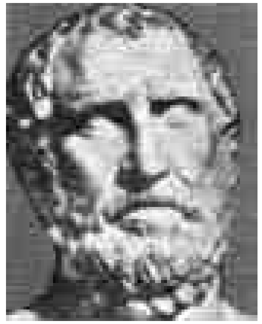
IN THE NAME OF GOD

QUANTUM ZENO EFFECT

QZE : Quantum Zeno Effect

- Zeno of Elea
- Introduction
- QZE fundamentals
- Incomplete measurements
- Nonselective measurements
- Unitary kicks
- Continuous coupling
- Quantum Zeno dynamics

Zeno of Elea



Zeno was an Eleatic philosopher, a native of Elea in Italy,
He was born about 488 BC.

Achilles and the tortoise

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

The flying arrow is at rest

At any given moment it is in a space equal to its own length, and therefore is at rest at that moment. So, it is at rest at all moments. The sum of an infinite number of these positions of rest is not a motion.

Introduction

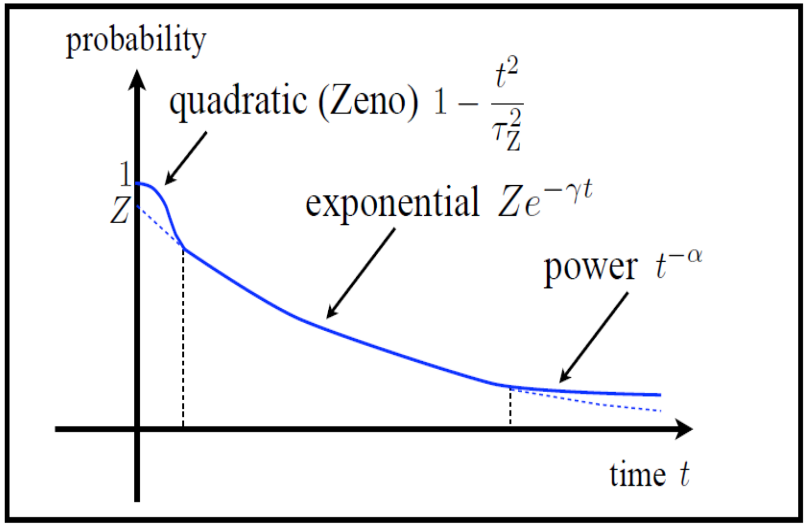
$$\mathcal{A}(t) = \langle\psi_0|\psi(t)\rangle = \langle\psi_0|e^{-iHt}|\psi_0\rangle$$
$$p(t) = |\mathcal{A}(t)|^2 = |\langle\psi_0|e^{-iHt}|\psi_0\rangle|^2$$

$$|\psi(\delta t)\rangle = e^{-iH\delta t}|\psi_0\rangle = |\psi_0\rangle + |\delta\psi\rangle$$

$$\mathcal{A}(\delta t) = 1 - i\langle H\rangle_0 \delta t - \frac{1}{2}\langle H^2\rangle_0(\delta t)^2$$

$$p(\delta t) = 1 - \frac{(\delta t)^2}{\tau_Z^2} + O((\delta t)^4)$$

$$\tau_Z^{-2} \equiv \langle H^2\rangle_0 - \langle H\rangle_0^2$$



$$p^{(N)}(t) = p(\tau)^N = p(t/N)^N$$
$$\overset{N \text{ large}}{\sim} [1-(t/N\tau_Z)^2]^N \sim \exp\left(-t^2/N\tau_Z^2\right) \overset{N\to\infty}{\longrightarrow} 1$$

P. Facchi1, S .Pascazio J. Phys. A: Math. Theor. 41 493001(2008).

Quantum Zeno effect: fundamentals

Consider a quantum system Q, whose states are in $\mathcal{H}$. Time evolution $U(t) = \exp(-iHt)$.

P a projection operator $[P, H] \neq 0$, and $P\mathcal{H} = \mathcal{H}_P$ its eigenspace.

