

## Mid-Term(1) Exam Solutions

1.

a.

$$e^{i\theta\sigma_x} = \sum_{n=0}^{\infty} \frac{(i\theta\sigma_x)^n}{n!}$$

On the other hand,

$$\sigma_x^2 = \mathbb{1} \quad \rightarrow \quad \sigma_x^{2k} = \mathbb{1}, \quad \sigma_x^{2k+1} = \sigma_x$$

hence,

$$\begin{aligned} e^{i\theta\sigma_x} &= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i\sigma_x \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ &= \mathbb{1} \cos \theta + i\sigma_x \sin \theta \end{aligned}$$

b.

$$\begin{aligned} e^{i\theta\sigma_x} \sigma_z e^{i\theta\sigma_x} &= (\mathbb{1} \cos \theta + i\sigma_x \sin \theta) \sigma_z (\mathbb{1} \cos \theta - i\sigma_x \sin \theta) \\ &= \sigma_z \cos^2 \theta + \sigma_x \sigma_z \sigma_x \sin^2 \theta + i[\sigma_x, \sigma_z] \sin \theta \cos \theta \\ &= \sigma_z \cos^2 \theta - \sigma_z \sigma_x^2 \sin^2 \theta + 2\sigma_y \sin \theta \cos \theta \\ &= \sigma_z \cos(2\theta) + \sigma_y \sin(2\theta) \end{aligned}$$

Thus

$$a_1 = 0, \quad a_2 = \sin(2\theta), \quad a_3 = \cos(2\theta).$$

2.

Firstly we define,

$$\begin{aligned} \vec{L} &= \vec{S}_1 + \vec{S}_3 \\ \vec{J} &= \vec{L} + \vec{S}_2 \end{aligned}$$

Notice that  $\ell$  has values 0 and 1. For  $\ell = 0$  we have  $j = 1/2$  and for  $\ell = 1$ ,  $j$  is  $1/2$  or  $3/2$ . Then we find

$$\begin{aligned} H &= \frac{\mathcal{E}_1}{\hbar^2} \vec{L} \cdot \vec{S}_2 + \frac{\mathcal{E}_2}{\hbar^2} (J_z)^2 \\ &= \frac{\mathcal{E}_1}{\hbar^2} \frac{1}{2} (J^2 - L^2 - S_2^2) + \frac{\mathcal{E}_2}{\hbar^2} (J_z)^2 \end{aligned}$$

In the basis which are the eigenstates of  $L^2$ ,  $S_z^2$ ,  $J^2$  and  $J_z$  we find the energies:

$$E_{\ell j m_j} = \mathcal{E}_1 \frac{1}{2} \left( j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) + \mathcal{E}_2 m_j^2$$

$$\begin{aligned} \ell = 0 \quad j = \frac{1}{2} \quad m_j = \pm \frac{1}{2} &\rightarrow E_{0 \frac{1}{2} \frac{1}{2}} = \mathcal{E}_2/4 \\ \ell = 1 \quad j = \frac{1}{2} \quad m_j = \pm \frac{1}{2} &\rightarrow E_{1 \frac{1}{2} \frac{1}{2}} = -\mathcal{E}_1 + \mathcal{E}_2/4 \\ \ell = 1 \quad j = \frac{3}{2} \quad m_j = \pm \frac{1}{2} &\rightarrow E_{1 \frac{3}{2} \frac{1}{2}} = \mathcal{E}_1/2 + \mathcal{E}_2/4 \\ \ell = 1 \quad j = \frac{3}{2} \quad m_j = \pm \frac{3}{2} &\rightarrow E_{1 \frac{3}{2} \frac{3}{2}} = \mathcal{E}_1/2 + 9\mathcal{E}_2/4 \end{aligned}$$

For each energy we have a double degeneracy.

For the case where  $S_{1z} = +\hbar/2$  and  $S_{3z} = +\hbar/2$ , we know  $\ell = 1$ , then  $j$  is either  $1/2$  or  $3/2$  with  $m_j = 1/2$ , thus energies can be  $E_{1 \frac{1}{2} \frac{1}{2}}$  and  $E_{1 \frac{3}{2} \frac{1}{2}}$ . This means that the state  $|+ - +\rangle$  can be written as a combination of  $|\ell = 1, j = \frac{1}{2}, m_j = \frac{1}{2}\rangle$  and  $|\ell = 1, j = \frac{3}{2}, m_j = \frac{1}{2}\rangle$ .

