Mid-Term(1) Exam Solutions

1.

a.

$$e^{i\theta\sigma_x} = \sum_{n=0} \frac{(i\theta\sigma_x)^n}{n!}$$

On the other hand,

$$\sigma_x^2 = \mathbb{1} \qquad \rightarrow \qquad \sigma_x^{2k} = \mathbb{1}\,, \quad \sigma_x^{2k+1} = \sigma_x$$

hence,

$$\begin{aligned} e^{i\theta\sigma_x} &= \sum_{k=0}(-1)^k \frac{\theta^{2k}}{(2k)!} + i\sigma_x \sum_{k=0}(-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ &= \mathbb{1}\cos\theta + i\sigma_x \sin\theta \end{aligned}$$

b.

$$\begin{aligned} e^{i\theta\sigma_x}\sigma_z e^{i\theta\sigma_x} &= (\mathbb{1}\cos\theta + i\sigma_x\sin\theta)\sigma_z(\mathbb{1}\cos\theta - i\sigma_x\sin\theta) \\ &= \sigma_z\cos^2\theta + \sigma_x\sigma_z\sigma_x\sin^2\theta + i[\sigma_x,\sigma_z]\sin\theta\cos\theta \\ &= \sigma_z\cos^2\theta - \sigma_z\sigma_x^2\sin^2\theta + 2\sigma_y\sin\theta\cos\theta \\ &= \sigma_z\cos(2\theta) + \sigma_y\sin(2\theta) \end{aligned}$$

Thus

$$a_1 = 0\,, \quad a_2 = \sin(2\theta)\,, \quad a_3 = \cos(2\theta).$$

2.

Firstly we define,

$$\begin{aligned} \overrightarrow{L} &= \overrightarrow{S_1} + \overrightarrow{S_3} \\ \overrightarrow{J} &= \overrightarrow{L} + \overrightarrow{S_2} \end{aligned}$$

Notice that ℓ has values 0 and 1. For $\ell = 0$ we have $j = 1/2$ and for $\ell = 1$, j is $1/2$ or $3/2$. Then we find

$$\begin{aligned} H &= \frac{\mathcal{E}_1}{\hbar^2}\overrightarrow{L}\cdot\overrightarrow{S_2} + \frac{\mathcal{E}_2}{\hbar^2}(J_z)^2 \\ &= \frac{\mathcal{E}_1}{\hbar^2}\frac{1}{2}\left(J^2 - L^2 - S_2^2\right) + \frac{\mathcal{E}_2}{\hbar^2}(J_z)^2 \end{aligned}$$

In the basis which are the eigenstates of L^2, S_2^2, J^2 and J_z we find the energies:

$$E_{\ell j m_j} = \mathcal{E}_1 \frac{1}{2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4}\right) + \mathcal{E}_2 m_j^2$$

$$\begin{array}{llllr} \ell=0 & j=\frac{1}{2} & m_j=\pm\frac{1}{2} & \rightarrow & E_{0\frac{1}{2}\frac{1}{2}} = \mathcal{E}_2/4 \\ \ell=1 & j=\frac{1}{2} & m_j=\pm\frac{1}{2} & \rightarrow & E_{1\frac{1}{2}\frac{1}{2}} = -\mathcal{E}_1 + \mathcal{E}_2/4 \\ \ell=1 & j=\frac{3}{2} & m_j=\pm\frac{1}{2} & \rightarrow & E_{1\frac{3}{2}\frac{1}{2}} = \mathcal{E}_1/2 + \mathcal{E}_2/4 \\ \ell=1 & j=\frac{3}{2} & m_j=\pm\frac{3}{2} & \rightarrow & E_{1\frac{3}{2}\frac{3}{2}} = \mathcal{E}_1/2 + 9\mathcal{E}_2/4 \end{array}$$

For each energy we have a double degeneracy.

For the case where $S_{1z} = +\hbar/2$ and $S_{3z} = +\hbar/2$, we know $\ell = 1$, then j is either 1/2 or 3/2 with $m_j = 1/2$, thus energies can be $E_{1\frac{1}{2}\frac{1}{2}}$ and $E_{1\frac{3}{2}\frac{1}{2}}$. This means that the state $|+-+\rangle$ can be written as a combination of $|\ell = 1, j = \frac{1}{2}, m_j = \frac{1}{2}\rangle$ and $|\ell = 1, j = \frac{3}{2}, m_j = \frac{1}{2}\rangle$.

3.