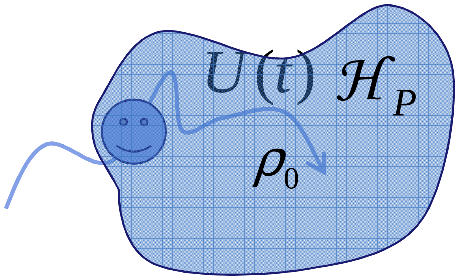
Initial density matrix ρ_0 in $\mathcal{H}_P$: $\rho_0 = P\rho_0 P$, $\mathrm{Tr}[\rho_0 P] = 1$

Perform a measurement at time τ, in order tocheck whether Q has "survived".

$$\rho_0 \longrightarrow \rho(\tau) = PU(\tau)\rho_0 U(\tau)^\dagger P,$$

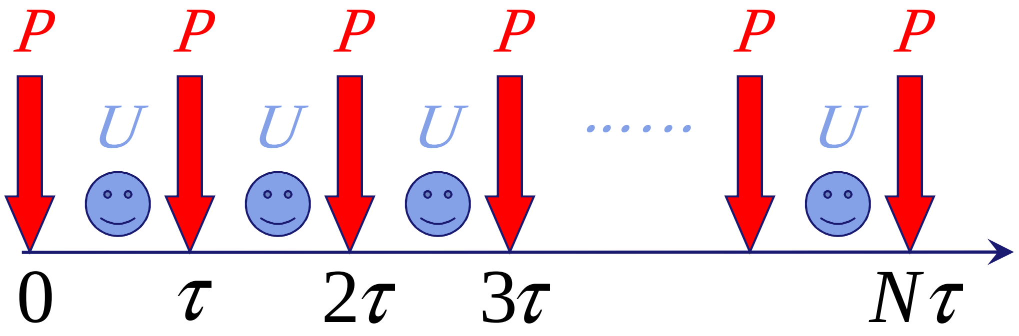
with probability

$$p(\tau) = \mathrm{Tr}[U(\tau)\rho_0 U(\tau)^\dagger P] = \mathrm{Tr}[V(\tau)\rho_0 V(\tau)^\dagger], \qquad V(\tau) = PU(\tau)P$$



QZE: fundamentals

Prepare Q in the initial state ρ_0 at time 0 and perform a series of P-observations at times $\tau_j = jt/N = j\tau, \quad (j = 1, \ldots, N)$.

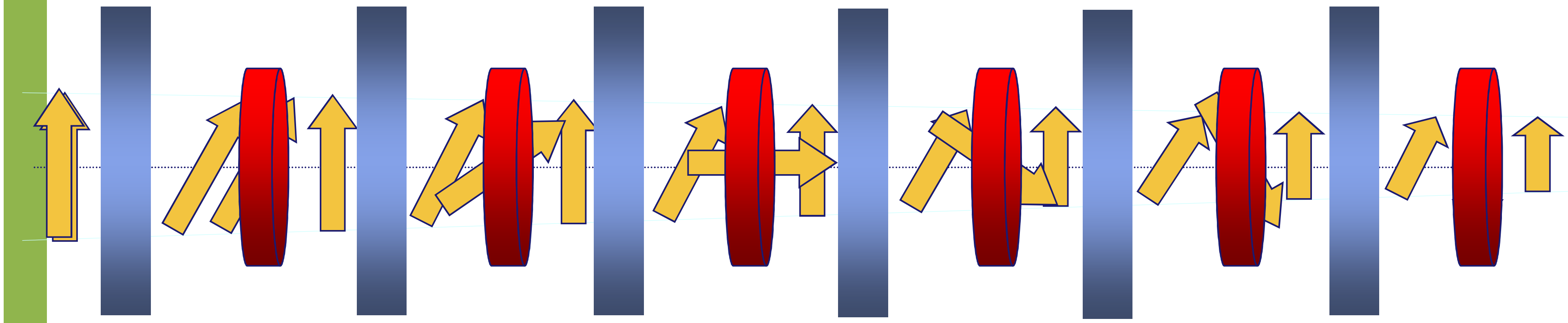


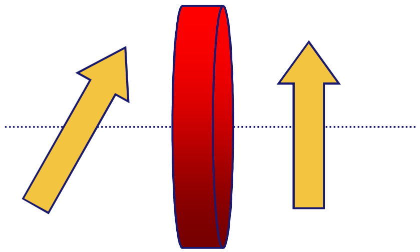
$$\rho^{(N)}(t) = V_N(t)\rho_0 V_N^{\dagger}(t), \qquad V_N(t) = [PU(t/N)P]^N$$

The probability to find the system in $\mathcal{H}_P$ reads

$$p^{(N)}(t) = \mathrm{Tr}[V_N(t)\rho_0 V_N^{\dagger}(t)] \xrightarrow{N\to\infty} \mathrm{Tr}[P\rho_0] = 1$$

