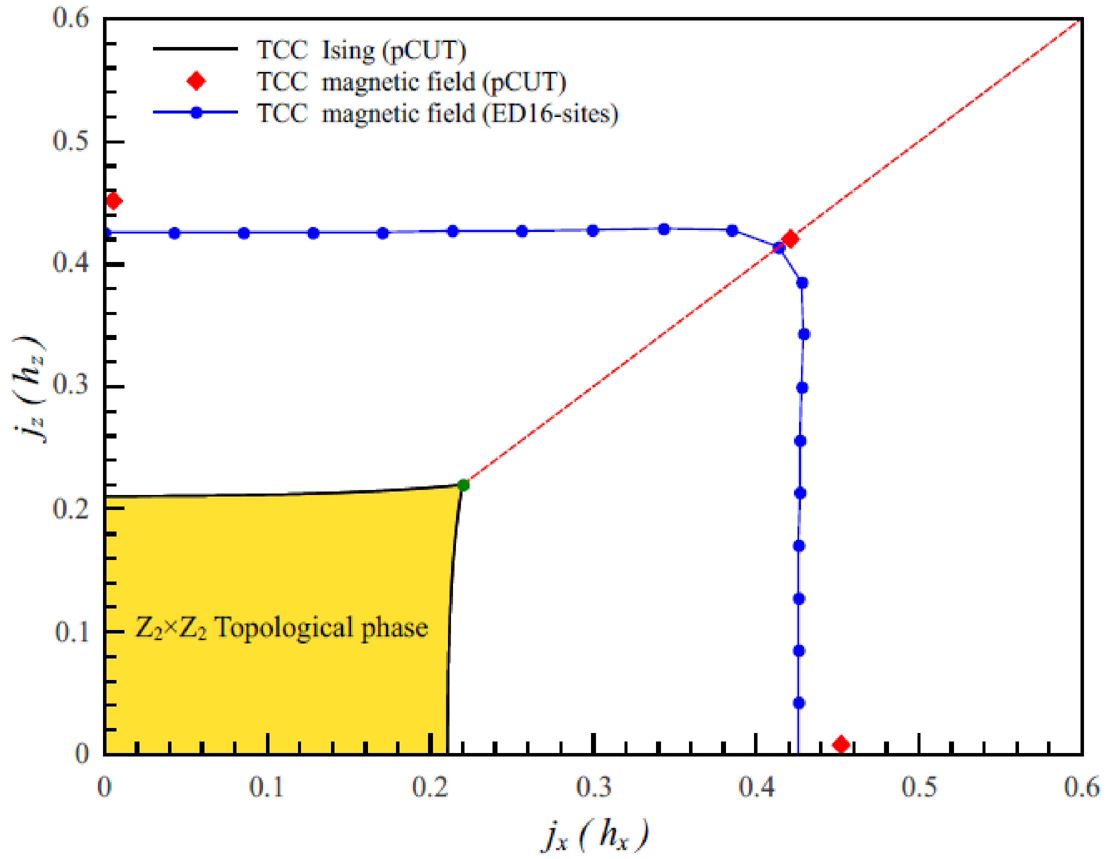


magnetization $(m = d\epsilon_0/dh_x)$

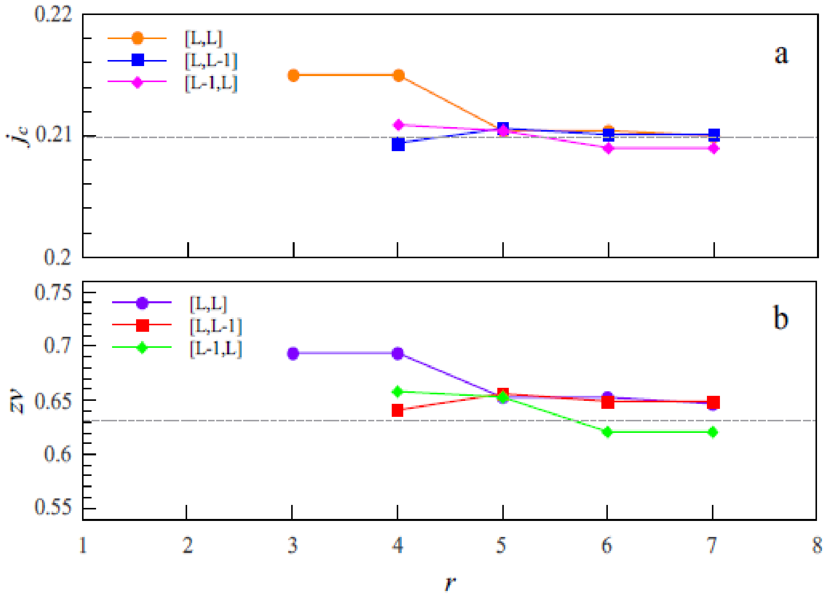
ground state susceptibility $(m = -d^2\epsilon_0/dh_x^2)$