The $n = 2$ level in the ideal Hydrogen atom has 4 degeneracy for ϕ_{200}, ϕ_{211}, ϕ_{210}, ϕ_{21-1}. Thus we should find the eigenvalues and eigenstates of the 4×4 potential matrix:

$$\begin{aligned} \langle\phi_{2\ell m}|\mathcal{E}(x^2+y^2-z^2)|\phi_{2\ell' m'}\rangle &= \langle\phi_{2\ell m}|\mathcal{E}r^2(\sin^2\theta - \cos^2\theta)|\phi_{2\ell' m'}\rangle \\ &= \langle\phi_{2\ell m}|\mathcal{E}r^2(1 - 2\cos^2\theta)|\phi_{2\ell' m'}\rangle \\ &= \mathcal{E}\int r^4 dr \int d\varphi \int d\cos\theta \phi^*_{2\ell m}(1-2\cos^2\theta)\phi_{2\ell' m'} \\ &= \mathcal{E}\int r^4 R_{2\ell}R_{2\ell'} dr \int d\varphi \int d\cos\theta Y^*_{\ell m} Y_{\ell' m'}(1-2\cos^2\theta) \end{aligned}$$

From φ integration we find $m = m'$ otherwise the integral gives zero. Thus the only non-vanishing non-diagonal element can be $\langle\phi_{200}|\mathcal{E}(x^2+y^2-z^2)|\phi_{210}\rangle$, however this element is zero since ϕ_{210} is an odd function while ϕ_{200} and $(x^2+y^2-z^2)$ are even functions. So, we have to calculate only 4 diagonal elements.

We use the following relations:

$$\langle r^2\rangle_{n\ell} = \int_0^\infty r^4 (R_{n\ell}(r))^2 dr = \left(\frac{a_0}{Z}\right)^2 \frac{n^2}{2}\left[5n^2 + 1 - 3\ell(\ell+1)\right]$$

thus,

$$\begin{aligned} \langle r^2\rangle_{20} &= \int_0^\infty r^4 (R_{20}(r))^2 dr = 42\left(\frac{a_0}{Z}\right)^2 , \\ \langle r^2\rangle_{21} &= \int_0^\infty r^4 (R_{21}(r))^2 dr = 30\left(\frac{a_0}{Z}\right)^2 . \end{aligned}$$

Then we have,

$$\begin{aligned}
\langle\phi_{200}|\mathcal{E}(x^2+y^2-z^2)|\phi_{200}\rangle &= 2\pi\mathcal{E}\int r^4 dr \int d(\cos\theta)\phi^*_{200}(1-2\cos^2\theta)\phi_{200} \\
&= \frac{1}{2}\mathcal{E}\int r^4 R_{20}^2 dr \int_{-1}^{1} d(\cos\theta)(1-2\cos^2\theta) \\
&= \frac{1}{3}\mathcal{E}\int r^4 R_{20}^2 dr \\
&= 14\mathcal{E}\left(\frac{a_0}{Z}\right)^2 ,
\end{aligned}$$

$$\begin{aligned}
\langle\phi_{210}|\mathcal{E}(x^2+y^2-z^2)|\phi_{210}\rangle &= 2\pi\mathcal{E}\int r^4 dr \int d(\cos\theta)\phi^*_{210}(1-2\cos^2\theta)\phi_{210} \\
&= \frac{3}{2}\mathcal{E}\int r^4 R_{21}^2 dr \int_{-1}^{1} d(\cos\theta)\cos^2\theta(1-2\cos^2\theta) \\
&= -\frac{1}{5}\mathcal{E}\int r^4 R_{21}^2 dr \\
&= -6\mathcal{E}\left(\frac{a_0}{Z}\right)^2 ,
\end{aligned}$$

$$\begin{aligned}
\langle\phi_{21-1}|\mathcal{E}(x^2+y^2-z^2)|\phi_{21-1}\rangle &= \langle\phi_{211}|\mathcal{E}(x^2+y^2-z^2)|\phi_{211}\rangle \\
&= 2\pi\mathcal{E}\int r^4 dr \int d(\cos\theta)\phi^*_{211}(1-2\cos^2\theta)\phi_{211} \\
&= \frac{3}{4}\mathcal{E}\int r^4 R_{21}^2 dr \int_{-1}^{1} d(\cos\theta)\sin^2\theta(1-2\cos^2\theta) \\
&= \frac{3}{5}\mathcal{E}\int r^4 R_{21}^2 dr \\
&= 18\mathcal{E}\left(\frac{a_0}{Z}\right)^2 .
\end{aligned}$$

Thus the perturbation matrix is given by,

$$h_{ij} = \mathcal{E}\left(\frac{a_0}{Z}\right)^2 \begin{pmatrix} 14 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 18 \end{pmatrix}$$

States corresponding to these eigenvalues are as before ϕ_{200}, ϕ_{211}, ϕ_{210}, ϕ_{21-1} with a degeneracy between ϕ_{211} and ϕ_{21-1}.

4.