Quantum Zeno effect: repeated observations in succession inhibit transitions outside $\mathcal{H}_P$



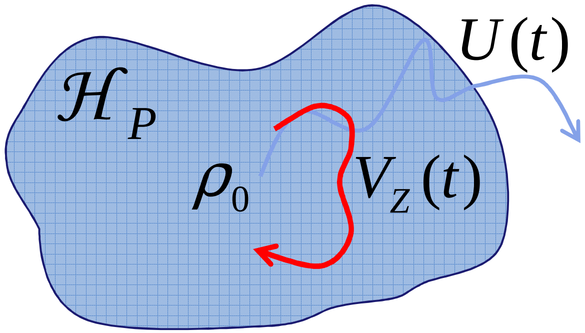


$$p(\theta) = \cos^2 \theta$$

$$p^{(N)}(t) = p\left(\frac{t}{N}\right)^N = \left[\cos^2\left(\frac{t}{\tau_Z N}\right)\right]^N \approx 1 - \frac{t^2}{\tau_Z^2 N} \xrightarrow{N\to\infty} 1$$

S. Pascazio, M. Namiki, G. Badurek and H. Rauch. Physics Letters A 179 (1993) 155

Incomplete measurements



Multidimensional $\mathcal{H}_P = P\mathcal{H}, \; \mathrm{Tr}\, P > 1,$

bounded Hamiltonian, $\|H\| < \infty,$

Limiting time evolution

$$V_Z(t) = \lim_{N\to\infty} V_N(t) = \lim_{N\to\infty} [PU(t/N)P]^N = P\exp(-iPHPt)$$

unitary in $\mathcal{H}_P$, i.e. PHP self-adjoint in $\mathcal{H}_P$.

The state of the system Q is not *completely* specified but can *evolve* in a suitable subspace $\mathcal{H}_P$ (instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$).

Nonselective measurement

Partition $\mathcal{H} = \bigoplus_n \mathcal{H}_{P_n}, \qquad \sum_n P_n = 1$

$$\hat{P}\rho_0 = \sum_n P_n \rho_0 P_n \qquad \hat{P}\rho = \sum_n P_n \rho P_n$$

$$\rho(t) = \hat{V}_t^{(N)} \rho_0$$

$$\rho(t) = \hat{U}_Z(t)\rho_0 = \sum_n U_Z^{(n)}(t)\rho_0 U_Z^{(n)\dagger}(t)$$

$$U_Z^{(n)}(t) = P_n \exp(-\mathrm{i} P_n H P_n t)$$

Time evolution operator for the n-th subspace

$$H_Z = \hat{P}H = \sum_n P_n H P_n$$

Zeno Hamiltonian

$$p_n(t) = \mathrm{Tr}[\rho(t) P_n] = \mathrm{Tr}\left[U_Z^{(n)}(t)\rho_0 U_Z^{(n)\dagger}(t)\right] = \mathrm{Tr}[\rho_0 P_n] = p_n(0)$$

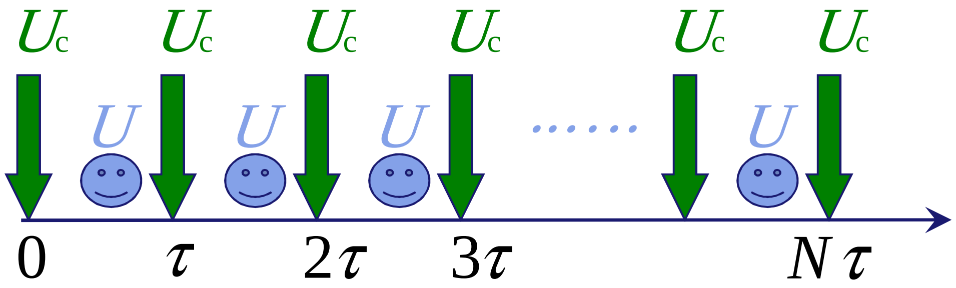
Schwinger, Proc. Nat. Acad. Sc. 45, 1552 (1959)