Quantum Phase transition

parallel perturbations (j_x, j_z) or (h_x, h_z) setting $J = 1$

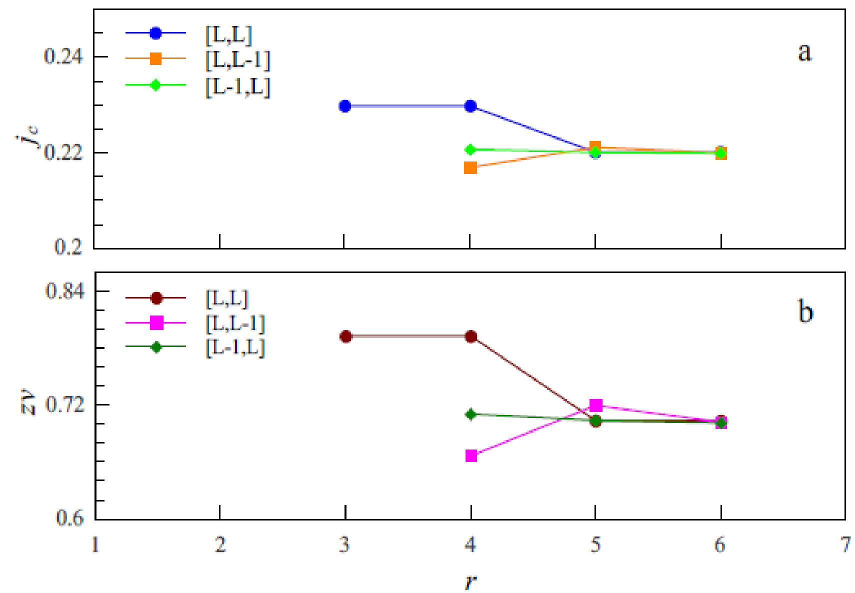


Quantum Phase transition