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2+y^2) + \lambda(x^3y + y^3x)$$

We need the following relations,

$$\begin{aligned}
E_{\vec{n}} &= E_{n_x n_y} = \hbar\omega(n_x + n_y + 1) \\
x &= \sqrt{\frac{\hbar}{m\omega}}\left(A + A^\dagger\right) \\
(A^\dagger A + AA^\dagger) &= 2A^\dagger A + 1 \\
A^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\
A|n\rangle &= \sqrt{n}|n-1\rangle \\
A^\dagger A|n\rangle &= n|n\rangle \\
\langle 0|x|n\rangle &= \sqrt{\frac{\hbar}{m\omega}}\langle 0|\left(A + A^\dagger\right)|n\rangle \\
&= \sqrt{\frac{\hbar}{m\omega}}\langle 0|A|n\rangle = \sqrt{\frac{\hbar}{m\omega}}\delta_{n1}\ , \\
\langle 0|x^3|n\rangle &= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|\left(A + A^\dagger\right)^3|n\rangle \\
&= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|A\left(A + A^\dagger\right)^2|n\rangle \\
&= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|\left(A^3 + A(A^\dagger A + AA^\dagger) + AA^\dagger A^\dagger\right)|n\rangle \\
&= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|\left(A^3 + A(2A^\dagger A + 1) + AA^\dagger A^\dagger\right)|n\rangle \\
&= \left(\frac{\hbar}{m\omega}\right)^{3/2}\left(\sqrt{3!}\delta_{n3} + 3\delta_{n1}\right)\ , \\
\langle 00|x^3y|n_x n_y\rangle &= \langle 0|x^3|n_x\rangle\langle 0|y|n_y\rangle \\
&= \left(\frac{\hbar}{m\omega}\right)^2\left(\sqrt{3!}\delta_{n_x3} + 3\delta_{n_x1}\right)\delta_{n_y1} \\
\langle 00|(x^3y + y^3x)|n_x n_y\rangle &= \left(\frac{\hbar}{m\omega}\right)^2\left[\left(\sqrt{3!}\delta_{n_x3} + 3\delta_{n_x1}\right)\delta_{n_y1} + \left(\sqrt{3!}\delta_{n_y3} + 3\delta_{n_y1}\right)\delta_{n_x1}\right] \\
&= \left(\frac{\hbar}{m\omega}\right)^2\left[\sqrt{6}\delta_{n_x3}\delta_{n_y1} + 6\delta_{n_x1}\delta_{n_y1} + \sqrt{6}\delta_{n_y3}\delta_{n_x1}\right] \\
|\langle 00|(x^3y + y^3x)|n_x n_y\rangle|^2 &= \left(\frac{\hbar}{m\omega}\right)^4\left[\sqrt{6}\delta_{n_x3}\delta_{n_y1} + 6\delta_{n_x1}\delta_{n_y1} + \sqrt{6}\delta_{n_y3}\delta_{n_x1}\right]^2 \\
&= \left(\frac{\hbar}{m\omega}\right)^4\left[6\delta_{n_x3}\delta_{n_y1} + 36\delta_{n_x1}\delta_{n_y1} + 6\delta_{n_y3}\delta_{n_x1}\right]
\end{aligned}$$

a.

The first order energy:

$$\begin{aligned} E_{00}^{(1)} &= \langle 00|(x^3y+y^3x)|00\rangle \\ &= 0 \end{aligned}$$

The second order energy:

$$\begin{aligned} E_{00}^{(2)} &= \sum_{\vec{n}\neq 0} \frac{|\langle 00|(x^3y+y^3x)|n_xn_y\rangle|^2}{E_{00}^{(0)}-E_{n_xn_y}^{(0)}} \\ &= \left(\frac{\hbar}{m\omega}\right)^4 \sum_{\vec{n}\neq 0} \frac{\left[6\delta_{n_x3}\delta_{n_y1}+36\delta_{n_x1}\delta_{n_y1}+6\delta_{n_y3}\delta_{n_x1}\right]}{E_{00}^{(0)}-E_{n_xn_y}^{(0)}} \\ &= \frac{1}{\hbar\omega}\left(\frac{\hbar}{m\omega}\right)^4 \left[\frac{6}{1-5}+\frac{36}{1-3}+\frac{6}{1-5}\right] \\ &= -\frac{21}{\hbar\omega}\left(\frac{\hbar}{m\omega}\right)^4 \end{aligned}$$

b.

For the first excited state, we have a double degeneracy $n_x = 1, n_y = 0$ and $n_x = 0, n_y = 1$, thus we should find the following matrix elements,

$$\langle 10|(x^3y+y^3x)|10\rangle\ ,\quad \langle 01|(x^3y+y^3x)|01\rangle\ ,\quad \langle 10|(x^3y+y^3x)|01\rangle$$

It is obvious that we have only the following nonzero element,

$$\begin{aligned} \langle 10|(x^3y+y^3x)|01\rangle &= \langle 1|x^3|0\rangle\langle 0|y|1\rangle + \langle 1|x|0\rangle\langle 0|y^3|1\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^2 (3\times 1 + 1\times 3) \\ &= 6\left(\frac{\hbar}{m\omega}\right)^2 \end{aligned}$$

The matrix can be written as,

$$h_{ij} = \begin{pmatrix} 0 & 6\left(\frac{\hbar}{m\omega}\right)^2 \\ 6\left(\frac{\hbar}{m\omega}\right)^2 & 0 \end{pmatrix}$$

Eigenvalues and eigenstates are,

$$\begin{aligned} E_1 = 6\left(\tfrac{\hbar}{m\omega}\right)^2 &\quad\rightarrow\quad |\psi_1\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ E_2 = -6\left(\tfrac{\hbar}{m\omega}\right)^2 &\quad\rightarrow\quad |\psi_2\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$