Unitary kicks

Q undergoes N kicks U_{c} (instantaneous unitary transformations) in a time interval t $(P \rightarrow U_{\mathrm{c}})$

$$V_N(t) = \left[U_{\mathrm{c}} U\left(\frac{t}{N}\right)\right]^N,$$

$$U(t) = \exp(-iHt)$$



Limiting evolution $N \rightarrow \infty$

$$V_{\mathrm{Z}}(t) = \lim_{N\rightarrow\infty} U_{\mathrm{c}}^{\dagger N} V_N(t) = \exp(-iH_{\mathrm{Z}}t)$$

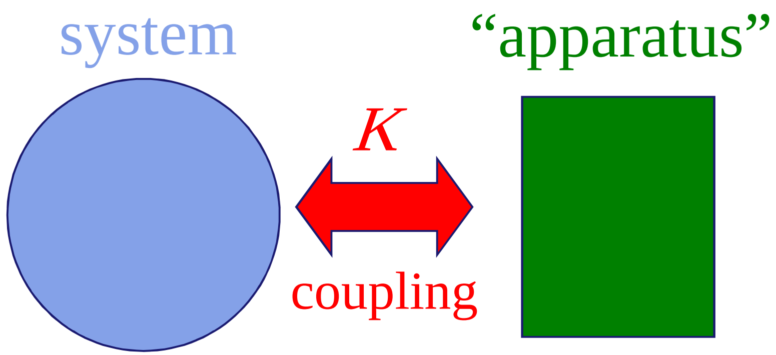
$$H_{\mathrm{Z}} = \sum_n P_n H P_n, \qquad U_{\mathrm{c}} = \sum_n e^{-i\lambda_n} P_n$$

$$V_{\mathrm{Z}}(t) = \sum_n P_n \exp(-iP_n H P_n t)$$

Dynamical decoupling
"Bang-bang" control

Viola and Lloyd, Phys. Rev. A **58**, 2733 (1998).Viola, Knill and Lloyd, Phys. Rev. Lett. **82**, 2417 (1999)
Byrd and Lidar, Quant. Inf. Proc. **1**, 19 (2002).P.F., Lidar, Pascazio, Phys. Rev. A **69**, 032314 (2004)

Continuous coupling



Observed system + "apparatus"

$$H_K = H + KH_{\rm c}, \qquad U_K = \exp(-iH_K t)$$

H system Hamiltonian (+ apparatus free Hamiltonian)

H_c interaction between system and apparatus

K strength of the "measurement".

Strong coupling limit $K \to \infty$

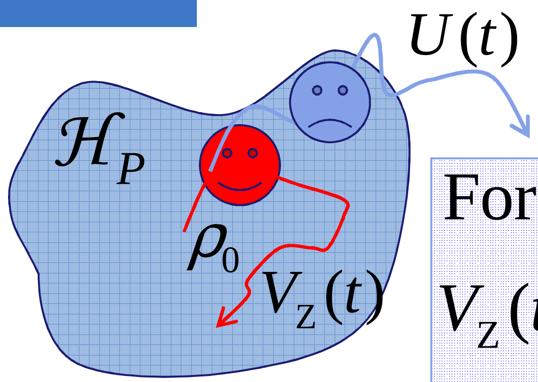
$$V_{\rm Z}(t) = \lim_{K\to\infty} \exp(iKH_{\rm c}t)U_K(t) = \exp(-iH_{\rm Z}t)$$

$$H_{\rm Z} = \sum_n P_n H P_n, \qquad H_{\rm c} = \sum_n \lambda_n P_n$$

$$V_{\rm Z}(t) = \sum_n P_n \exp(-iP_n H P_n t)$$

P. F. and Pascazio, Phys. Rev. Lett. **89**, 080401 (2002)

The quantum Zeno dynamics



Is the Zeno dynamics unitary?
(Misra and Sudarshan 1977: semigroup)

For H bounded,

$$V_Z(t) = \lim_{N\to\infty}[PU(t/N)P]^N = P\exp(-iH_Z t)$$

unitary in $\mathcal{H}_P$, i.e. $H_Z = PHP$ self-adjoint in $\mathcal{H}_P$.

For H unbounded,

$$V_Z(t) = \lim_{N\to\infty}[PU(t/N)P]^N \overset{?}{=} P\exp(-iH_Z t)$$

PHP in general is not self-adjoint, nor even closed

Answer: under general hypotheses YES

$$H_Z = \left(H^{1/2}P\right)^\dagger \left(H^{1/2}P\right),$$

if H_Z densely defined.

