Quantum algorithmic complexity

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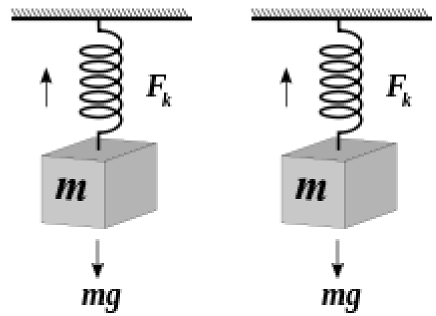
Ergodic Theory



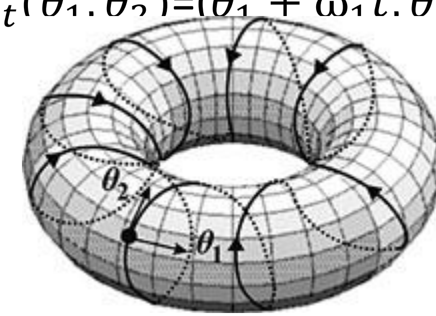
$$H(r) = H_1(r) + H_2(r), \qquad H_i(q_i, p_i) = \frac{p_i^2}{2m_1} + \frac{m_i \omega_i^2}{2} q_i$$

- Phase spase $\quad R^4 \leftrightarrow T^2 \coloneqq \{(\theta_1, \theta_2):\ 0 \le \theta_1, \theta_2 < 2\pi\}$

- Dynamical system $\quad (\mathbb{T}^2,\ T = T_1,\ \frac{d\theta_1\, d\theta_2}{4\pi^2})$

Where the Hamiltonian flux is $T_t(\theta_1, \theta_2)$=$(\theta_1 + \omega_1 t, \theta_2 + \omega_2 t)$.

