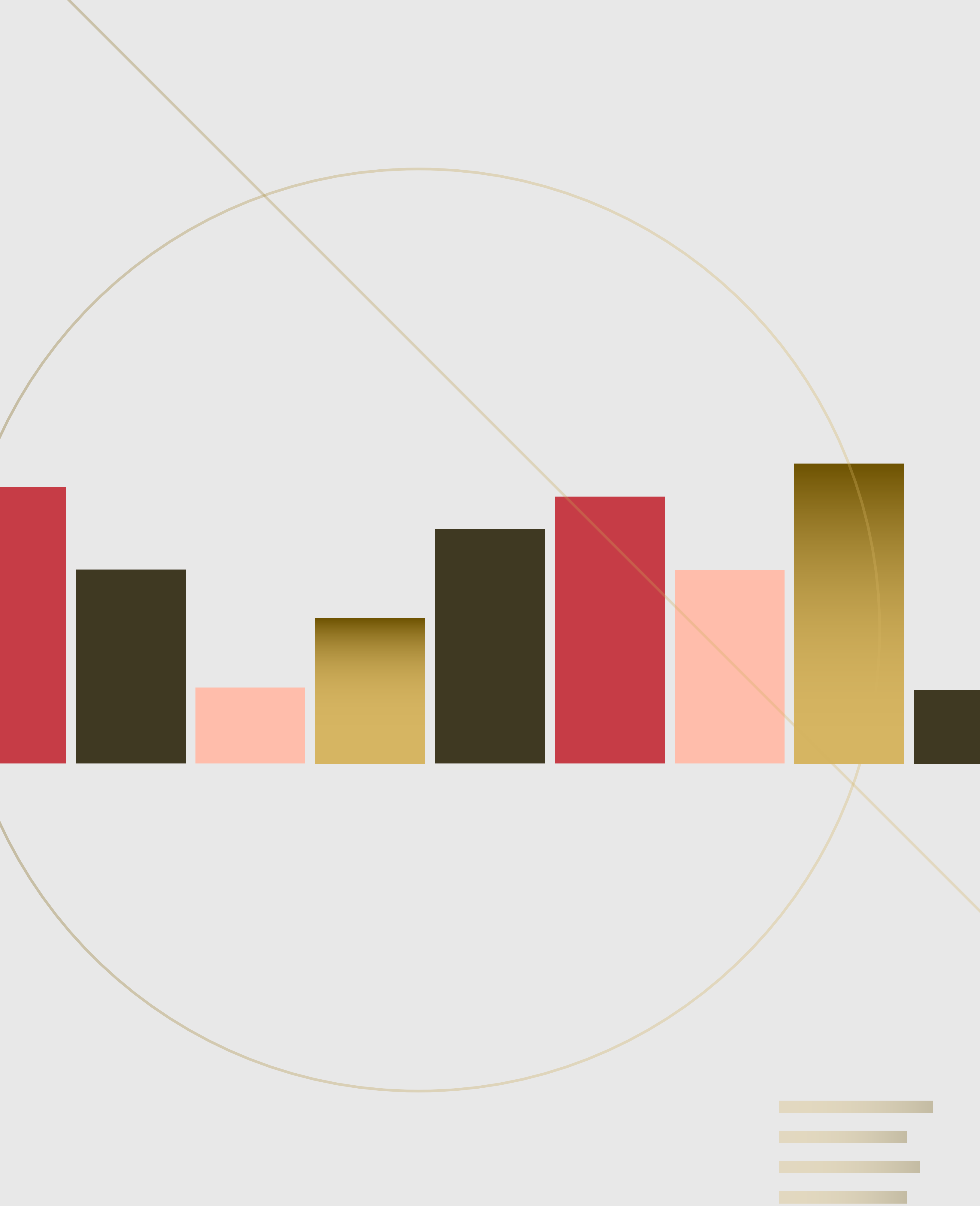
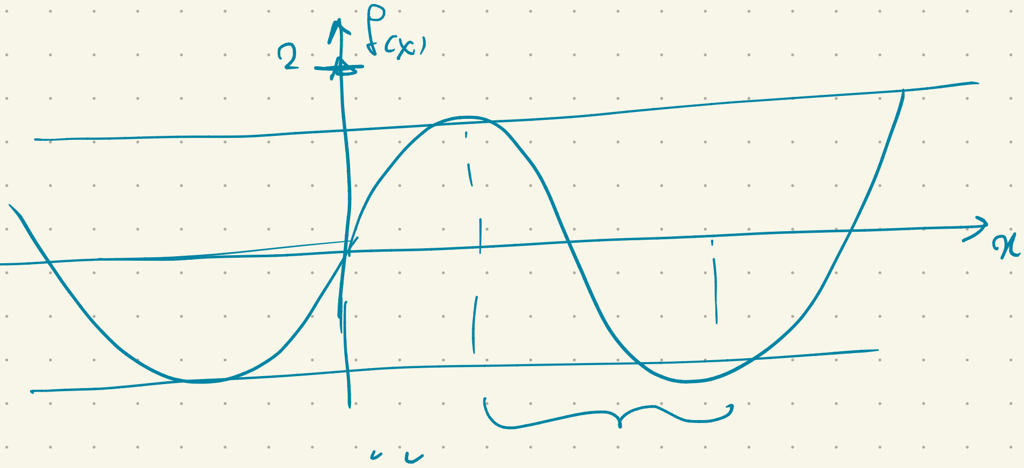


$f(x) = \sin x$

$R \longrightarrow R$

$[-1,1]$



$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

$$g \circ f = f' \circ g'$$

$$X \xrightarrow{f} Y \qquad f \ \& \ f^{-1}$$

پیوسته

X homeomorphic to Y.

همریخت

$$G \xrightarrow{\phi} G'$$

$$\Phi(g \cdot g') = \phi(g) * \phi(g')$$

Homomorphism.

QUATERNIONS $\{R, +, \cdot\}$

$+ \to$ 5 Axioms, $\cdot \to$ 5 Axioms

$a(b+c) = ab + ac$

$a < b$

$a+b \in R$

$a+b = b+a$

$(a+b)+c = a+(b+c)$

$\exists 0 \quad 0+a = a$

$\exists -a \quad a+(-a) = 0$

$$\boxed{\begin{gathered} R \subset C \quad \{C, +, \cdot\} \\ (x,y)+(x',y') = (x+x', y+y') \\ (x,y)(x',y') = (xx'-yy', \; xy'+x'y) \\ (\underbrace{0,1}_{i})(0,1) = (-1,0) \qquad i^2 = -1 \end{gathered}}$$

$(x,y) < (x',y')$

$C \subset H$

$\{H, +, \cdot\}$

$q \in H \qquad q = q_0 + \underbrace{q_1 i + q_2 j + q_3 k}$

$$\boxed{\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = k = -ji \quad \text{cycl.} \end{gathered}}$$

$qq' \neq q'q$

GR

Real part

Imaginary part.

$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$

$q\bar{q} = |q|^2$

$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$

$i \to \sigma_x$

$j \to \sigma_y$

$k \to \sigma_z$

$H \cong M_{2\times 2}$ (Hermits) = Pauli Matrices

$$q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + i\,\vec{q}\cdot\vec{\sigma} = |q|\left[\frac{q_0}{|q|} + \frac{\vec{q}}{|q|}\cdot\vec{\sigma}\right]$$

$$= |q|\left[\cos\theta + \sin\theta\cdot\hat{n}\cdot\vec{\sigma}\right] \quad ?$$

$\mathbb{C}$ $\quad z \longrightarrow w = \dfrac{az+b}{cz+d}$ $\quad ad-bc \neq 0$ $\quad$ Mobius

$$z \longrightarrow w = \frac{az+b}{cz+d} \longrightarrow w' = \frac{a'w+b'}{c'w+d'}$$

$$\boxed{w' \overset{?}{=} w'(z)}$$

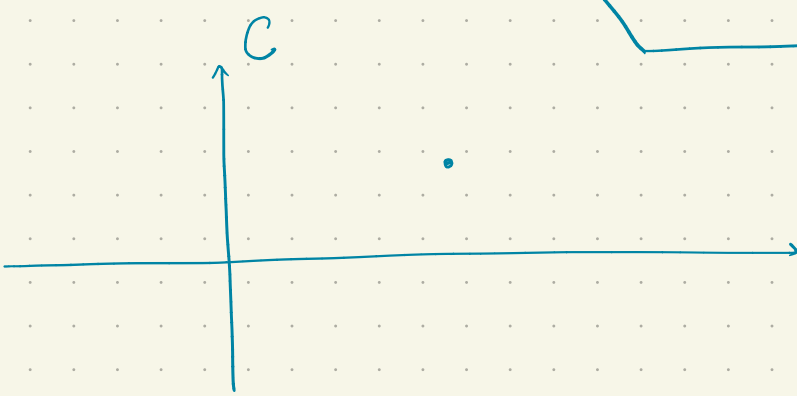
Linear

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad GL_2(C)$ = Group.

general

$SL_2(C) \subset GL_2(C)$

$\det(g \in SL_2(C)) = 1$



Mobius Transf. $C \longrightarrow C$

$$z \longrightarrow w = \frac{az+b}{cz+d}$$

$$wcz + wd = az + b$$

$$z = ?\cdot w$$

$ad - bc \neq 0$

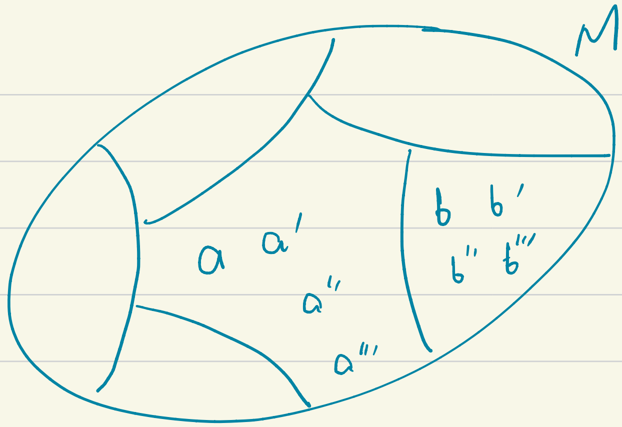
$a, b, c, d \longrightarrow \lambda a, \lambda b, \lambda c, \lambda d$

$ad - bc = 1$

Mobius $\longleftrightarrow (a\ b\ c\ d)$ $\quad ad-bc=1$

$SL_2(C) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

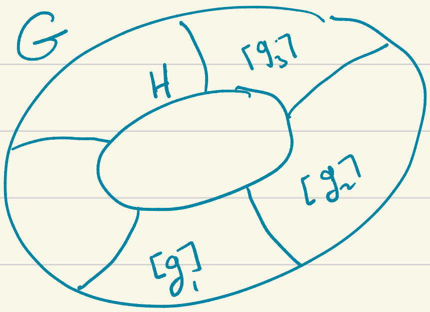
Equivalence Relation:



$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ $\quad a \sim b$ if $a - b = 2k$.

$\mathbb{Z}_{even} = \{-2, 0, 2, \dots\}$

$\mathbb{Z}_{odd} = \{-3, -1, +1, +3 \dots\}$



$g \sim g'$ if $g' = gh$ for some $h \in H$

(1) $g \sim g$

(2) $g \sim g' \longrightarrow g' \sim g$

(3) $\rightarrow$ $g \sim g'$ $\quad g' \sim g''$

$\downarrow \qquad \downarrow$

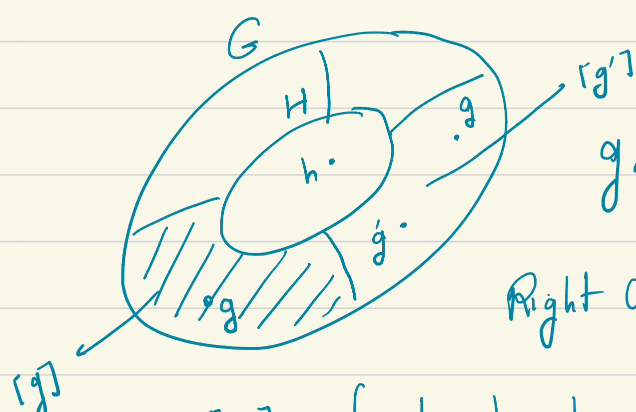
$g' = gh_1 \qquad g'' = g'h_2$

$g'' = g'h_2 = g h_1 h_2$

$\downarrow$

$g'' \sim g$

$[g_1] = \{ g_1 h_1, g_1 h_2, g_1 h_3 \dots g_1 h_N \} = g_1 H$



$g \underset{R}{\sim} g'$ if $g' = gh$

Right Cosets $\quad [g] = \{ g' \mid g' = gh \text{ for some } h \}$

$[g]_R = \{ gh_1, gh_2, gh_3 \dots gh_N \} = gH$

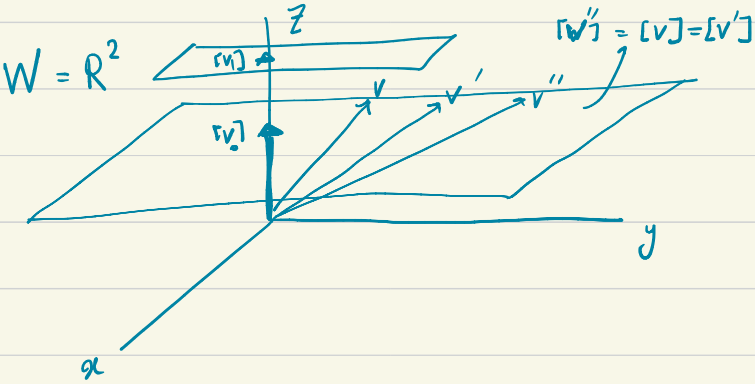
$G/H = \{ [g], [g'], [g''] \dots \}$

$g \underset{L}{\sim} g'$ if $\exists h \mid g' = hg$ $\qquad [g]_L = Hg$

$W \subset V$ $\qquad v \sim v'$ if $v' - v = w$ for some w.