Exner P and Ichinose T 2005 Ann. H. Poincar´e 6 195. Misra and Sudarshan, J. Math. Phys. **18**, 756 (1977)

Free particle in D dimensions

$$H=\frac{p^2}{2M}=-\frac{\Delta^2}{2M},\quad U(t)=\exp(-iHt)\quad \text{in } L^2(\mathbf{R}^D)$$

$\Omega\subset\mathbf{R}^D$, $P=\chi_\Omega(x)$ spatial projection

$$\text{Zeno dynamics}\quad V_Z(t)=\lim_{N\to\infty}\left[V\left(\frac{t}{N}\right)\right]^N,\quad V(s)=PU(s)P$$

How does the particle move inside Ω? Does it leak out?

THEOREM

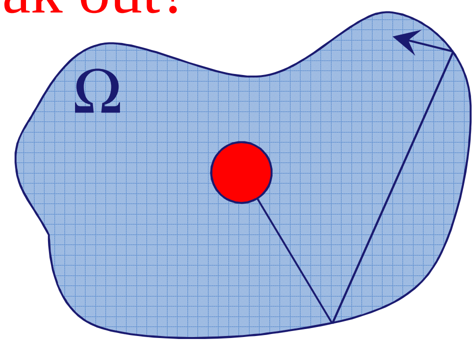
The limit $V_Z(t)$ exists and yields

$V_Z(t)=P\exp(-iH_Z t)$, a unitary group in $L^2(\Omega)$

where the Zeno Hamiltonian reads

$$H_Z=-\frac{\Delta^2}{2M},\quad D[H_Z]=\left\{\psi\in H^2(\Omega)\,|\,\psi(\partial\Omega)=0\right\}$$

Dirichlet boundary conditions

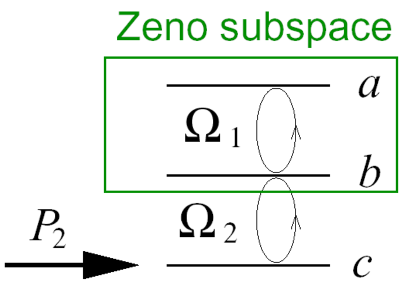


Free particle in a box with perfectly reflecting hard walls

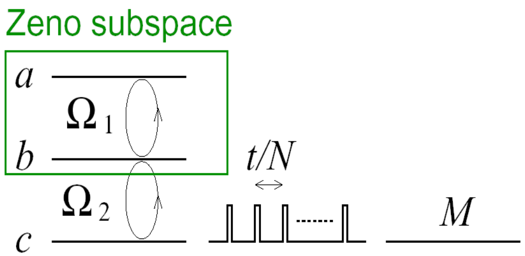
…although there is NO wall!

Facchi P, Pascazio S, Scardicchio A and Schulman L S 2001 Phys. Rev. A 65 012108

Different manifestations of the same physical phenomenon

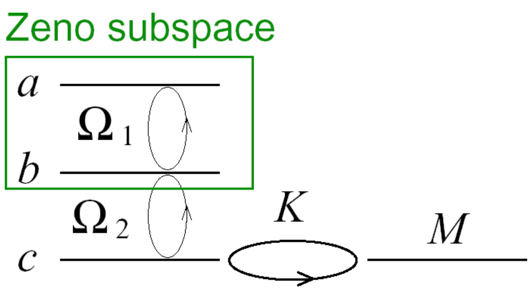


Von Neumann Measurements
(projections)



Unitary kicks

(bang-bang control)



Continuous coupling

Nonselective measurement

EXAMPLE:

$$H = \Omega_1(|a\rangle\langle b| + |b\rangle\langle a|) + \Omega_2(|b\rangle\langle c| + |c\rangle\langle b|) = \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & \Omega_2 \\ 0 & \Omega_2 & 0 \end{pmatrix}.$$