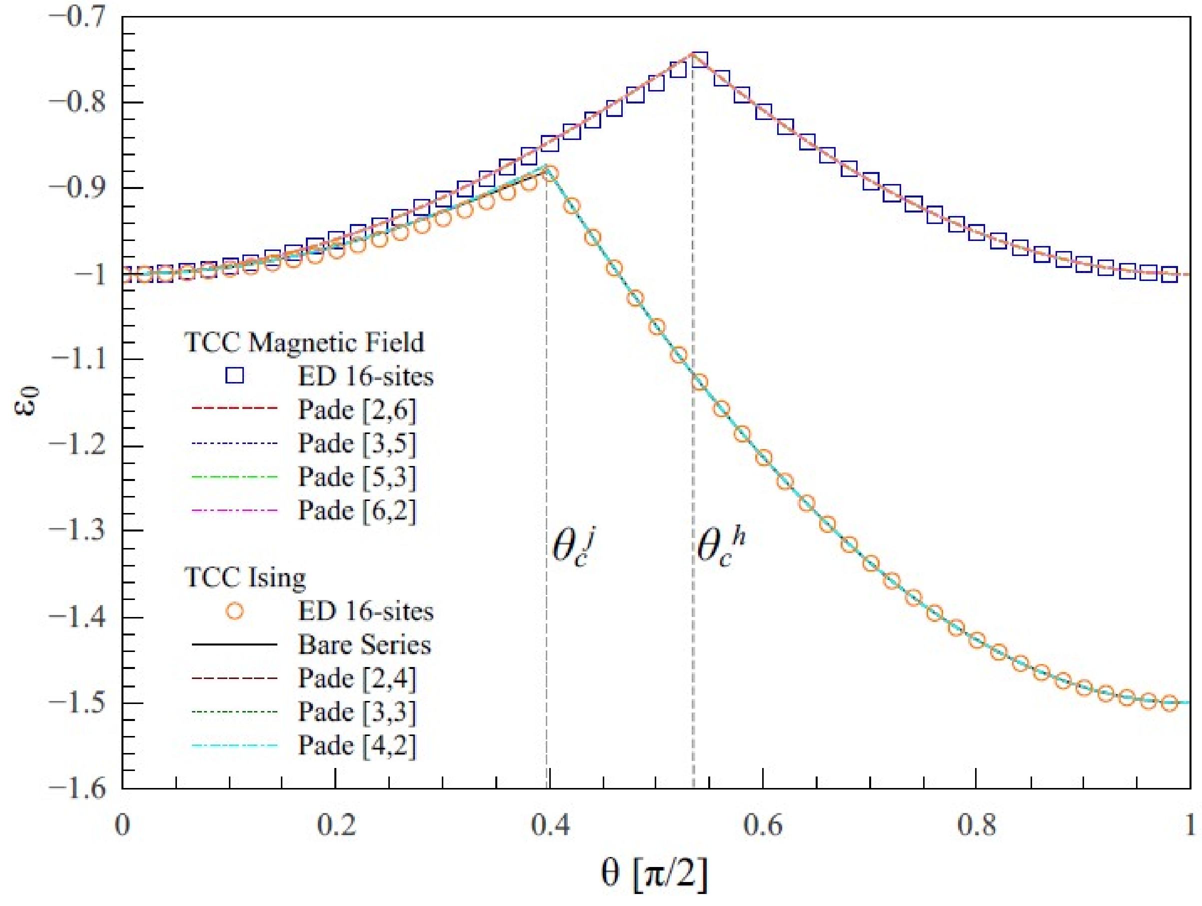


$j_c = 0.209$ and $z\nu = 0.630$

$(j_x = j_z, j_y = 0)$

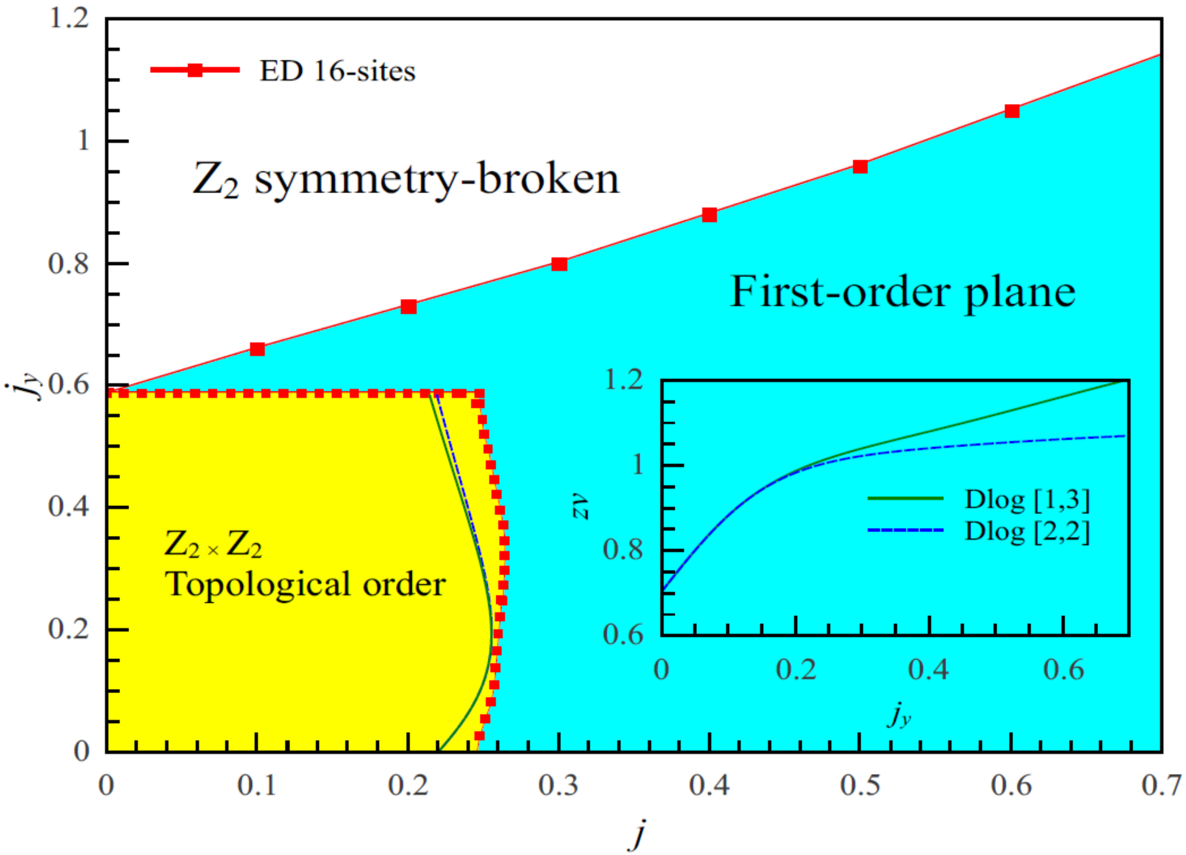


Quantum Phase transition