$V/W = \{[v]\}$ → فضای برداری

$V = \mathbb{R}^3 \qquad W = \mathbb{R}^2$

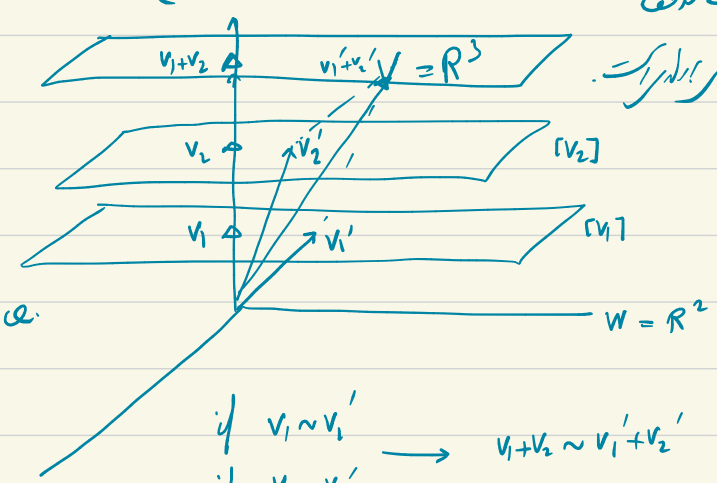


$$\begin{cases} [v_1] + [v_2] = ? \\ c\,[v_1] = ? \end{cases} \qquad \begin{cases} [v_1]+[v_2] := [v_1+v_2] \\ c\,[v_1] := [cv_1] \end{cases}$$

→ تمرین: نشان دهید V/W با این تعاریف واقعاً یک فضای برداری است.

$\mathbb{R}^3 / \mathbb{R}^2 = \mathbb{R}$

Questien Vector Space



if $v_1 \sim v_1'$
if $v_2 \sim v_2'$ $\longrightarrow v_1+v_2 \sim v_1'+v_2'$ $\qquad$ خوش تعریفی:

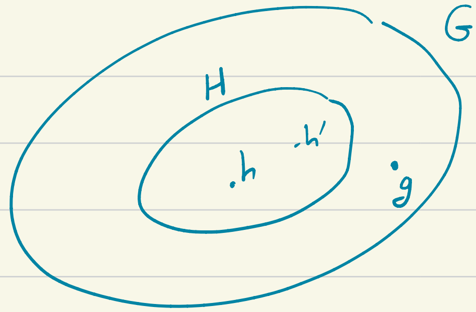
$[g_1]_R [g_2]_R := [g_1 g_2]_R$ $\qquad$ آیا G/H یک گروه هست یا نه؟

$[g_1][g_2] := [g_1 g_2]$ ← از این به بعد [illegible] R را نمی‌نویسیم.

① if $g_1 \sim g_1'$
② $g_2 \sim g_2'$ $\xrightarrow{?} g_1 g_2 \sim g_1' g_2'$ $\qquad$ خوش تعریفی؟

① → $g_1' = g_1 h_1$
② → $g_2' = g_2 h_2$ → $g_1' g_2' = (g_1 h_1)(g_2 h_2) \neq g_1 g_2 (h_3)$

Normal subgroup.



Def: H is a normal subgroup of G

if $\quad ghg^{-1} \in H \quad \forall g \,\&\, h.$

$\hookrightarrow \quad ghg^{-1} = h' \longrightarrow gh = h'g \qquad$ Normal subgroup. For Normal subgroups.

if $g_1 \sim g_1' \qquad g_2 \sim g_2'$

$\downarrow \; g_1' = g_1 h_1 \qquad g_2' = g_2 h_2 \longrightarrow g_1' g_2' = g_1 \underline{h_1 g_2} h_2 = g_1 g_2 h_1' h_2$

$= g_1 g_2 h_3.$

نشان بدهید ترکیب واقعاً خوش تعریف است.
نشان دهید G/H واقعاً یک گروه است.
} تمرین: $\longleftarrow \quad g_1' g_2' \sim g_1 g_2$

مثال:

$G = U(n) \qquad H = SU(n)$

ماتریس‌های یکانی $n \times n$ $\qquad$ ماتریس‌های یکانی $n \times n$

$\det g = 1$

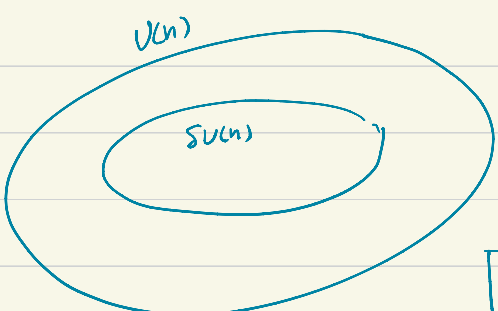
$SU(n) \subset U(n)$

Normal.

let $h \in SU(n) \quad g \in U(n) \longrightarrow ghg^{-1} \in SU(n)$?

$\longleftarrow \det(ghg^{-1}) \stackrel{?}{=} 1$

$\det g \det h (\det g^{-1}) = \det h = 1$

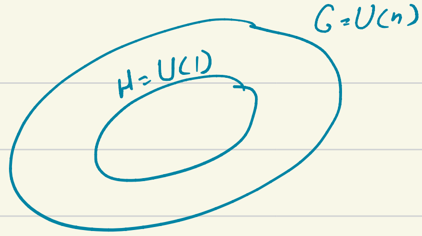


$$\boxed{U(n)/SU(n) \equiv U(1)} \longleftarrow \text{تمرین}$$

مثال ۲)

$G = U(n)$ $\qquad$ $H = U(1)$

$\downarrow$ $n \times n$ $\qquad$ $\downarrow$ 1×1



$U(1) \subset U(n)$

$$U(1) = \left\{ e^{i\theta} I_{n\times n} = \begin{bmatrix} e^{i\theta} & & \\ & e^{i\theta} & \\ & & e^{i\theta} \end{bmatrix}_{n\times n} \right\}$$

$g \in U(n)$ $\quad$ $h \in U(1)$ $\qquad$ آیا $U(1) \underset{\text{Normal}}{\subset} U(n)$ ؟

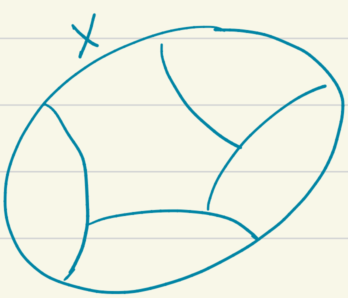
$$\underbrace{g h g^{-1}}_{e^{i\theta} I} \in U(1) \; ?$$

تمرین: نشان دهید $U(n)/U(1) = ?$

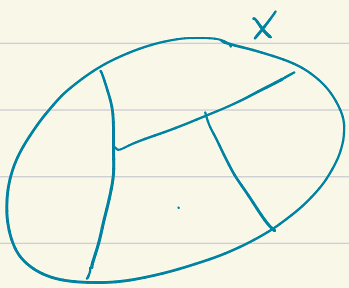
غلط $\quad$! $U(n)/U(1) = SU(n)$

! $U(n)/U(1) = SU(n)$ $\quad$ پس $\leftarrow$ $U(n)/SU(n) = U(1)$ $\quad$ تمرین - حتماً

یادآوری: تعریف همبندی

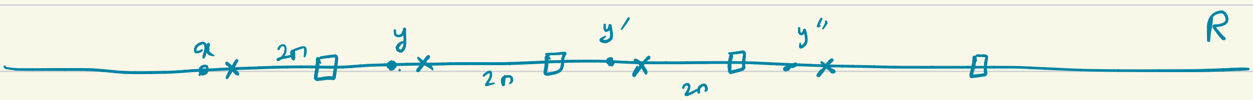


$\sim \rightarrow$ افراز



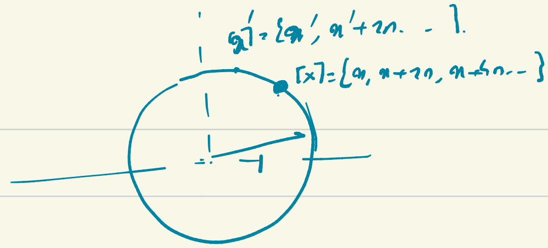
افراز $\rightarrow \sim$

$R/_{2\pi} = S^1$, $\quad R^2/_{2\pi\times 2\pi} = T^2$; $\quad \underline{S^n = \{ \vec{x} \in R^{n+1} \mid |\vec{x}|^2 = 1 \}}$ $\quad D^n = \{ \vec{x} \in R^n \mid |x|^2 \le 1 \}$. $\quad \partial D^n = S^{n-1}$.



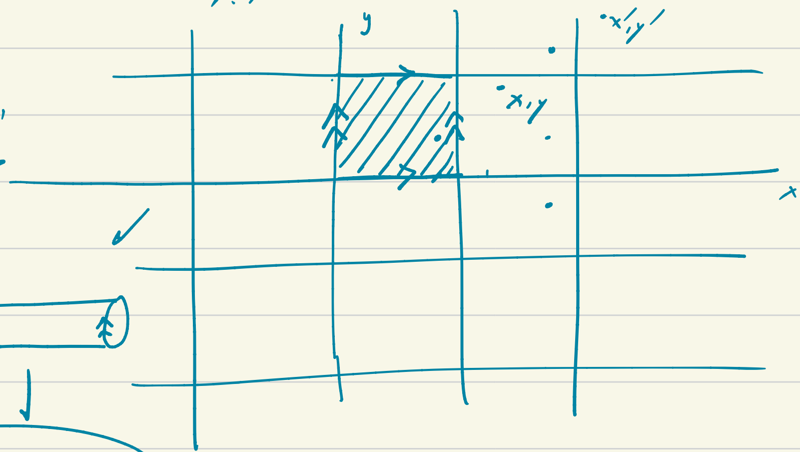
$x \sim y$ if $y - x = 2\pi n$

$$\mathbb{R}/_{2\pi} = \{ [x], [x'], [x''], \dots \}$$

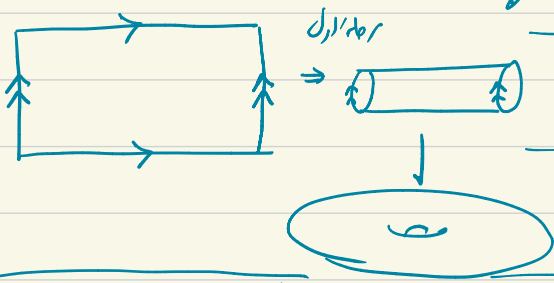


$\mathbb{R}/_{2\pi} = S^1$ $\quad \mathbb{R}^2/_{2\pi \times 2\pi} = T^2 \rightarrow$ چنبره

$(x,y) \sim (x',y')$ if $x' = x + 2\pi n_1$, $y' = y + 2\pi n_2$



$T^2 =$



$\mathbb{R}$ — به جای یک نمونه.

$H = \{ 2\pi n \mid n \in \mathbb{Z} \}$. $\quad H \subset \mathbb{R}$

سوال امتحان :

$\mathbb{R}/_{2\pi} = S^1$

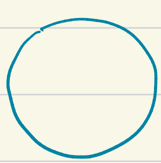
$2\pi n \in H \qquad 2\pi n' \in H$

$2\pi n + 2\pi n' = 2\pi(n+n') \in H$

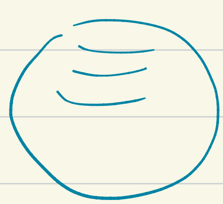
$S^n := \{ \vec{x} \in \mathbb{R}^{n+1} \mid x \cdot x = 1 \}$ n-dimensional sphere.

$D^n = \{ \vec{x} \in \mathbb{R}^n \mid x \cdot x \leq 1 \}$ n-dimensional Ball.

$S^1 =$



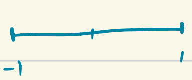
$S^2 =$



توخالی

$$\boxed{\partial D^n = S^{n-1}}$$

$D^1 =$



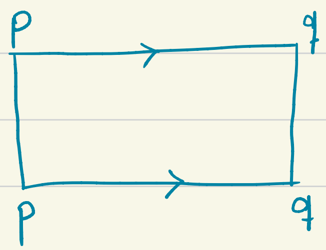
$D^2 =$

$\partial D^2 = S^1$, $\partial D^3 = S^2$ etc.

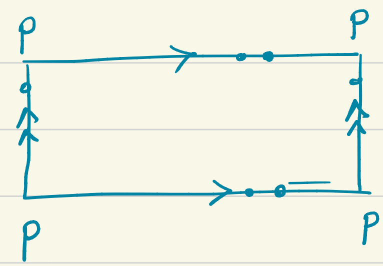
- Identification: Cylinder; Torus; Mobius Strip; In D^2 identify all the boundary →

$D^2/S^1 = S^2$; $D^n/S^{n-1} = S^n$; projective plane $\mathbb{RP}^2 = \mathbb{R}^3/\sim$ when $x \sim \lambda x$.