### 3.

The  $n = 2$  level in the ideal Hydrogen atom has 4 degeneracy for  $\phi_{200}$ ,  $\phi_{211}$ ,  $\phi_{210}$ ,  $\phi_{21-1}$ . Thus we should find the eigenvalues and eigenstates of the  $4 \times 4$  potential matrix:

$$\begin{aligned} \langle \phi_{2\ell m} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{2\ell' m'} \rangle &= \langle \phi_{2\ell m} | \mathcal{E}r^2(\sin^2 \theta - \cos^2 \theta) | \phi_{2\ell' m'} \rangle \\ &= \langle \phi_{2\ell m} | \mathcal{E}r^2(1 - 2\cos^2 \theta) | \phi_{2\ell' m'} \rangle \\ &= \mathcal{E} \int r^4 dr \int d\varphi \int d\cos \theta \phi_{2\ell m}^* (1 - 2\cos^2 \theta) \phi_{2\ell' m'} \\ &= \mathcal{E} \int r^4 R_{2\ell} R_{2\ell'} dr \int d\varphi \int d\cos \theta Y_{\ell m}^* Y_{\ell' m'} (1 - 2\cos^2 \theta) \end{aligned}$$

From  $\varphi$  integration we find  $m = m'$  otherwise the integral gives zero. Thus the only non-vanishing non-diagonal element can be  $\langle \phi_{200} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{210} \rangle$ , however this element is zero since  $\phi_{210}$  is an odd function while  $\phi_{200}$  and  $(x^2 + y^2 - z^2)$  are even functions. So, we have to calculate only 4 diagonal elements.

We use the following relations:

$$\langle r^2 \rangle_{n\ell} = \int_0^\infty r^4 (R_{n\ell}(r))^2 dr = \left( \frac{a_0}{Z} \right)^2 \frac{n^2}{2} [5n^2 + 1 - 3\ell(\ell + 1)]$$

thus,

$$\begin{aligned} \langle r^2 \rangle_{20} &= \int_0^\infty r^4 (R_{20}(r))^2 dr = 42 \left( \frac{a_0}{Z} \right)^2, \\ \langle r^2 \rangle_{21} &= \int_0^\infty r^4 (R_{21}(r))^2 dr = 30 \left( \frac{a_0}{Z} \right)^2. \end{aligned}$$

Then we have,

$$\begin{aligned}
\langle \phi_{200} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{200} \rangle &= 2\pi\mathcal{E} \int r^4 dr \int d(\cos \theta) \phi_{200}^* (1 - 2 \cos^2 \theta) \phi_{200} \\
&= \frac{1}{2} \mathcal{E} \int r^4 R_{20}^2 dr \int_{-1}^1 d(\cos \theta) (1 - 2 \cos^2 \theta) \\
&= \frac{1}{3} \mathcal{E} \int r^4 R_{20}^2 dr \\
&= 14\mathcal{E} \left( \frac{a_0}{Z} \right)^2,
\end{aligned}$$

$$\begin{aligned}
\langle \phi_{210} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{210} \rangle &= 2\pi\mathcal{E} \int r^4 dr \int d(\cos \theta) \phi_{210}^* (1 - 2 \cos^2 \theta) \phi_{210} \\
&= \frac{3}{2} \mathcal{E} \int r^4 R_{21}^2 dr \int_{-1}^1 d(\cos \theta) \cos^2 \theta (1 - 2 \cos^2 \theta) \\
&= -\frac{1}{5} \mathcal{E} \int r^4 R_{21}^2 dr \\
&= -6\mathcal{E} \left( \frac{a_0}{Z} \right)^2,
\end{aligned}$$

$$\begin{aligned}
\langle \phi_{21-1} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{21-1} \rangle &= \langle \phi_{211} | \mathcal{E}(x^2 + y^2 - z^2) | \phi_{211} \rangle \\
&= 2\pi\mathcal{E} \int r^4 dr \int d(\cos \theta) \phi_{211}^* (1 - 2 \cos^2 \theta) \phi_{211} \\
&= \frac{3}{4} \mathcal{E} \int r^4 R_{21}^2 dr \int_{-1}^1 d(\cos \theta) \sin^2 \theta (1 - 2 \cos^2 \theta) \\
&= \frac{3}{5} \mathcal{E} \int r^4 R_{21}^2 dr \\
&= 18\mathcal{E} \left( \frac{a_0}{Z} \right)^2.
\end{aligned}$$

Thus the perturbation matrix is given by,

$$h_{ij} = \mathcal{E} \left( \frac{a_0}{Z} \right)^2 \begin{pmatrix} 14 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 18 \end{pmatrix}$$

States corresponding to these eigenvalues are as before  $\phi_{200}$ ,  $\phi_{211}$ ,  $\phi_{210}$ ,  $\phi_{21-1}$  with a degeneracy between  $\phi_{211}$  and  $\phi_{21-1}$ .

4.