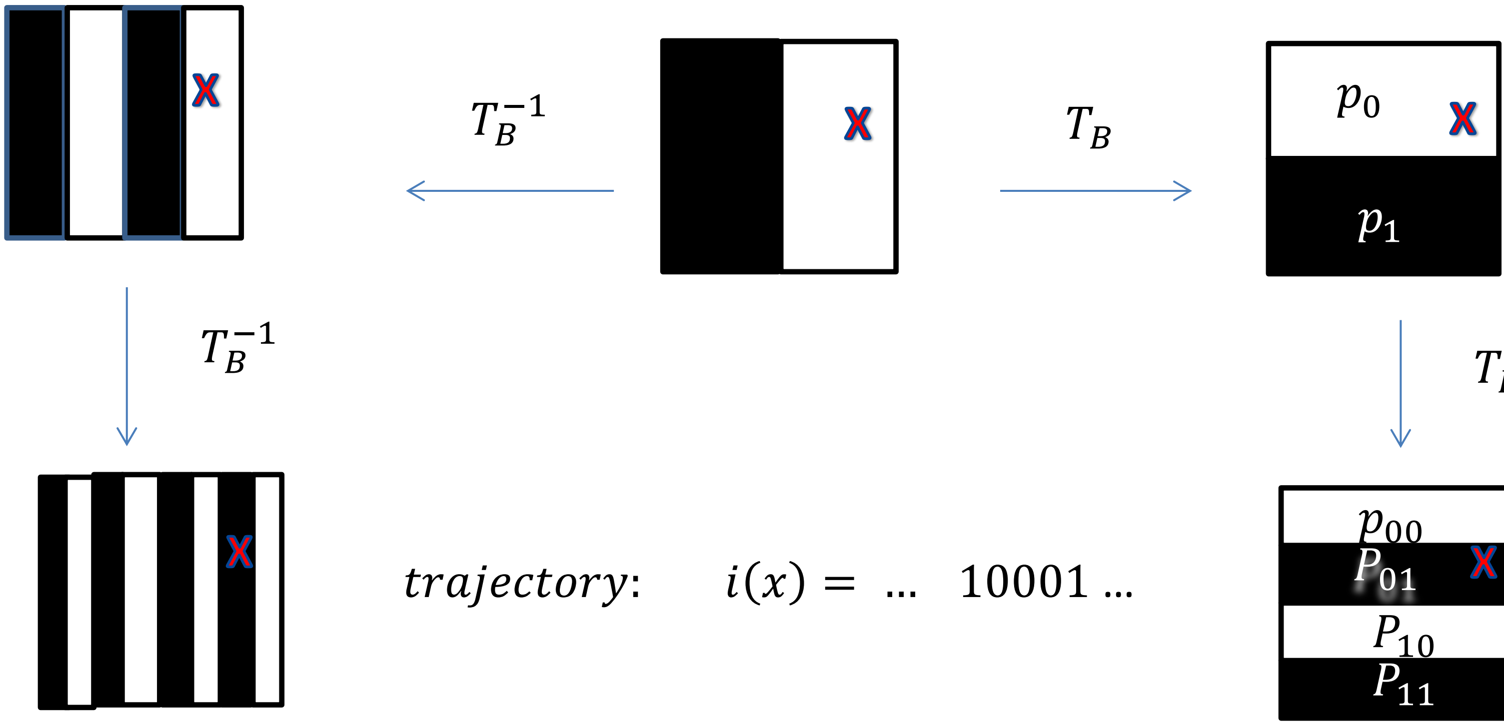
- Ergodic dynamical system



Symbolic dynamical system and Baker map

$(\mathbb{T}^2, T_B, dx\,dy)$

- $T_B(x,y) = \begin{cases} \left(2x, \frac{y}{2}\right) & 0 \le x < \frac{1}{2} \\ \left(2x-1, \frac{1+y}{2}\right) & \frac{1}{2} \le x < 1 \end{cases}$
- $T_B^{-1}(x,y) = \begin{cases} \left(\frac{x}{2}, 2y\right) & 0 \le y < \frac{1}{2} \\ \left(\frac{1+x}{2}, 2y-1\right) & \frac{1}{2} \le y < 1 \end{cases}$



Symbolic dynamical system $(\Omega_2, T_\sigma, \pi)$

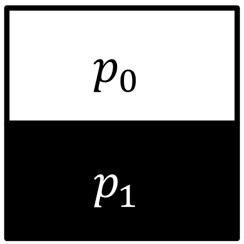
$$i(x) = \quad \dots.10001\dots$$

$$T_\sigma(i(x)) = \quad \dots10001\ \dots$$

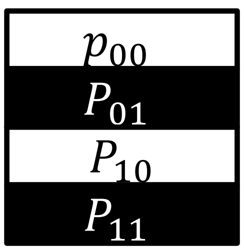
$\pi^{(n)}(i^{(n)}) = \frac{1}{2^n}$ where $\pi^{(n)}(0i_1 \dots i_{n-1}) + \pi^{(n)}(1i_1 \dots i_{n-1}) = \pi^{(n-1)}(i_1 \dots i_{n-2})$

KS-Entropy

Symbolic dynamical system $(\Omega_2, T_\sigma, \pi)$



$$H\left(P^{(1)}\right) = -p(0)\log p(0) - p(1)\log p(1)$$



$$H\left(P^{(2)}\right) = -\sum_{i_1 i_2 \in \Omega_2^2} p(i_1 i_2)\log p(i_1 i_2)$$

- Entropy rate

$$h(\pi, P) = \lim_{n\to\infty} \frac{1}{n} H(P^{(n)})$$

- KS-entropy

$$h(\pi) = \sup_{\mathrm{P}} \mathrm{h}(\pi, \mathrm{P})$$

Turing machine and binary coding

0	0	1	1	1	0	0	0

$(q_1, 1) \rightarrow (q_2, 0, R)$

A= 01001001010 11

0	0	0	1	1	0	0	0

$(q_2, 1) \rightarrow (q_3, 0, R)$

B= 0010010001010 11

0	0	0	0	1	0	0	0

$(q_3, 1) \rightarrow (q_4, 1, L)$

C =00010010000100100 11

0	0	0	0	1	0	0	0

$0 \rightarrow 0$

$1 \rightarrow 00$

$q_i \rightarrow 00 \ldots 0 \quad i\ times$

$R \rightarrow 0$

$L \rightarrow 00$

Program: 1 $\underbrace{0000}_{number\ of\ internal\ states}$ 11 $\underbrace{00}_{number\ of\ alphabets}$ 11 A B C 111

Kolmogorov complexity

- 1111111111111111111111
  Print **1** 20 times

- 111000111000111000111000
- Print 111000 4 times

- 1010100011101011010100

- Kolmogorov is defined as algorithmic (descriptive) complexity of an object $i_1 i_2 \ldots i_n$ as the length of the shortest binary computer program that describes the object $i_1 i_2 \ldots i_n$.

Brudno Theorem

- If $(\Omega_2, T_\sigma, \pi)$ be a binary ergodic source with entropy rate $h(\pi)$. Then,

$$\mathrm{K}(i) = lim_{n\to\infty} \frac{1}{n} K\left(i^{(n)}\right) = h(\pi) \ ,$$

for almost all trajectories $i \in \Omega_2$.