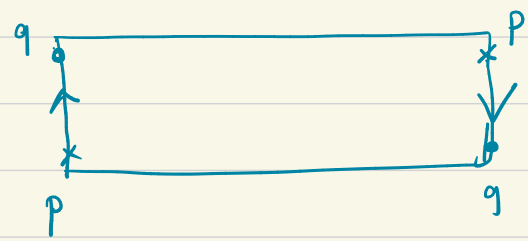
Cylinder: =



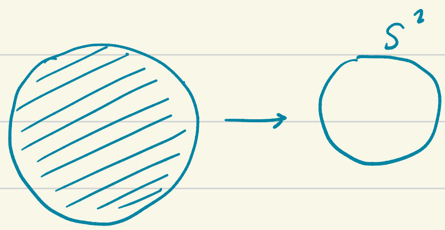
Torus $= T^2$



Mobius Strip:



D^2

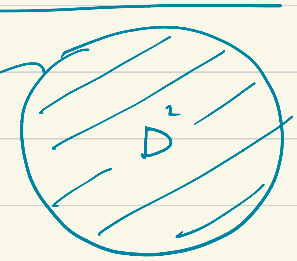


تمام نقاط مرز = یک نقطه

تمام نقاط مرز به هم چسبیده می‌شوند

$$\boxed{D^2/_{\partial D^2} = S^2} \to D^2/_{S^1} = S^2$$

$S^1 = \partial D^2$



تعمیم → $D^n/_{\partial D^n} = S^n \to D^n/_{S^{n-1}} = S^n$.

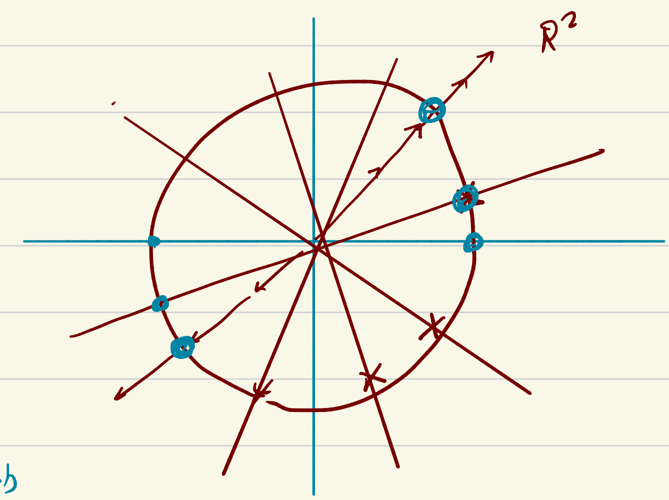
Real projective plane: $= \mathbb{RP}^2$

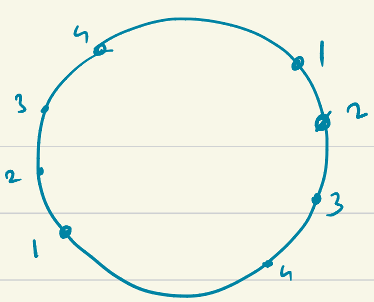
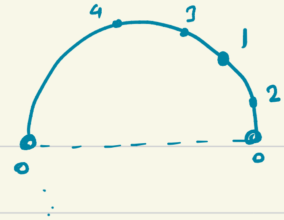
$\vec{x} \sim \vec{y}$ if $\vec{y} = k\vec{x}$ for some $k \neq 0$

$\mathbb{R}^2 - \{0\}$

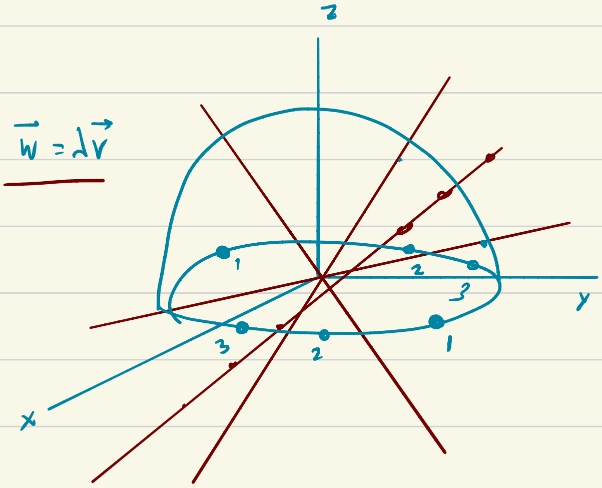
کلاس = امتدادها

$\mathbb{RP}^2 =$ دایره‌ای با نقاط روبروی هم قطری یکی شده است.

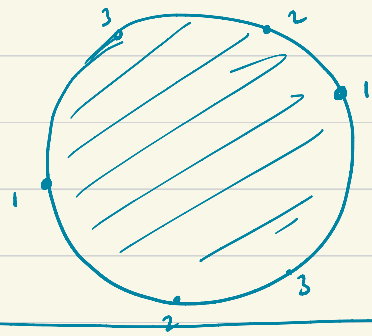


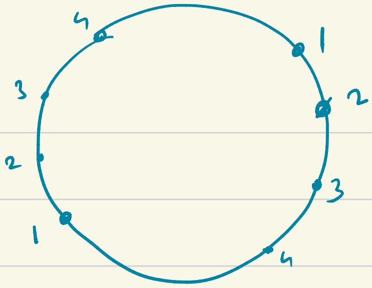
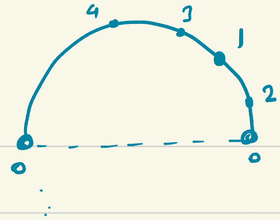
$\mathbb{R}P^2 =$  $=$  $= S^1$

$\mathbb{R}P^n = \mathbb{R}^n_{-\{0\}} / \sim$ $\quad$ $\underline{\vec{v} \sim \vec{w}}$ iff $\underline{\vec{w} = \lambda \vec{v}}$

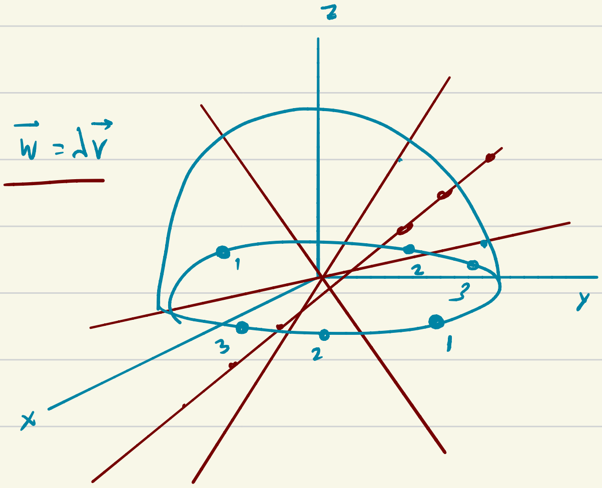


$\mathbb{R}P^3 = \mathbb{R}^3 / \sim$

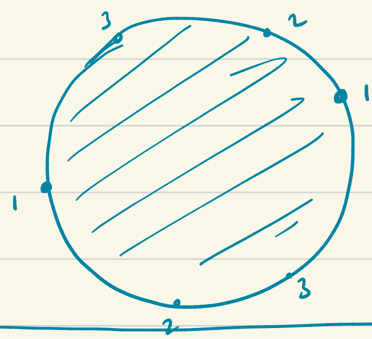


$RP^2 =$  $=$  $= S^1$

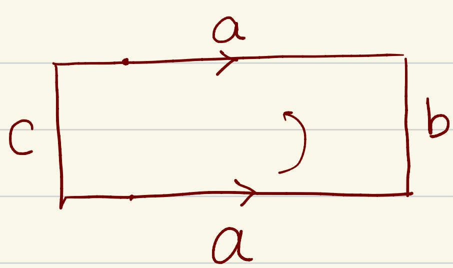
$RP^n = \mathbb{R}^n_{-\{0\}}/\sim$ $\qquad$ $\vec{v} \sim \vec{w}$ iff $\vec{w} = \lambda \vec{v}$



$RP^3 = R^3/\sim$

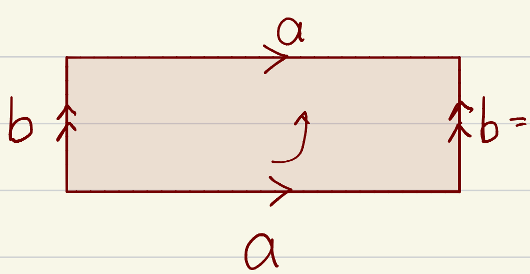


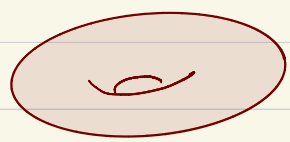
Cylinder



$\longleftrightarrow\ aba^{-1}c$

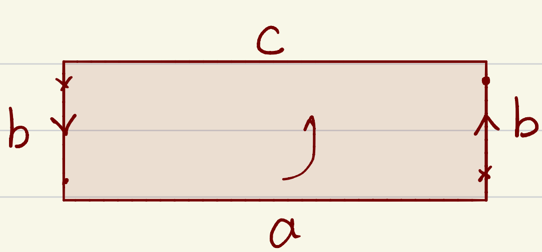
Torus



$=$  $= aba^{-1}b^{-1}$

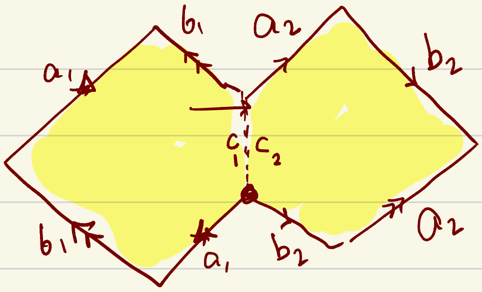
word description

Mobius strip



$= abcb$

Exercise: write the word description of RP^2 & Klein Bottle.



$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$$

Exercise: Show that the word description of the torus with genus g is equal to:

$$\prod_{i=1}^{g} (a_i \cdot b_i \cdot a_i^{-1} \cdot b_i^{-1})$$

Vectors & Tensors.

$T: V \longrightarrow W \qquad Ker(T) \qquad Im(T)$

$Ker(T) : \{ x \in V \quad Tx = 0 \} \qquad Im(T) = \{ y \in W \mid y = Tx \}$ for some x

Ex. $\dim Ker(T) + \dim(Im(T)) = \dim(V)$

Dual vector space.

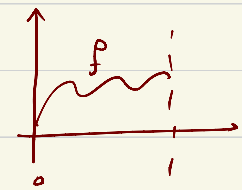
$V^*: \qquad \alpha : V \longrightarrow R \qquad \alpha(v) \in R$

α: linear functional

$$\begin{cases} \alpha(v+v') = \alpha(v) + \alpha(v') \\ \alpha(cv) = c\,\alpha(v) \end{cases}$$

$c \in R$

Ex: $V =$ فضای توابع از $[0,1]$ به $[0,1]$



$I(f) = \int_0^1 f(x)dx \qquad I(f+g) = I(f) \dots$

$I(cf) = c\, I(f)$

$V = R^3 \qquad \alpha(v) = v_1$

$\beta(v) = w \cdot v$ for a fixed w.

$\gamma(v) = w + v$ for a fixed w. سوال: γ تابعی خطی هست یا نه؟

if α is a linear functional &
β " " .

$$\begin{cases} (\alpha+\beta)(v) := \alpha(v) + \beta(v). \\ (c\alpha)(v) := c \times \alpha(v). \end{cases}$$

Test: مجموعه(ها) تابعی‌های خطی تشکیل یک فضای برداری می‌دهند

V^*

$$(\alpha+\beta)(v+w) = \alpha(v+w) + \beta(v+w) = (\alpha(v)+\alpha(w)) + (\beta(v)+\beta(w))$$
$$= (\alpha(v)+\beta(v)) + (\alpha(w)+\beta(w))$$
$$= (\alpha+\beta)(v) + (\alpha+\beta)(w)$$

$V^* :=$ the space of linear functionals over V.

$\rightarrow v = v^i e_i$ قرارداد

فضای محدودبعد :

Basis of $V = \{ e_1, e_2, \ \ e_N \}$ $\qquad$ لم: $\underline{\dim V = \dim V^*}$

Basis of $V^* := \{ e^1, e^2, \dots e^n \}$

$e^i(v) = .. \qquad e^i(e_j) :=$ ~~با دکرونکر نشان~~ $= \delta^i_j$

$e^i(v) = e^i(v^j e_j) = v^j \underbrace{e^i(e_j)}$

$e^1(e_1) = 1 \qquad e^1(e_2, e_3, \dots) = 0$ etc

$e^i(v) = e^i(v^j e_j) = v^j e^i(e_j) = v^j \delta^i_j = v^i$

Linear independence: Assume $\sum_i c_i e^i = 0 \rightarrow (\sum_i c_i e^i)(e_j) = \sum_i c_i \delta^i_j = c_j$!!

$c_i = 0$

Let β be a linear functional: $\beta(e_i) = \beta_i \leftarrow$ جزء β است یعنی ضرایب

ادعا $\beta = \sum_i \beta_i e^i \qquad \beta(e_j) = \sum_i \beta_i e^i(e_j) = \beta_j$

$\{e^i\}$ is a basis of V^* & $\dim V = \dim V^*$. No inner product.

pairing between two different spaces

Tensor product of two spaces: $\qquad V^*$ is the dual of V.

V is the dual of V^*.