TCC plus transverse perturbations j_y or h_y

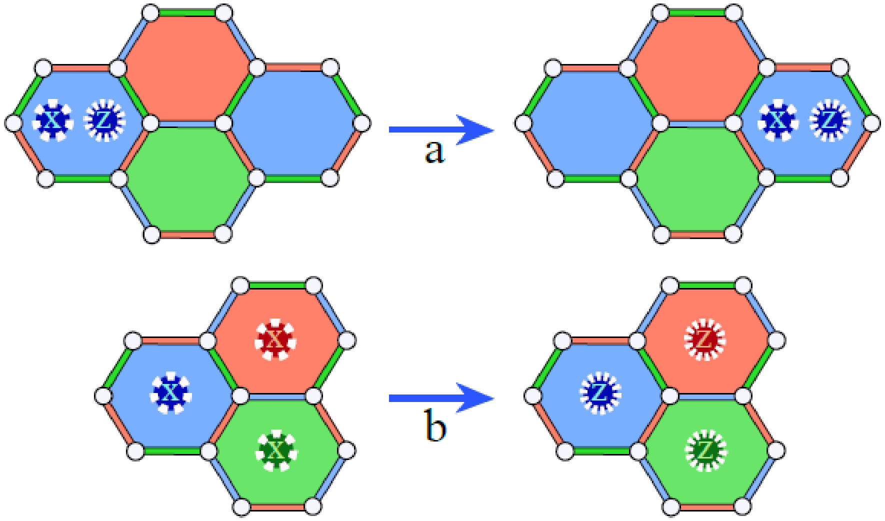
Quantum Phase transition



interactions (j, j_y) with $j \equiv j_x = j_z$ and $J = 1$

Transverse field (y)