$\boldsymbol{K(i)}$

- Quantum qubit complexity — Berthiaume, van Dam, Laplante
- Quantum bit complexity — Vitany
- *circuit quantum complexity* — Mora, Briegel
- *Gacs quantum algorithmic complexity* — Gacs

KS-entropy

- Non-commutative generation of KS-entropy — Alicki, Fannes
- Decomposition of states — Connes, Narnhofer, Thirring

semi-computable Semi-measure

- A positive function $f\colon \Omega_2^* \to R$ is called a semi-measure if

$$\sum_{i \in \Omega_2^*} f(i) \leq 1.$$

- It is a semi-computable function if there is a monotonically increasing sequence $g_n(x)$ of functions converging to it such that $(n, x) \to g_n(x)$ is a computable function mapping into rational numbers.

- **Theorem.** There is a semi-measure semi-computable $\mu\colon \Omega_2^* \to R$ such that for any semi-measure semi-computable f there is a constant $c > 0$ such that

$$cf(i) \leq \mu(i).$$

- Relation between semi-measure and Kolmogorov complexity

$$K\left(i^{(n)}\right) + c_1 \leq -\log \mu\left(i^{(n)}\right) \leq K\left(i^{(n)}\right) + 2\log \mathrm{n} + \mathrm{c}_2$$

Quantum algorithmic complexity

- Elemetary semi-computable

$$\rho_n = \sum_{i^{(n)}} a_{i^{(n)}} \ |i^{(n)} >< i^{(n)}|$$

- Semi-computable semi-density matrix

$$Tr\, \rho \leq 1 \quad ,$$

there is a sequence of quasi –increasing of elementary semi-computable $\rho_n \nearrow \rho$

- **Theorem.** There is a universal semi-density semi-computable matrix $\hat{\mu}$ such that for any semi-density semi-computable matrix ρ there is a constant $c > 0$ such that

$$c\rho \leq \hat{\mu}.$$

- Gacs algorithmic complexity

$$\overline{H}(\rho) = -Tr(\rho \log \hat{\mu}) \ , \ \ \underline{H}(\rho) = -\log Tr(\rho\, \hat{\mu})$$

One dimensional Quantum spin chain

- $M_{[-n,n]} = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots \otimes M_2\ (\mathbb{C})\ \ , 2n + 1\ times$

- Quantum spin chain $(M,\ \Theta_\sigma,\ \omega)$

- $M = \lim_{n\to\infty} M_{[-n,n]}$

- Dynamic Θ_σ:

$$\dots 1_{-n-2} \otimes 1_{-n-1}\ \otimes A \otimes 1_{n+1} \otimes 1_{n+2}$$
$$\to \cdots 1_{-n-1} \otimes 1_{-n}\ \otimes A \otimes 1_{n+2} \otimes 1_{n+3}$$

- State ω: $M \to\ \mathbb{C}$ is a translation invariant over M where $\omega \circ \Theta_\sigma = \omega$

Example for translation invariant state

- Let $A_{ijk} = A_i \otimes A_j \otimes A_k$

- $\rho^{(2)} = a\, A_{00} + \left(\frac{1}{2} - a\right) A_{01} + \left(\frac{1}{2} - a\right) A_{10} + a\, A_{11}$
- $Tr_{\{1\}}\rho^{(2)} = Tr_{\{2\}}\rho^{(2)} = \frac{1}{2} A_0 + \frac{1}{2} A_1 = \rho^{(1)}$

- $\rho^{(3)} = bA_{000} + (a-b)A_{001} + \left(c - 2a + \frac{1}{2}\right) A_{011} + (a-b)A_{100} + \left(b - 2a + \frac{1}{2}\right) A_{101} + (a-c)A_{110} + c\, A_{111}$
- $Tr_{\{1\}}\rho^{(3)} = Tr_{\{3\}}\rho^{(3)} = \rho^{(2)}$

- $\omega(A) = Tr(\rho\, A),$ where $\rho = \lim \rho^{(n)}$

Result

- **Theorem.** Let $\rho^{(n)} \in H_{[-n,n]}$ be a computable sequence of semi-computable density matrices giving rise to a shift-invariant state ω on the quantum spin chain M. Then

$$\lim_{n\to\infty} \frac{\overline{H}\left(\rho^{(n)}\right)}{2n+1} = \lim_{n\to\infty} \frac{S\left(\rho^{(n)}\right)}{2n+1} = h_{\omega}{}^{CNT}(\Theta_{\sigma})$$

References

- Fabio Benatti, Dynamics, information and complexity in quantum systems, Theoretical and Mathematical Physics, Springer, 2009.

- Peter Gacs, Quantum algorithmic entropy, J. Phys. A 34 (2001), 6859-6880.

- M.Li and P.M.P. Vitany, Introduction to Kolmogorov complexity and its applications, Springer, New York, NY, USA, 2008.