Def: A bilinear functional on $V \times V$ $\rightarrow$ ضرب داخلی

$(x, y) \in V \times V$ $\qquad\qquad$ $\alpha : V \times V \longrightarrow R$

α should be bi-linear.

$$\begin{cases} \alpha(x+x', y) = \alpha(x,y) + \alpha(x', y) \\ \alpha(cx, y) = c\,\alpha(x,y) \end{cases}$$

also for y.

$\xrightarrow[\text{تعمیم}]{}$ Multilinear.

if α is a bi-linear functional on $V \times V$
β " " " " :

$\alpha + \beta$ is also a bilinear functional ?

بله

the space of all bi-linear functionals form a vector space.

we want a basis for $(V \times V)^*$

is $V \times V$ a vector space ?
$$\begin{cases} (x,y) + (x', y') := (x+x', y+y') \\ c(x,y) = (cx, cy) \end{cases}$$

What is the basis of $V \times V$? $\qquad (x,y) \in V \times V$

$(x,y) \in V \times V \qquad (x,y) = (x,0) + (0,y) = (\sum_i x^i e_i, 0) +$

$(0, \sum_j y^j e_j) = \sum_i x^i (e_i, 0) + \sum_j y^j (0, e_j) \rightarrow$

Basis of $V \times V = \{ (e_i, 0), (0, e_j) \}$ $\qquad \dim V \times V = \dim V + \dim V$

در حالت کلی $\dim V \times W = \dim V + \dim W$.

$V \oplus W$ direct sum of 2-spaces. $\qquad \underline{R^3 = R^2 \oplus R = R \oplus R \oplus R}$

تعداد ضرب با جمع برابر است ضرب دکارتی وکتور اسپیس‌ها فرق کند ← این کار را برای تعریف ضرب تانسوری

$$(x,y) = (2e_1 + 3e_2,\ e_1 + e_2) =$$

$$\begin{cases} (x,y) + (x',y') = (x+x',\ y+y') \\ c(x,y) = (cx,\ cy) \end{cases}$$

ترکیب خطی جمع در $V \times V$

$$\neq\ 2(e_1,e_1) + 3(e_2,e_1) + 2(e_1,e_2) + 3(e_2,e_2)$$

$$(2e_1, 2e_1) + (3e_1, 3e_1) + \cdots \neq (x,y)$$

if

$$(x,y) = \sum x^i y^j (e_i, e_j)$$

$$\downarrow$$

$$(x,0) = 0$$

Vafa's proof:

جلسه سوم:

V, $W = V$ $\qquad V \otimes V \qquad \underbrace{e_i \otimes e_j}_{} = ?$

$V \to$ Basis of $V = \{e_i\}$

$\omega \in V \otimes V \qquad \omega = \sum_{i,j} \omega^{ij} \underbrace{e_i \otimes e_j}_{?}$

V is a vector space. $\qquad V \times V =$ مجموعه

$V \times V$ مجموعه تمام تابعی دو خطی در $= W$ $\qquad$ if $\alpha \in W \qquad \alpha : V \times V \longrightarrow \mathbb{R}$

$\alpha(x,y) \in \mathbb{R}$ $\qquad$ دو خطی بودن = bilinear

$$\begin{cases} \alpha(x_1 + x_2, y) = \alpha(x_1, y) + \alpha(x_2, y) \\ \alpha(cx, y) = c\,\alpha(x,y) \end{cases}$$

$V \times V$ لزوما تعریف جمع در نیست $\leftarrow \alpha\big((x,y) + (x',y')\big) = ? \leftarrow$ Also for y.

یعنی اینکه $V \times V$ لزومی ندارد که ساختار فضای برداری داشته باشد.

if $\alpha \in W$, $\beta \in W$ $\qquad$ آیا خود W یک فضای برداری هست یا خیر؟

$$\begin{cases} \alpha + \beta \\ c\alpha \end{cases} \in W\ ?$$

$$\begin{cases} (\alpha+\beta)(x,y) := \alpha(x,y) + \beta(x,y) & (1) \\ (c\alpha)(x,y) := c\,\alpha(x,y) \end{cases}$$

Check: $$\underbrace{(\alpha+\beta)}_{\gamma}\left[(x_1+x_2, y)\right] \overset{?}{=} \underbrace{(\alpha+\beta)}_{\gamma}(x_1,y) + \underbrace{(\alpha+\beta)}_{\gamma}(x_2,y)$$

(1)
$$\gamma(x_1+x_2, y) = \alpha[x_1+x_2, y] + \beta[x_1+x_2, y] =$$

$$= \alpha(x_1,y) + \alpha(x_2,y) + \beta(x_1,y) + \beta(x_2,y)$$

$$= \underbrace{\alpha(x_1,y) + \beta(x_1,y)}_{\gamma(x_1,y)} + \underbrace{\alpha(x_2,y) + \beta(x_2,y)}_{+\ \gamma(x_2,y)}$$

✓

W is a vector space.

Basis for W: if $\alpha \in W$ $\quad \alpha(x,y)$?

$$\alpha(x,y) = \alpha\Big(\sum_i x^i e_i, \sum_j y^j e_j\Big) = \sum_i x^i \alpha\Big(e_i, \sum_j y^j e_j\Big)$$

$$= \sum_{i,j} x^i y^j \alpha(e_i, e_j) \longrightarrow$$ پس کافی‌ست α رو روی (e_i, e_j) تعریف کنیم.

Basis of W: $\quad \tau^{ij}(e_k, e_l) := \delta^i_k \delta^j_l$.

$$\tau^{ij}(x,y) = \tau^{ij}\Big(\sum_k x^k e_k, \sum_l y^l e_l\Big) = x^i y^j$$

باید نشان بدیم: (1) $\{\tau^{ij}\}$ مستقل خطی‌اند. (2) می‌توان هر عضو W را بر اساس آن‌ها بسط داد.

① Linear independence: let $\sum_{i,j} c_{ij}\, \tau^{ij} = 0 \longrightarrow$

$$\sum_{i,j} c_{ij}\, \tau^{ij}(e_k, e_l) = 0 \rightarrow \sum_{i,j} c_{ij}\, \delta^i_k \delta^j_l = 0 \rightarrow c_{kl} = 0$$

$\blacksquare$

② let $\alpha \in W$ we should prove that α can be expanded in terms of τ^{ij}'s! Since $\alpha \in W \rightarrow$

$$\alpha(e_i, e_j) = \alpha_{ij} \qquad \text{we set } \alpha := \sum_{i,j} \alpha_{ij}\, \tau^{ij}$$

why? $\alpha(e_k, e_l) = \sum_{i,j} \alpha_{ij}\, \underbrace{\tau^{ij}(e_k, e_l)}_{\delta^i_k \delta^j_l} = \alpha_{kl}.$

$\dim(W) = ?$ if $\dim V = n$ $\dim(W) = n^2$

○ Definition: let $\alpha \in V^*$ $\beta \in V^*$

یک نام برای W: $(V^* \otimes V^*)$

نشان می‌دهیم که $V^* \otimes V^*$ یک فضای برداری است با بعد n^2.

let $\alpha \in V^*$ $\beta \in V^*$ $\alpha \otimes \beta = ?$

$$(\alpha \otimes \beta)(x, y) := \alpha(x)\, \beta(y)$$

تمرین: ثابت کنید که $\alpha \otimes \beta$ واقعاً یک نگاشت دوخطی است.

به همین ترتیب $V^* \otimes V^* \otimes V^* \cdots \otimes V^*$ تعریف می‌شود.

سوال: $V \otimes V$ چگونه تعریف می‌کنیم؟

V & V^* are dual to each other

if $\alpha \in V^*$, $x \in V$ $\qquad \alpha(x) \in \mathbb{R}$ $\qquad \begin{cases} \alpha \text{ is linear.} \\ \alpha+\beta \in V^* \end{cases}$

V^* is the space of linear functionals on V or

V " " " " V^*.

$\mathcal{X}(\alpha) := \alpha(x) \longrightarrow$ نمادگذاری مرسوم $\langle \alpha, x \rangle$

$\alpha(x)$

$V \otimes V$ is the space of bilinear functionals on $V^* \times V^*$.

$V^* := L(V, \mathbb{R}) \qquad V^* \otimes V^* := L(\underbrace{V \times V}_{} \longrightarrow \mathbb{R})$

we have not used the linear structure of $V \times V$.

---

if $\tau \in V \otimes V$ $\quad \tau = \tau^{ij} e_i \otimes e_j$. $\quad$ if $\tau \in V^* \otimes V^*$ $\quad \tau = \tau_{ij} e^i \otimes e^j$

if $\tau \in V \otimes V^*$ $\quad \tau = \tau^i{}_j e_i \otimes e^j$ $\quad$ etc.

تعریف: $V^* \otimes V^* :=$ The space of bi-linear functionals on $V \times V$.

① prove that this is a vector space.

② . Show that $\tau^{ij}(e_k, e_l) = \delta^i_k \delta^j_l$ is a basis for $V^* \otimes V^*$.

③ define $(\alpha \otimes \beta)$ as: $(\alpha \otimes \beta)(x,y) = \alpha(x)\beta(y)$.

تمرین ④ prove that: $\begin{cases} \alpha \otimes (\beta + \gamma) = \alpha \otimes \beta + \alpha \otimes \gamma. \\ \alpha \otimes (c\beta) = c(\alpha \otimes \beta). \end{cases}$

⑤ In the same way: $V \otimes V :=$ the space of bilinear functionals on $V^* \times V^*$.

⑥ Note: We have never endowed $V \times V$ with a linear structure. only V is a vector space.

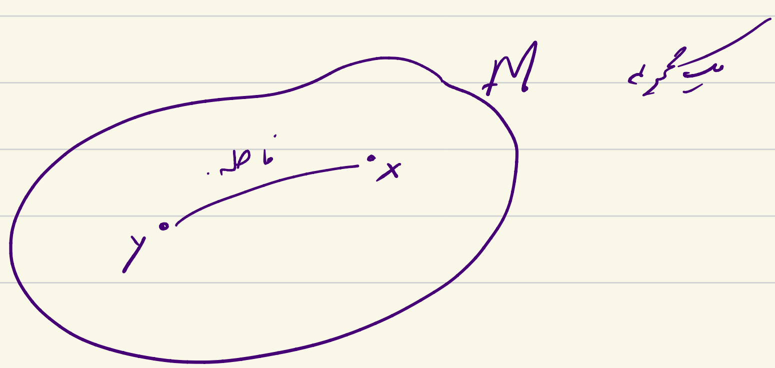
⑦ $\dim(V \otimes W) = \dim V \cdot \dim W$.

⑧ Mixed Tensors can defined. $\tau = \tau_{ij}{}^{k}\, e^i \otimes e^j \otimes e_k$ etc.

$\in V^* \otimes V^* \otimes V$.

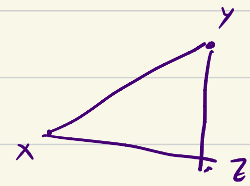
Metric & Topological Spaces: (M, d) is a Metric space.

$d: M \times M \longrightarrow R_+$



① $d(x,y) \geq 0$ ② $d(x,y) = d(y,x)$.

③ $\begin{cases} d(x,x) = 0 \\ \text{if } d(x,y) = 0 \longrightarrow x = y. \end{cases}$ ④ $d(x,y) \leq d(x,z) + d(z,x)$



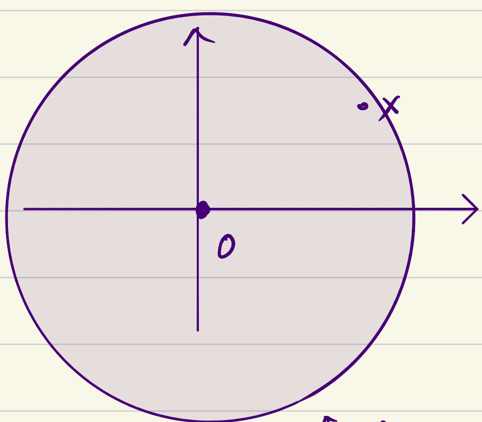
EXAMPLES: if $M = \mathbb{R}^n$

$$d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i|$$

$$d_2(x,y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_p(x,y) := \sqrt[1/p]{\sum_{i=1}^{n} |x_i - y_i|^p}$$

تمرین: اینها را تحویل دهید. در فضای دو بعدی R^2، یک نقطه مبدأ بگیرید.



S^2 ↑ for d_2

$$S^1 := \{ x \mid d(x,0) = 1 \}$$

دایره

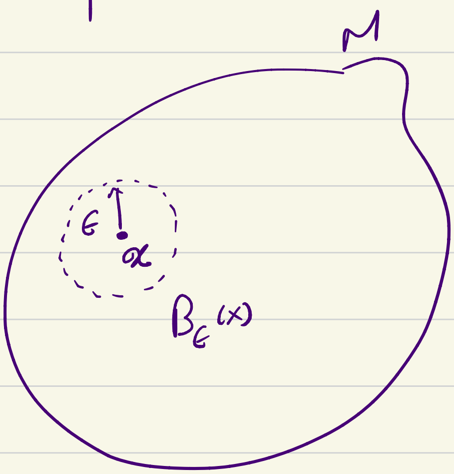
سوال: شکل دایره را برای d_1، d_∞ رسم کنید.

Another example: let M be any set.