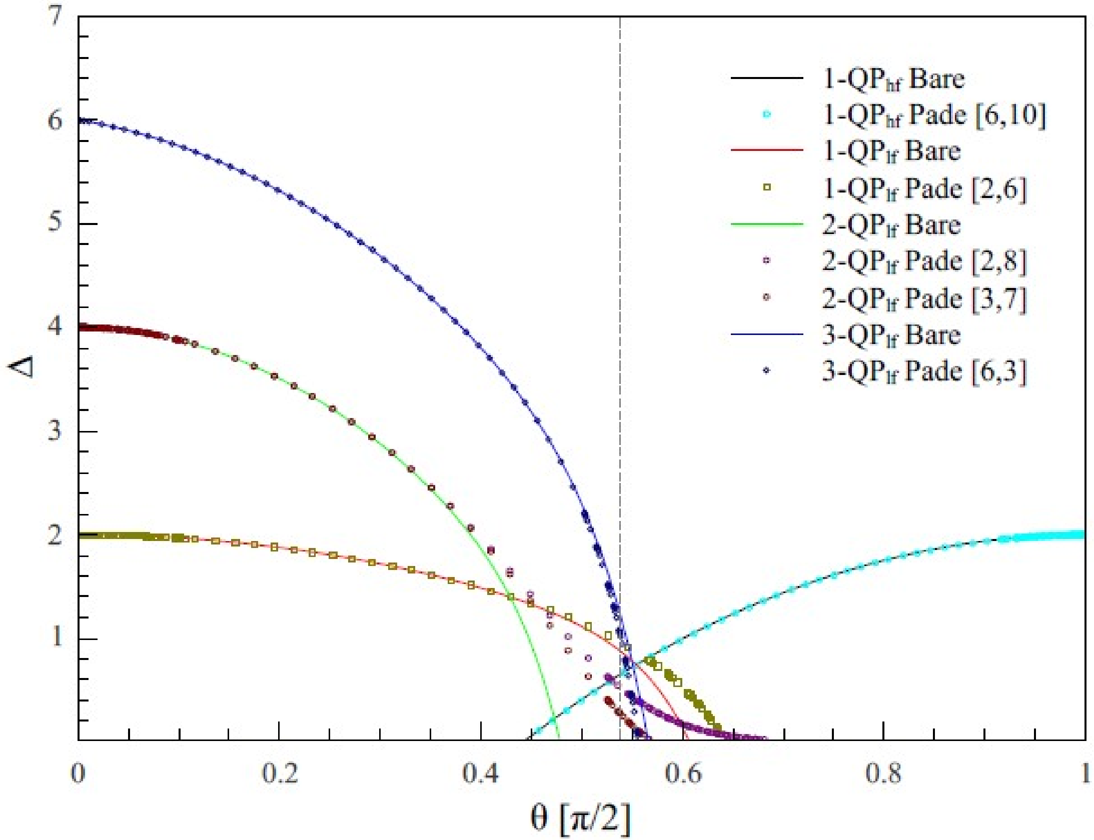
$$H_{\text{trans}} = -J\sum_p (X_p + Z_p) - h_y \sum_i \sigma_i^y$$

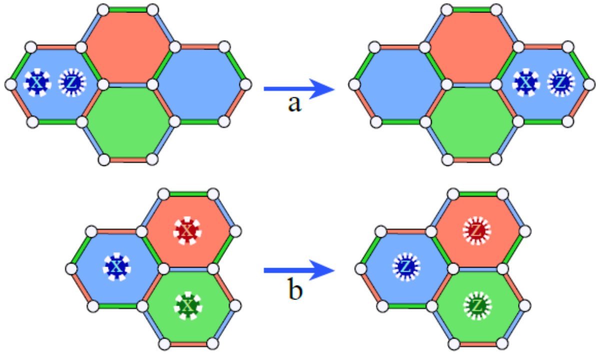


$$\sigma^z = i\sigma^y\sigma^x$$

$$H = -h_y \sum_i \sigma_i^y - J\sum_p X_p(1 - Y_p)$$

Transverse field





S. S. Jahromi, M. Kargarian, S. F. Masoudi, K. P. Schmidt, arXiv:1308.1407

Thanks For Your Attention