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) + \lambda(x^3y + y^3x)$$

We need the following relations,

$$\begin{aligned} E_{\vec{n}} &= E_{n_x n_y} = \hbar\omega(n_x + n_y + 1) \\ x &= \sqrt{\frac{\hbar}{m\omega}}(A + A^\dagger) \\ (A^\dagger A + AA^\dagger) &= 2A^\dagger A + 1 \\ A^\dagger |n\rangle &= \sqrt{n+1}|n+1\rangle \\ A |n\rangle &= \sqrt{n}|n-1\rangle \\ A^\dagger A |n\rangle &= n|n\rangle \\ \langle 0|x|n\rangle &= \sqrt{\frac{\hbar}{m\omega}}\langle 0|(A + A^\dagger)|n\rangle \\ &= \sqrt{\frac{\hbar}{m\omega}}\langle 0|A|n\rangle = \sqrt{\frac{\hbar}{m\omega}}\delta_{n1}, \\ \langle 0|x^3|n\rangle &= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|(A + A^\dagger)^3|n\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|A(A + A^\dagger)^2|n\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|(A^3 + A(A^\dagger A + AA^\dagger) + AA^\dagger A^\dagger)|n\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^{3/2}\langle 0|(A^3 + A(2A^\dagger A + 1) + AA^\dagger A^\dagger)|n\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^{3/2}(\sqrt{3!}\delta_{n3} + 3\delta_{n1}), \\ \langle 00|x^3y|n_x n_y\rangle &= \langle 0|x^3|n_x\rangle\langle 0|y|n_y\rangle \\ &= \left(\frac{\hbar}{m\omega}\right)^2(\sqrt{3!}\delta_{n_x 3} + 3\delta_{n_x 1})\delta_{n_y 1} \\ \langle 00|(x^3y + y^3x)|n_x n_y\rangle &= \left(\frac{\hbar}{m\omega}\right)^2\left[(\sqrt{3!}\delta_{n_x 3} + 3\delta_{n_x 1})\delta_{n_y 1} + (\sqrt{3!}\delta_{n_y 3} + 3\delta_{n_y 1})\delta_{n_x 1}\right] \\ &= \left(\frac{\hbar}{m\omega}\right)^2\left[\sqrt{6}\delta_{n_x 3}\delta_{n_y 1} + 6\delta_{n_x 1}\delta_{n_y 1} + \sqrt{6}\delta_{n_y 3}\delta_{n_x 1}\right] \\ |\langle 00|(x^3y + y^3x)|n_x n_y\rangle|^2 &= \left(\frac{\hbar}{m\omega}\right)^4\left[\sqrt{6}\delta_{n_x 3}\delta_{n_y 1} + 6\delta_{n_x 1}\delta_{n_y 1} + \sqrt{6}\delta_{n_y 3}\delta_{n_x 1}\right]^2 \\ &= \left(\frac{\hbar}{m\omega}\right)^4\left[6\delta_{n_x 3}\delta_{n_y 1} + 36\delta_{n_x 1}\delta_{n_y 1} + 6\delta_{n_y 3}\delta_{n_x 1}\right] \end{aligned}$$

a.

The first order energy:

$$\begin{aligned} E_{00}^{(1)} &= \langle 00 | (x^3 y + y^3 x) | 00 \rangle \\ &= 0 \end{aligned}$$

The second order energy:

$$\begin{aligned} E_{00}^{(2)} &= \sum_{\vec{n} \neq 0} \frac{|\langle 00 | (x^3 y + y^3 x) | n_x n_y \rangle|^2}{E_{00}^{(0)} - E_{n_x n_y}^{(0)}} \\ &= \left( \frac{\hbar}{m\omega} \right)^4 \sum_{\vec{n} \neq 0} \frac{[6\delta_{n_x 3} \delta_{n_y 1} + 36\delta_{n_x 1} \delta_{n_y 1} + 6\delta_{n_y 3} \delta_{n_x 1}]}{E_{00}^{(0)} - E_{n_x n_y}^{(0)}} \\ &= \frac{1}{\hbar\omega} \left( \frac{\hbar}{m\omega} \right)^4 \left[ \frac{6}{1-5} + \frac{36}{1-3} + \frac{6}{1-5} \right] \\ &= -\frac{21}{\hbar\omega} \left( \frac{\hbar}{m\omega} \right)^4 \end{aligned}$$

b.

For the first excited state, we have a double degeneracy  $n_x = 1, n_y = 0$  and  $n_x = 0, n_y = 1$ , thus we should find the following matrix elements,

$$\langle 10 | (x^3 y + y^3 x) | 10 \rangle, \quad \langle 01 | (x^3 y + y^3 x) | 01 \rangle, \quad \langle 10 | (x^3 y + y^3 x) | 01 \rangle$$

It is obvious that we have only the following nonzero element,

$$\begin{aligned} \langle 10 | (x^3 y + y^3 x) | 01 \rangle &= \langle 1 | x^3 | 0 \rangle \langle 0 | y | 1 \rangle + \langle 1 | x | 0 \rangle \langle 0 | y^3 | 1 \rangle \\ &= \left( \frac{\hbar}{m\omega} \right)^2 (3 \times 1 + 1 \times 3) \\ &= 6 \left( \frac{\hbar}{m\omega} \right)^2 \end{aligned}$$

The matrix can be written as,

$$h_{ij} = \begin{pmatrix} 0 & 6 \left( \frac{\hbar}{m\omega} \right)^2 \\ 6 \left( \frac{\hbar}{m\omega} \right)^2 & 0 \end{pmatrix}$$

Eigenvalues and eigenstates are,

$$\begin{aligned} E_1 = 6 \left( \frac{\hbar}{m\omega} \right)^2 &\rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ E_2 = -6 \left( \frac{\hbar}{m\omega} \right)^2 &\rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$