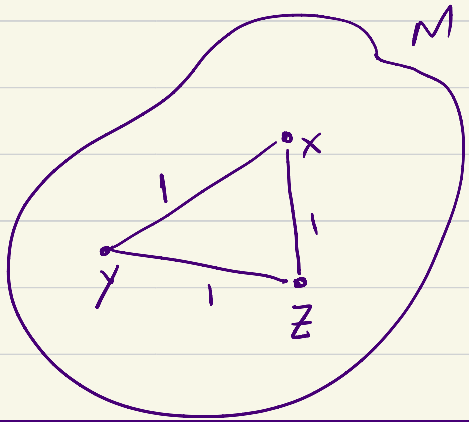
Define $d: M \times M \to \mathbb{R}$ $\begin{cases} d(x,x) = 0 \\ d(x,y) = 1 \end{cases}$

یک متریک خیلی خیلی زشت.

Def: Open ball.

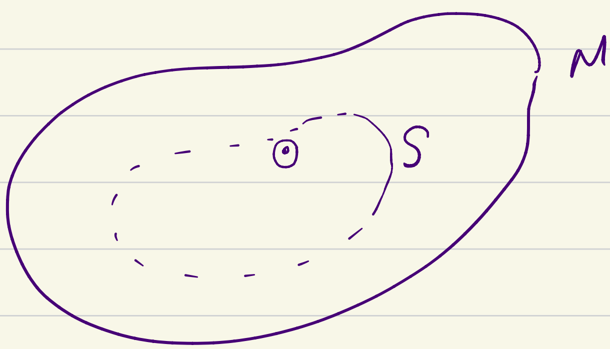


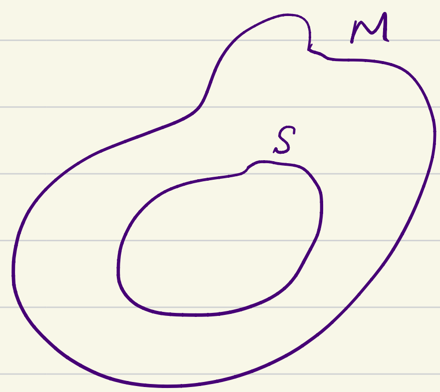
$1 \le 1+1$



$$B_\epsilon(x) = \{\ y \in M \mid d(x,y) < \epsilon\ \}.$$

Def: A subset $S \subset M$ is open when?





S is an open subset of M if $\forall\, x \in S$, $\exists\, \epsilon > 0 \mid$

$B_\epsilon(x) \subset S$.

$$V \quad , \quad V^*$$

$$\{e_i\} \quad \{e^i\} \longrightarrow \alpha \in V^* \qquad \underline{\alpha = \alpha_i e^i}$$

$$x \in V \to x = x^i e_i \qquad\qquad e^i(e_j) := \delta^i_j$$

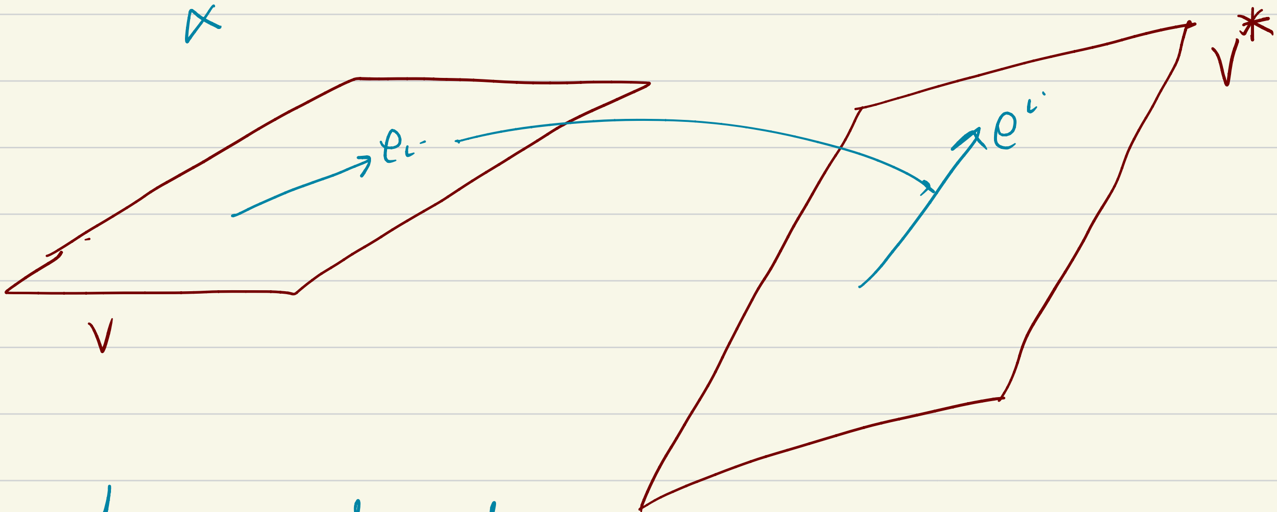
$$\alpha(v) = \alpha_i e^i(v^j e_j) = \alpha_i v^j e^i(e_j) = \alpha_i v^i$$

$V \longrightarrow V^*$ isomorphism آیا ?

$$\phi : \underbrace{e_i \longrightarrow e^i}$$

$$\phi(x) = \underbrace{\phi(x^i e_i)} = \underline{x^i e^i}$$

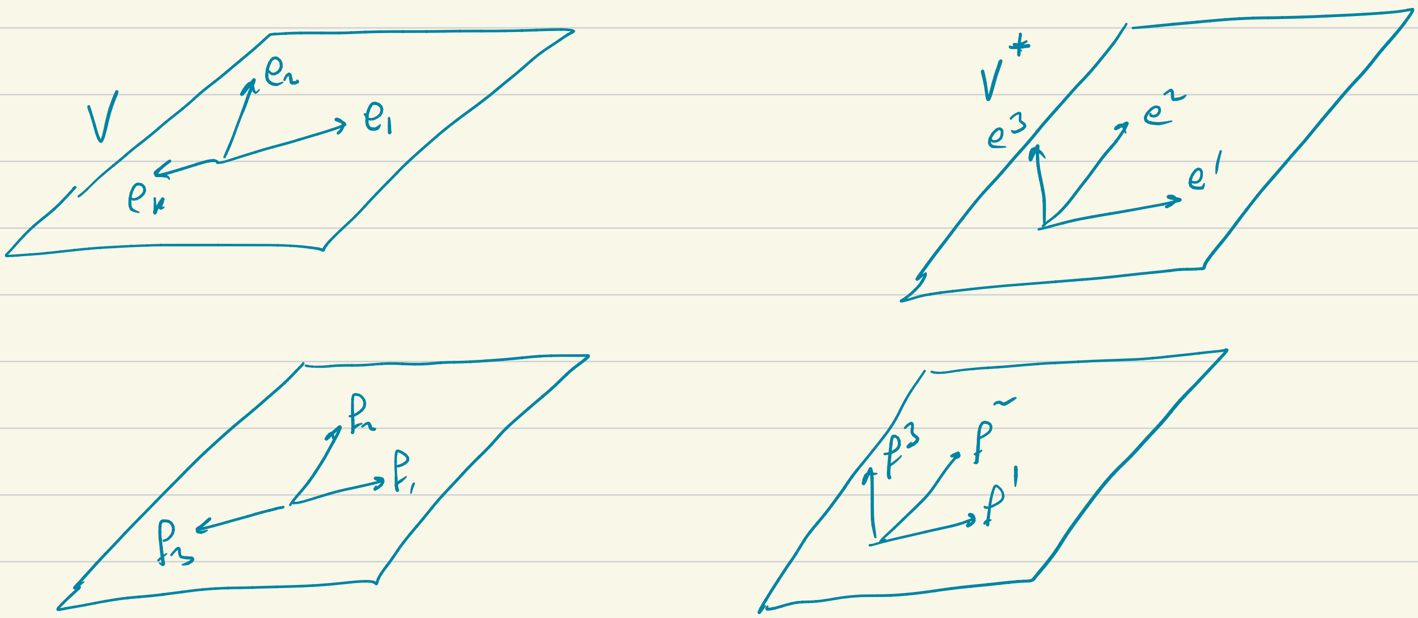
$$x = \underbrace{x^i e_i}_{x} \longrightarrow \phi(x) \in V^* \quad \phi(x) = \underline{x^i e^i}$$



$$\begin{cases} \phi(x+y) = \phi(x) + \phi(y) \\ \phi(cx) = c\,\phi(x) \end{cases}$$

$\phi : e_i \longrightarrow e^i$ Not Natural because the map changes under change of bases.

For V: if $\{e_i\}$ is changed to $\{f_i := S_i{}^j e_j\}$



$$f^i(f_j) = \delta^i_j$$

if $\quad f_i = S_i{}^j e_j \longrightarrow f^i = T^i{}_k e^k$

$$f^i(f_j) = \delta^i_j \qquad T^i{}_k e^k(S_j{}^l e_l) =$$

$$= T^i{}_k S_j{}^l \underbrace{e^k(e_l)}_{\delta^k_l} = T^i{}_k S_j{}^k = \delta^i_j$$

$$TS^T = 1$$

$$\boxed{T^{i}{}_{k}\, S_{j}{}^{k} = \delta^{i}_{j}} \qquad T = (S^{T})^{-1}$$

$$\phi : e_i \longrightarrow e^i \qquad \phi : f_i \not\longrightarrow f^i$$

$$x \xrightarrow{\{e_i\}} \alpha$$

$$x \xrightarrow{\{f_i\}} \beta$$

به این معنا این نگاشت وابسته به پایه است

و یکتاست از [illegible].

Assume that V is equipped with an inner product.

$$g : V \times V \longrightarrow R$$

$$\langle\ ,\ \rangle$$

$$\begin{cases} g(v,w) = g(w,v) \\ g(v,v) \geq 0 \qquad g(v,v) = 0 \longrightarrow v = 0 \\ g \text{ is bilinear} \end{cases}$$

$$g(e_i, e_j) = g_{ij}$$

↘ Metric

$$\phi : e_i \longrightarrow \overbrace{g_{ij}\, e^j}^{\phi(e_i)} \in V^*$$

$$\phi(x = x^i e_i) = x^i \phi(e_i) = x^i\, g_{ij}\, e^j = (g_{ji} x^i) e^j \equiv x_j e^j$$

$$x \in V \quad x = x^i e_i \xrightarrow{\ \phi\ } \phi(x) = x_i e^i \in V^*$$

تمرین: نشان دهید این نگاشت طبیعی است:

نشان دهید، تحت تغییر پایه $g_{ij}\, e^j$ و e_i

هر دو یک جور تبدیل می‌شوند.

$$\begin{aligned} V &\longrightarrow V^* \\ x &\longrightarrow \phi(x) \end{aligned} \qquad \langle \phi(x), y \rangle := g(x,y). \quad (*)$$

باید نشان دهیم:

① $\phi(x)$ یک تابعی خطی است.

② $\phi(x_1) + \phi(x_2) = \phi(x_1 + x_2)$

③ $c\,\phi(x) = \phi(cx)$

④ یک به یک و پوشاست.

⑤ نشان دهید $\phi(e_i) = g_{ij}\, e^j$ از $*$

Metric Spaces: (M, d)

$d \to$ distance. $d(x,y) =$ distance of x & y.

1) $d(x,y) \geq 0$

2) $d(x,y) = d(y,x)$

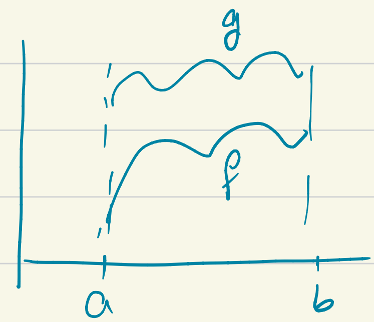
3) $d(x,x) = 0$ & $d(x,y) = 0 \longrightarrow x = y$

4) $d(x,y) \leq d(x,z) + d(z,y)$

Example ① in $\mathbb{R}^n$ $\quad d_p(x,y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$

$p = 1, 2, 3, \ldots \infty$.

Example ② $\quad M = C[a,b]$



توابع پیوسته در بازه [a,b] .

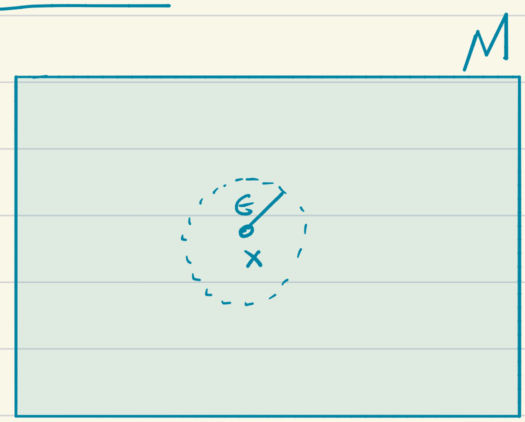
$$d(f,g) := \int_a^b |f(x) - g(x)|\, dx$$

Example ③. $\quad M = M_{n \times n}(\mathbb{R}) \qquad d(A,B) = \sum_{i,j=1}^{n} (A_{ij} - B_{ij})^2$

Def: ϵ-Ball

$$B_\epsilon(x)$$

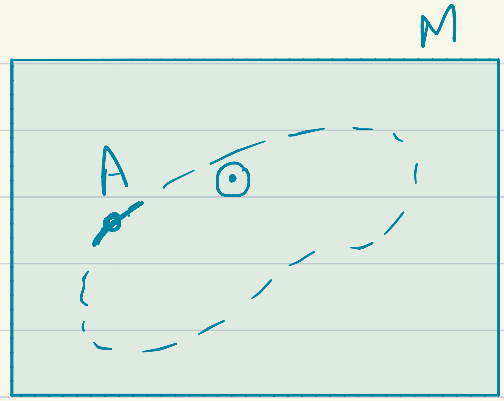
$$B_\epsilon(x) := \{ y \in M \mid d(x,y) < \epsilon \}$$



Def: Open set:

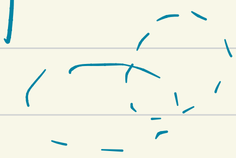
A is open if: $\forall x \in A$

$\exists \epsilon \mid B_\epsilon(x) \subset A$.



theorem: ① Φ is open. ② M is open.