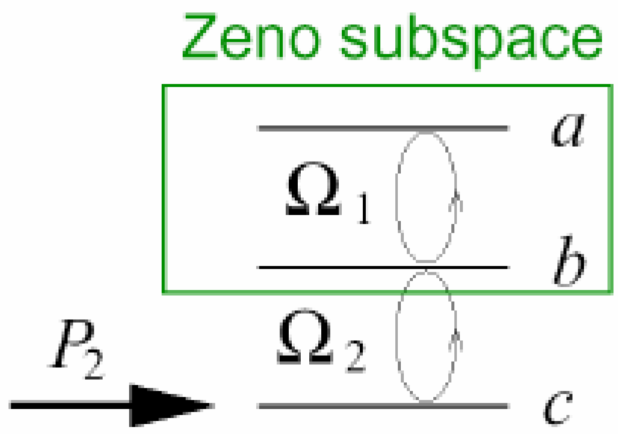
$$P_1 = |a\rangle\langle a| + |b\rangle\langle b| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = |c\rangle\langle c| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (P_1 + P_2 = \mathbf{1})$$

$$H_Z = P_1 H P_1 + P_2 H P_2 = \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$H = \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & \Omega_2 \\ 0 & \Omega_2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \Omega_1 & 0 \\ \Omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = H_Z$$

$\Uparrow$

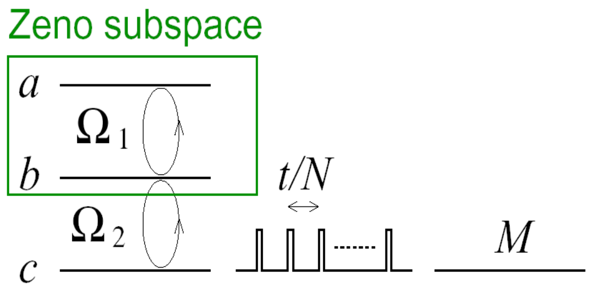
Zeno limit $(N \to \infty)$



Unitary kicks

EXAMPLE:

$$U_{\text{kick}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\lambda & -\mathrm{i}\sin\lambda \\ 0 & 0 & -\mathrm{i}\sin\lambda & \cos\lambda \end{pmatrix}$$



$$H = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ \Omega_1 & 0 & \Omega_2 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ \Omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = H_Z$$

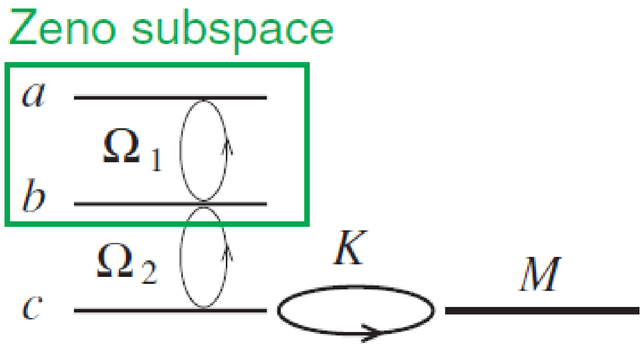
$\Uparrow$

Zeno limit $(N \rightarrow \infty)$

$$U_N(t) \sim \exp\left(-\mathrm{i}\sum_n N\lambda_n P_n + P_n H P_n t\right) = \begin{pmatrix} \cos\Omega_1 t & -\mathrm{i}\sin\Omega_1 t & 0 & 0 \\ -\mathrm{i}\sin\Omega_1 t & \cos\Omega_1 t & 0 & 0 \\ 0 & 0 & \cos N\lambda & -\mathrm{i}\sin N\lambda \\ 0 & 0 & -\mathrm{i}\sin N\lambda & \cos N\lambda \end{pmatrix}.$$

Continues strong coupling

$$K H_c = K(|c\rangle\langle M| + |M\rangle\langle c|) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K \\ 0 & 0 & K & 0 \end{pmatrix}$$



$$H_c = \eta_1 P_1 + \eta_- P_- + \eta_+ P_+, \qquad \eta_1 = 0, \eta_\pm = \pm 1$$

$$H = \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ \Omega_1 & 0 & \Omega_2 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \Omega_1 & 0 & 0 \\ \Omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = H_Z$$

Zeno limit $(K \rightarrow \infty)$

$$U_K(t) \sim \exp\left(-\mathrm{i} \sum_n K t \eta_n P_n + P_n H P_n t\right) = \begin{pmatrix} \cos \Omega_1 t & -\mathrm{i} \sin \Omega_1 t & 0 & 0 \\ -\mathrm{i} \sin \Omega_1 t & \cos \Omega_1 t & 0 & 0 \\ 0 & 0 & \cos K t & -\mathrm{i} \sin K t \\ 0 & 0 & -\mathrm{i} \sin K t & \cos K t \end{pmatrix}$$

Streed E W, Mun J, Boyd M, Campbell G K, Medley P, Ketterle W and Pritchard D E 2006 *Phys. Rev. Lett.* 97 260402

Superselection rule