③ if A & B are open $A \cup B$ is also open.



proof:

$$x \in A\cup B \longrightarrow \begin{cases} x \in A \to \exists \epsilon_1 \mid B_{\epsilon_1}(x) \subset A \to \underline{B_{\epsilon_1}(x) \subset A\cup B} \\ \text{or} \\ x \in B \to \exists \epsilon_2 \mid B_{\epsilon_2}(x) \subset B \to B_{\epsilon_2}(x) \subset A\cup B \end{cases}$$

$\downarrow$

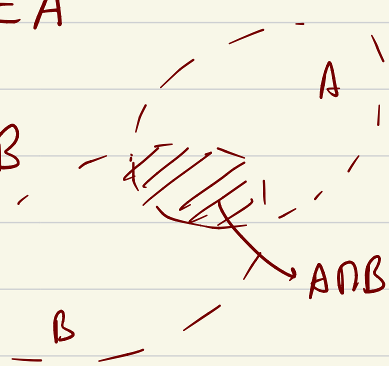
$A\cup B$ is open.

Union of any number of open sets is also open

④ the intersection of a finite number of open sets is open

proof:

$$x \in A\cap B \to \begin{cases} x \in A \to \exists \epsilon_1 \mid B_{\epsilon_1}(x) \subset A \\ \& \\ x \in B \to \exists \epsilon_2 \mid B_{\epsilon_2}(x) \in B \end{cases}$$



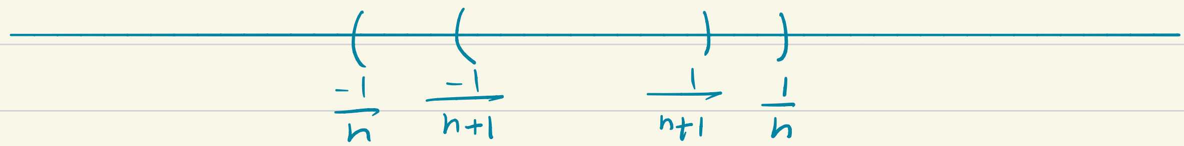
$\to$ take $\epsilon = \min(\epsilon_1, \epsilon_2)$

$$B_\epsilon(x) \in A, \quad B_\epsilon(x) \in B \longrightarrow B_\epsilon(x) \in A\cap B.$$

it can
$\downarrow$

finiteness is important. otherwise $\min(\epsilon_1, \epsilon_2, \dots) = 0$

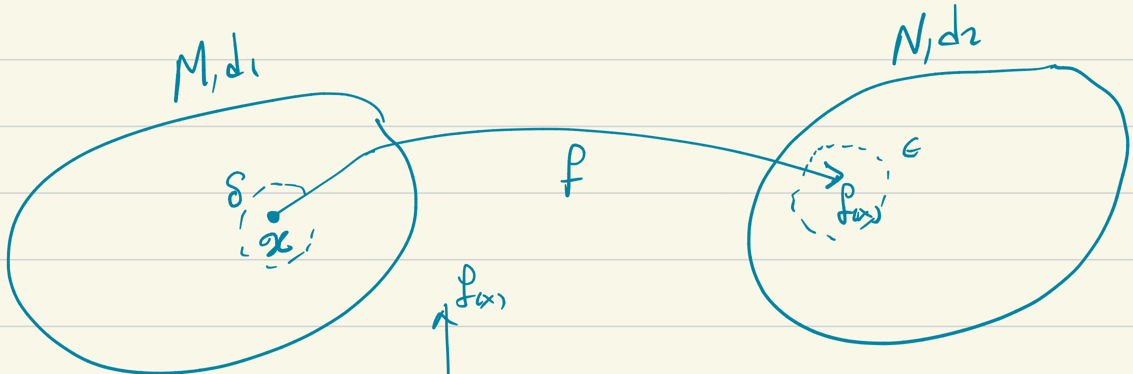
EXAMPLE: in R^1 $\quad \left(-\frac{1}{n}, \frac{1}{n}\right)$ $\quad n = 1, 2, 3, \ldots$



Def: Continous function: f is continous at x if

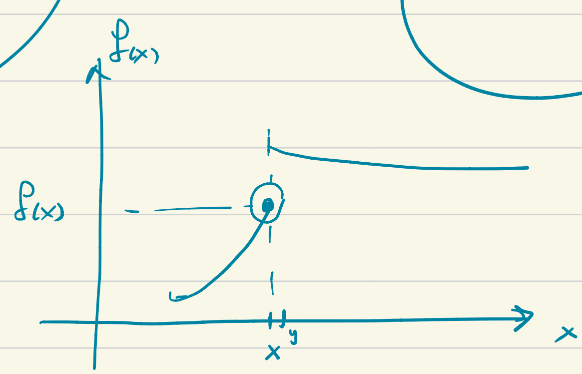
$$f : (M, d_1) \longrightarrow (N, d_2)$$

$$\forall \epsilon > 0 \mid \exists \delta > 0 \mid \text{ if } d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon$$



f پیوسته در M است اگر به کلی

پیوسته (در تمام) نقاط $x \in M$ باشد.



Theorem: (. تمرین مهم) .

prove that f is continous on M if $f^{-1}(U)$ is open for any open $U \in N$.

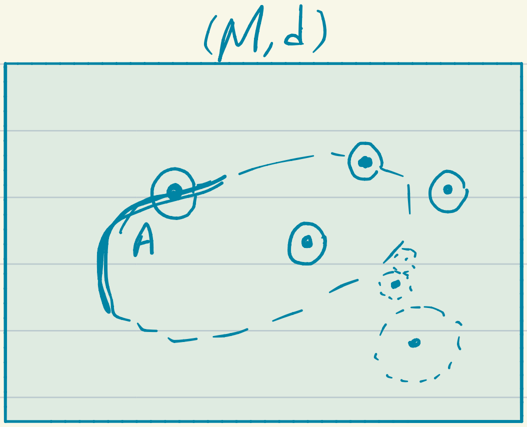
For any U which is open in (N, d_2) $f^{-1}(U)$ is open in (M, d_1).

Introduction to Metric & Topological spaces; by: W. A. Sutherland

180 pp.

A is closed if $A^c = X - A$ is open. (complement)

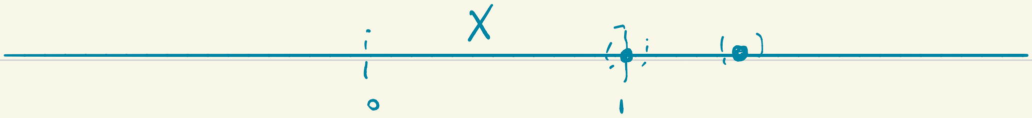
Def: limit point.



$x \in M$ is a limit point of $A \subset M$

if $\forall \epsilon > 0 \quad B_\epsilon(x) \cap A \neq \phi$

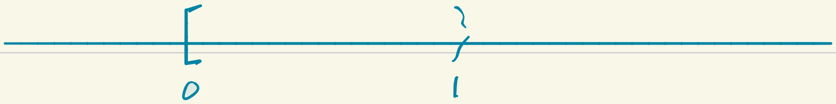
$M = \mathbb{R}$



$A = (0, 1)$

$1 \notin A \quad 1 \in \mathbb{R}$

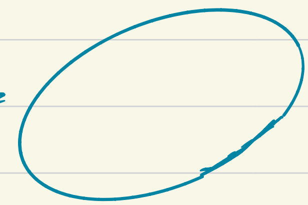
if $A = [0, 1)$



closure of $A = \bar{A} = A \cup \{\text{limit points of } A\}$.

$A = (0,1) \longrightarrow \bar{A} = [0,1]$

$A = [0,1) \longrightarrow \bar{A} = [0,1]$

if $A =$  $\bar{A} =$ 

$\partial A =$ مرز $A \neq \bar{A} - A$ مثال نقض: $A = [0,1)$

$\bar{A} = [0,1]$

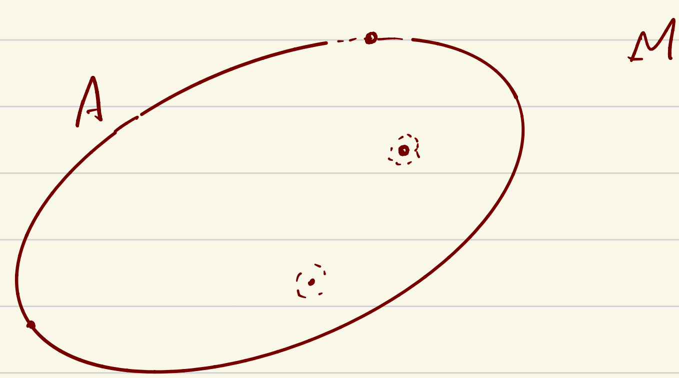
$\bar{A} - A = \{1\}$

$\partial A = \{0\} \cup \{1\}$.

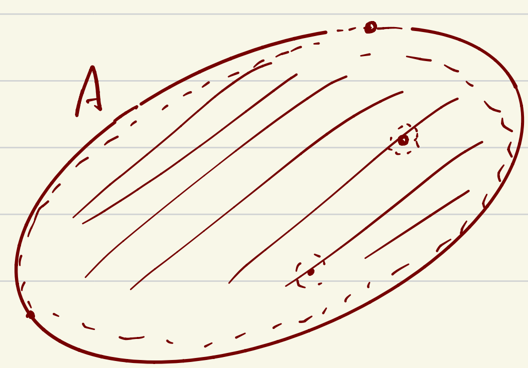
تمرین: آیا این گزاره درست است؟ ثابت کنید

$\partial A = \bar{A} \cap \overline{A^c}$?

Interior point. نقطه داخلی



$x \in A$ is an interior point of A if $\exists \epsilon > 0 \mid B_\epsilon(x) \subset A$.

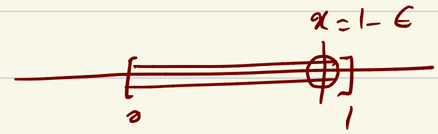


$A^\circ :=$ the set of all interior points of A.

EXAMPLE: if $A = [0,1]$ $A^\circ = (0,1)$

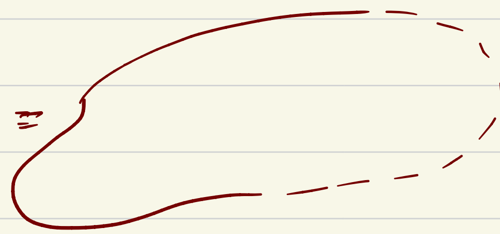
let $x = 1-\epsilon$ $\qquad$ $A = [0, 1]$

$\epsilon' = \epsilon/2$

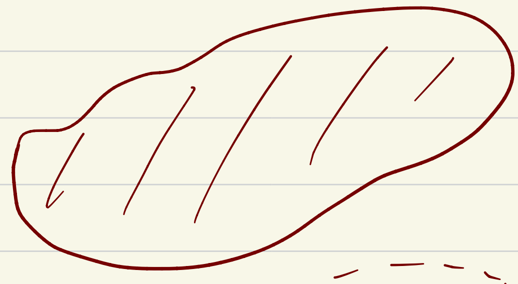


$\partial A = \bar{A} - A^\circ$

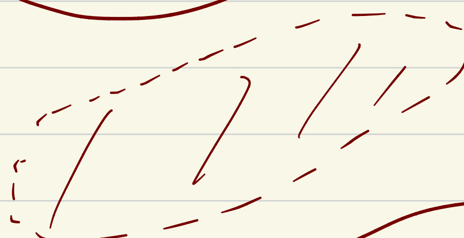
$A =$



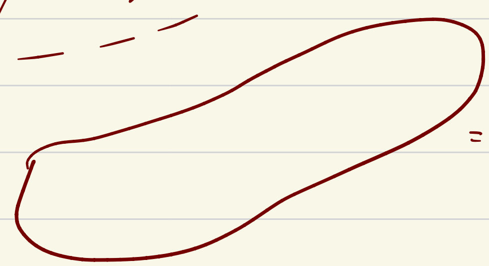
$\bar{A} =$



$A^\circ =$

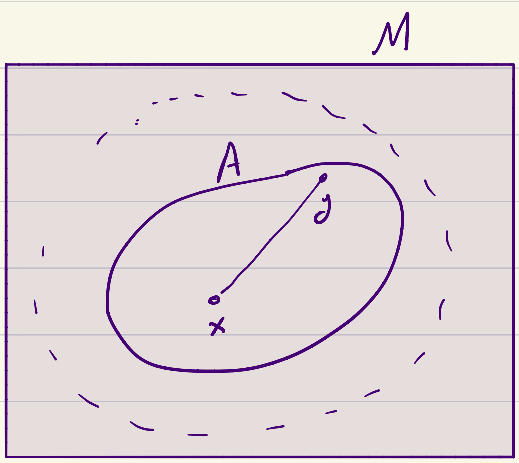


$\bar{A} - A^\circ =$



$= \partial A.$

- Bounded set



A is bounded? if $\exists\, x \in A$ & $M > 0 \mid d(x,y) < M \quad \forall y \in A$.

تمرین: نشان دهید که اگر S_1 , S_2 کراندار باشند، آنگاه. $S_1 \cup S_2$ نیز کراندار

Hint: Use triangle inequality

Topology & Topological Spaces: $d: M \times M \to R$ Not always necessary.

let X be a set. 2^X = the collection of all subsets of X.

$\tau \subset 2^X$ | ① $\phi \in \tau$ ✓

② $X \in \tau$ ✓

③ the intersection of any finite number of elements of $\tau \in \tau$

④ " union of any number of "
" " $\tau \in \tau$.

(X, τ) is a Topological space.

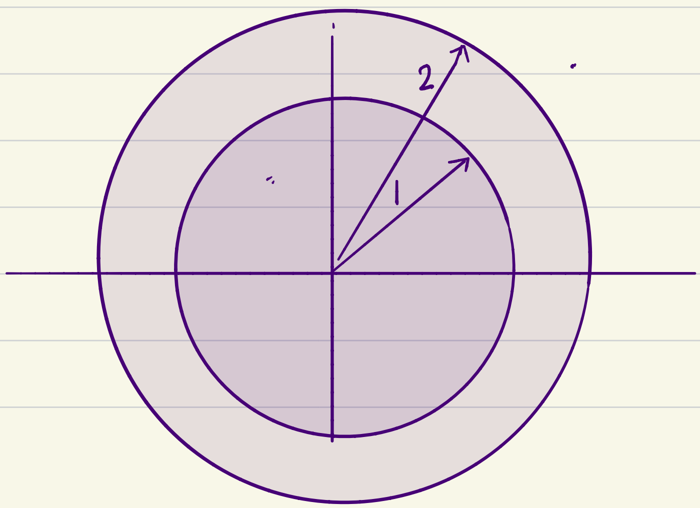
EXAMPLE 1) $X = \{a, b, c\}$ $\tau = \{\phi, \{a, b, c\}\}$

Example 2) $X = \{a, b, c\}$ $\tau = 2^X$

Example 3) $X = \{a, b, c\}$ $\tau = \{\phi, X, \{a, b\}\}$.

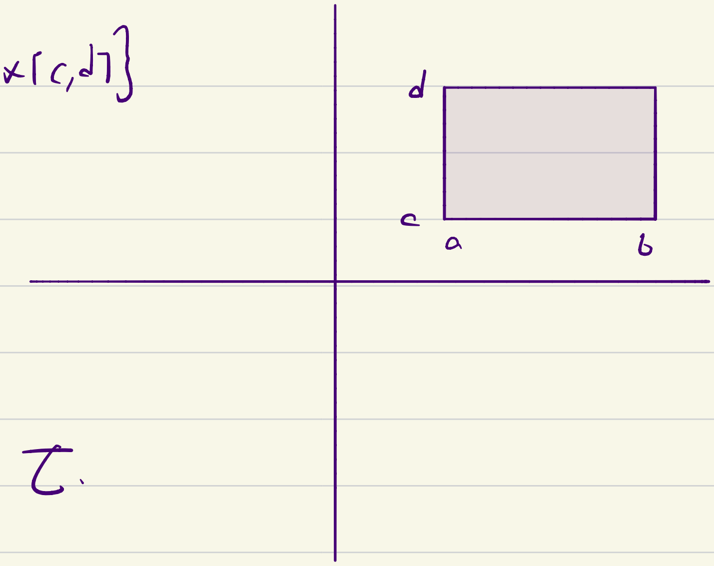
" 4) $X = R^2$

$\tau = \{\phi, R^2, D_n\}$



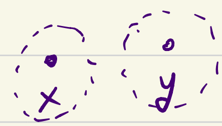
$M = \mathbb{R}^2 \qquad \tau = \{\phi, \mathbb{R}^2, [a,b]\times[c,d]\}$

تمرین: سعی کنید: تصحیح این توپولوژی τ را
تبدیل به یک توپولوژی کنید.



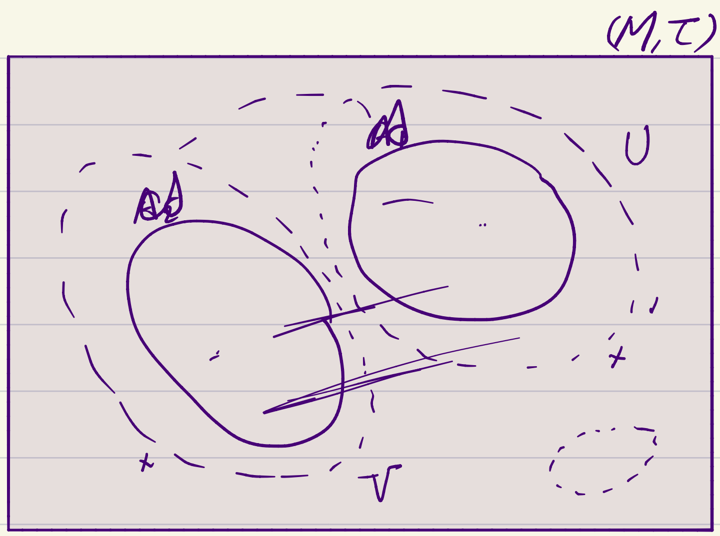
open set $\equiv$ any element of τ.

Hausdorf space:



if $\forall x \,\&\, y \;\exists\; U_1 \,\&\, U_2 \mid x \in U_1$
$y \in U_2 \;\&\; U_1 \cap U_2 = \phi$

Connectedness:



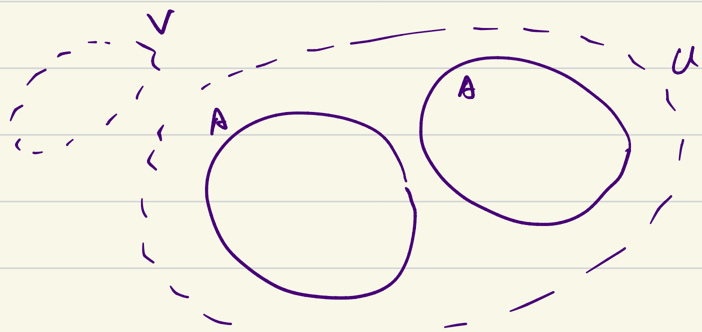
A is disconnected if we can find open sets U & V

① $A \subset U \cup V$ ✓

② $A \cap U \neq \phi$ ✓

③ $A \cap V \neq \phi$ ✓ ✗

④ $U \cap V = \phi$ ✓



~~cont.~~ → ~~p.d.~~ I.M.A., ~~simply connected~~, ~~Compact~~ Ex., Homeomorph., TI, ExAppl — Euler char. $\chi(\mathbb{R}, \mathbb{R}^2)$

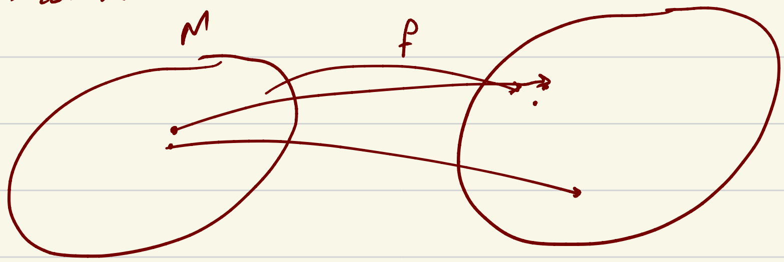
---

① Continuous function. $\qquad f : (M, d) \longrightarrow (M', d')$

f is continuous at x

$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \mid$

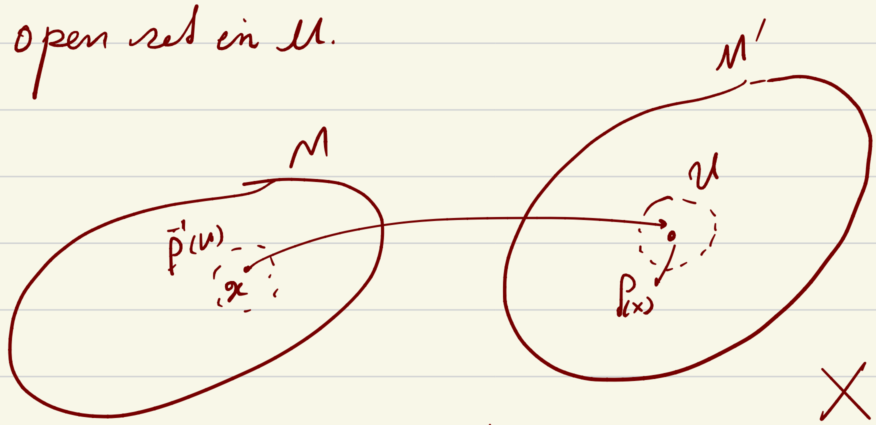
if $d(x, y) < \delta \rightarrow d'(f(x), f(y)) < \epsilon$ *



○ Def: f is Continuous in M if f is $\underbrace{\text{continuous at every}}$ $x \in M$.

○ f is Continuous at x iff. $\forall$ open set $\underset{f(x)}{U} \subset M' \Rightarrow$

$f^{-1}(U)$ is an open set in M.



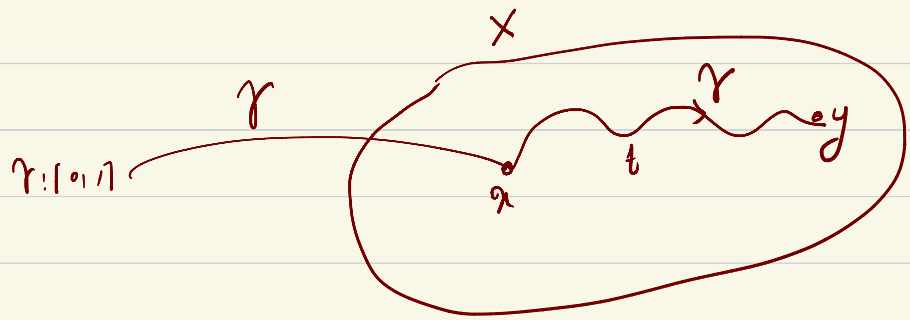
○ f is continuous in M if $\forall\, \underset{\text{open}}{U} \in M'$ $\; f^{-1}(U)$ is open in M.

$f : (X, \tau) \longrightarrow (Y, \tau')$ $\qquad$ f is Conti in X if

↑ topological space.

$\forall\, U \in \tau' \quad f^{-1}(U) \in \tau$.

path-Connectedness: $\gamma : [0,1] \longrightarrow (X, \tau)$



γ پیوسته است.

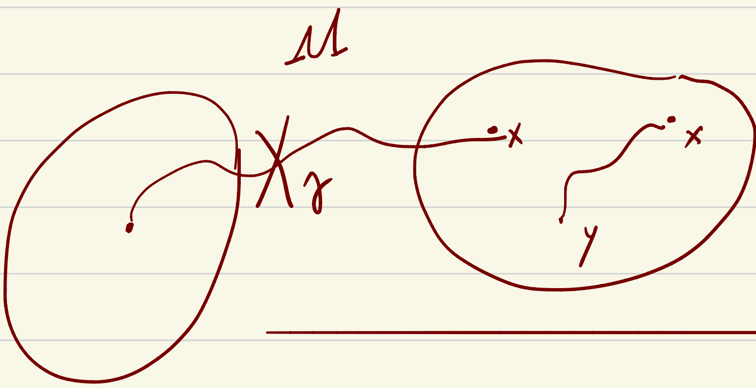
$\gamma(0) = x$

$\gamma(1) = y$

if $\gamma(0) = \gamma(1) \rightarrow \gamma$ is a loop

M is path connected if $\forall\, x, y \in M$ $\exists\, \gamma :$ $\gamma(0) = x$ $\gamma(1) = y$

Continuous



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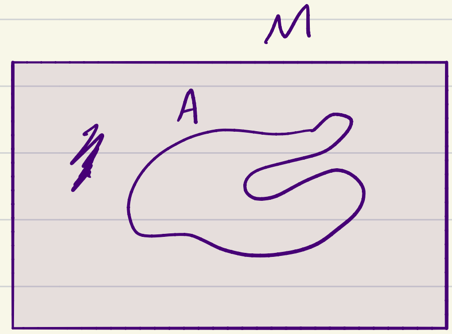
theorem: Any path-Connected $A \subset M \implies A$ is Connected.

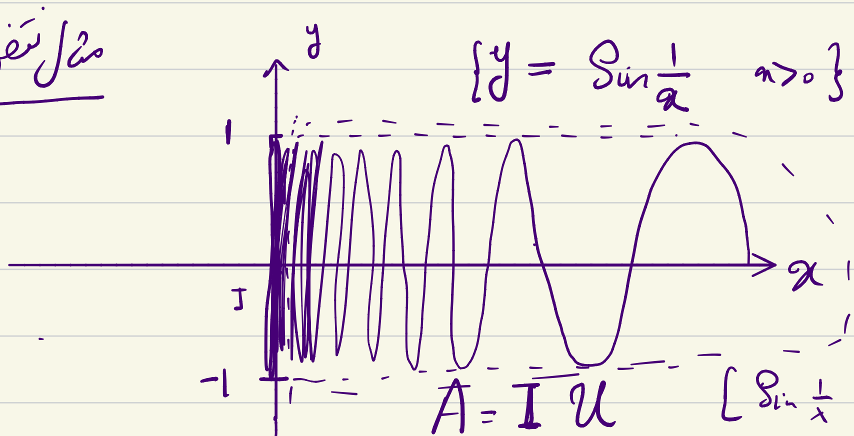
Topological space

the Converse is not necessarily true.



مثال نقض

$\{ y = \sin\frac{1}{x} \quad x > 0 \}$

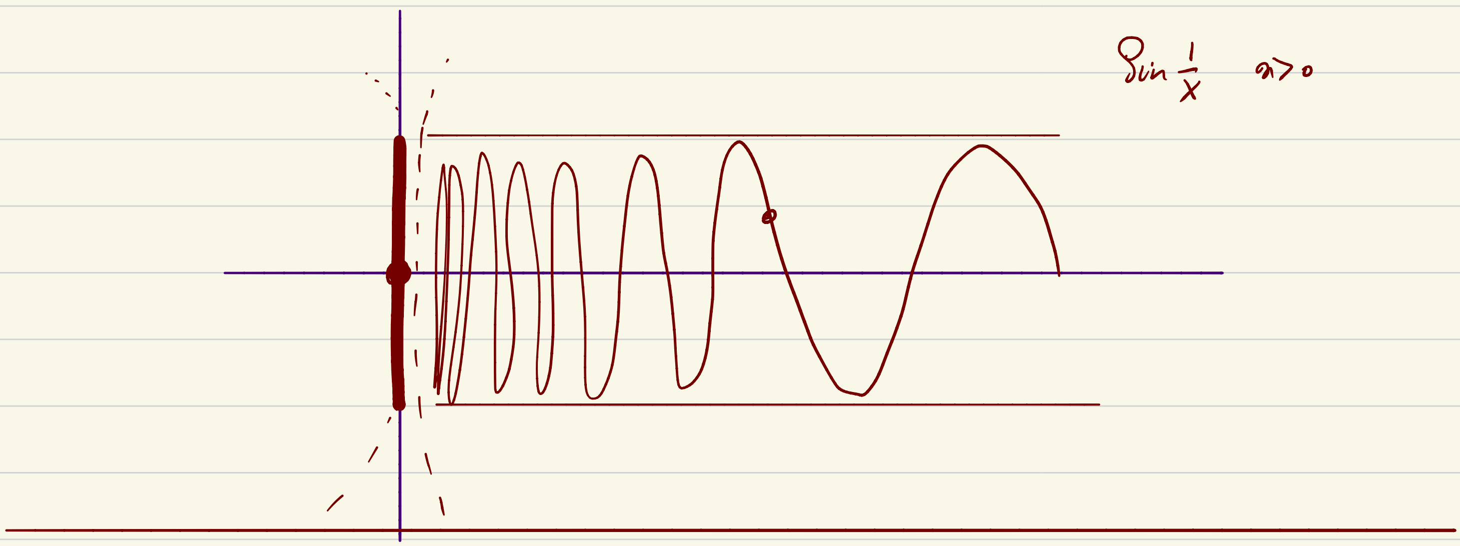


$A = \underset{\sim}{I} \cup \underbrace{\{\sin\frac{1}{x} \text{ گراف تابع}\}}_{B}$

I

$\boxed{M = R^2}$

A is connected $\longrightarrow$ But A is not path connected.

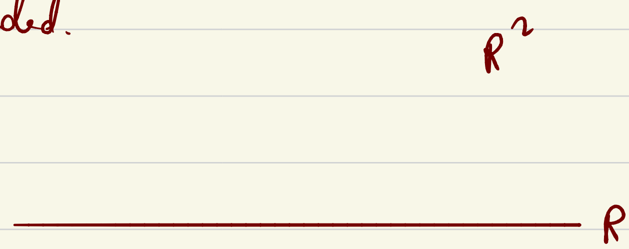


Compactness: in (M,d) A is compact: if

1) A is closed. 2) A is bounded.

Ex. R in R^2 is closed.

R is not bounded.



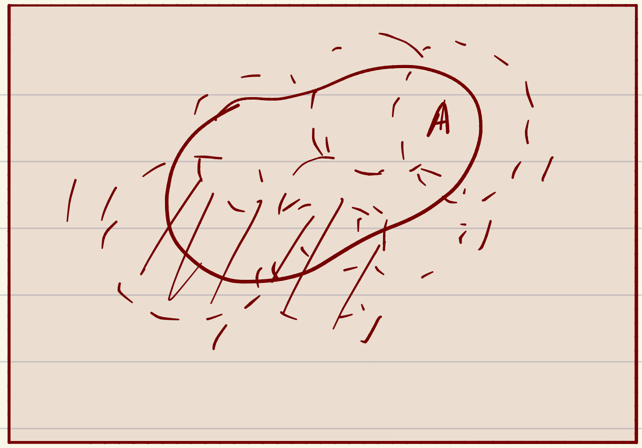
Ex. $(0,1) \subset R^1$ is bounded but is not closed.

therefore $(0,1)$ is not compact.

theorem: in (M,d) A is compact $\Longleftrightarrow$ $\forall$ open cover of A $\exists$ a finite sub-cover if A is compact.

Open cover of $A = \{ U_i \mid$

U_i's are open, $A \subset \bigcup U_i \}$



$(0,1)$ $\quad \{ (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, 2) \}$ Covers $(0,1)$

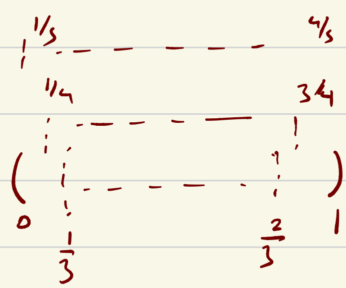
Def: if M is a topological space, $A \subset M$ is compact if $\underline{\forall \text{ cover}}$ of A, $\exists$ a finite subcover.

$A = (0,1) \quad M = \mathbb{R}$

گزاره ۱: A is compact because it has a finite cover

$\{-2, 2\}$

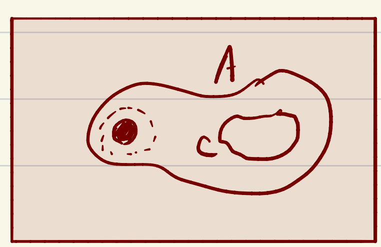
$\underline{\{ (\frac{1}{n}, 1 - \frac{1}{n}) \mid n = 3, 4, 5, \dots \}}$



Connected همبند , path-Connected همبند مسیری

Simply connected همبند ساده.

Def.: Simply Connected: A is simply connected



if every closed loop in A can be continuously contracted to a point.

Homeomorphism

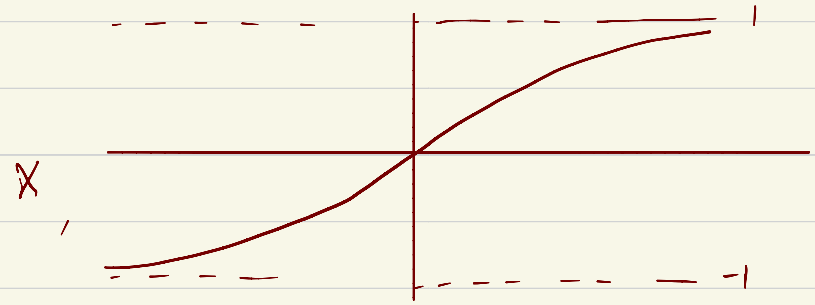
$f: X \longrightarrow Y$ $\quad X \sim Y$

$f^{-1}: Y \longrightarrow X$ $\quad$ Homeomorphic.

f & f^{-1} are continuous.

Ex: $X = (-1,1)$ $\quad Y = \mathbb{R}$ $\quad X \sim Y$

$f(x) = \tanh x$

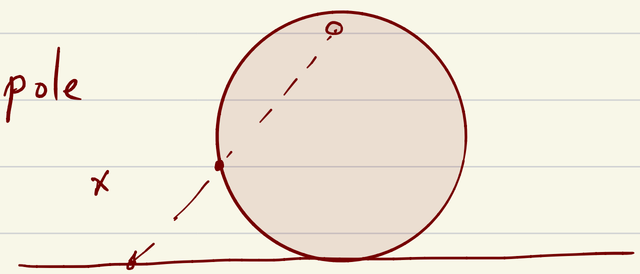


$(-1,1) \times (-1,1) \sim \mathbb{R}^2$

$S^2 \sim$ ellipsoid.

$S^1 \sim$ Rectangle

$\left\{ \begin{array}{l} S^2 - \text{North pole} \\ \sim \mathbb{R}^2 \end{array} \right.$



تمرین، نگاشت را صراحتاً بیابید

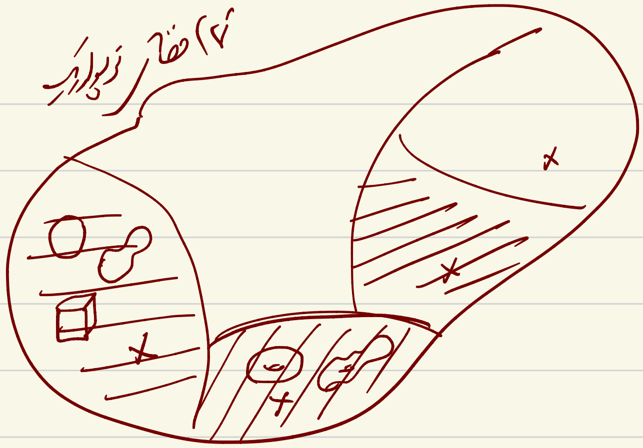
$S^N -$ North pole , $\mathbb{R}^n$ $\quad$ تمرین.

Homeomorphism is an equivalence relation.

$X \sim X$

$X \sim Y \rightarrow Y \sim X$

$\underset{f}{X \sim Y}, \underset{g}{Y \sim Z} \longrightarrow \underset{g \circ f}{X \sim Z}$



Definition: Topological Invariant.

$$M \longrightarrow I(M)$$

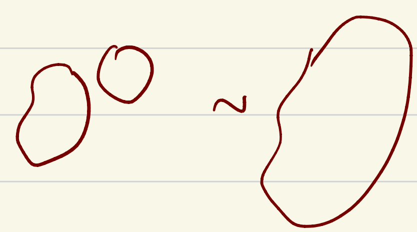
is the Mathematical object which we have defined or calculated for M.

I is a topological Invariant: if the following condition holds:

$*$ if $M \sim N \Longrightarrow I(M) = I(N)$

$M \not\sim N \xleftarrow{\text{[illegible]}} I(M) \neq I(N)$ نتیجه:

Ex: Connectedness is a topological invariant.



path connectedness

Compactness

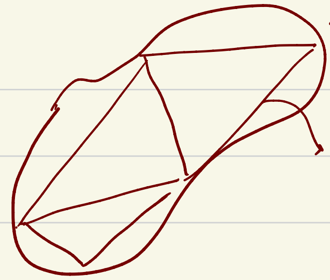
Homotopy ⟵ در دو بعد آنچه نگه‌دار توپولوژیک همومورفی است

Homology. خلاصه‌ای از

For two dimensions $\xrightarrow{\text{generalized to}}$ Higher dimensions.

Euler Characters:

X = A 2d surface



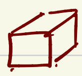
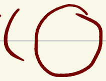
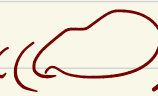
Triangulation

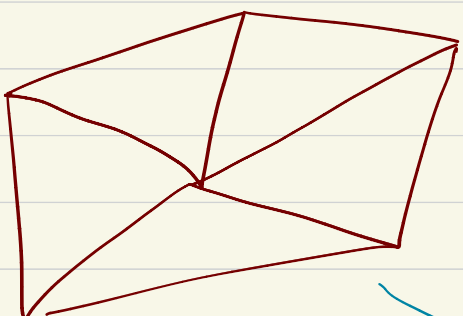
$$\chi = V - E + F$$

تعداد وجوه

تعداد رئوس

تعداد اضلاع

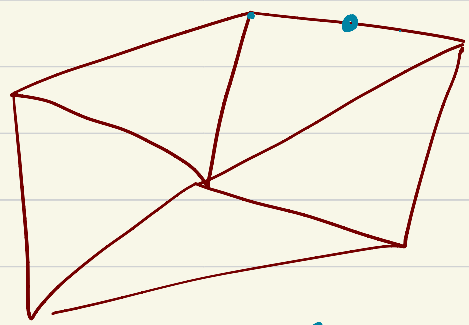
$\chi($  $) = 8 - 12 + 6 = 2.$ $\longrightarrow$ $\chi($  $) = \chi($  $) = 2$



تغییر مثلث‌بندی

① یک نقطه در وسط یک ضلع قرار دهیم.

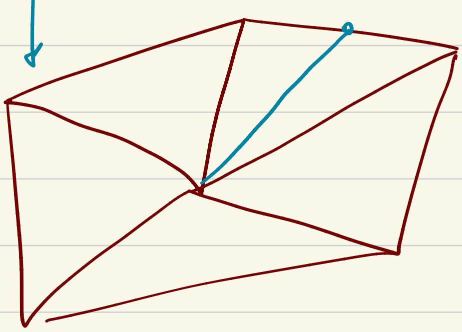
①



2

$V \to V+1$

$E \to E+1$



② یک وجه را نصف کنید.

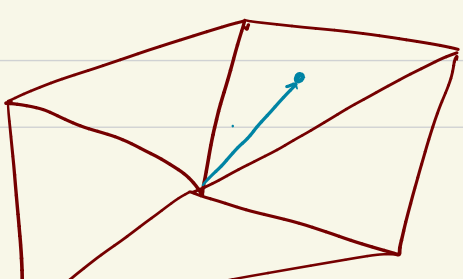
$V \longrightarrow V+1$

$E \longrightarrow E+2$

$F \longrightarrow F+1$

$\chi \longrightarrow \chi$

3



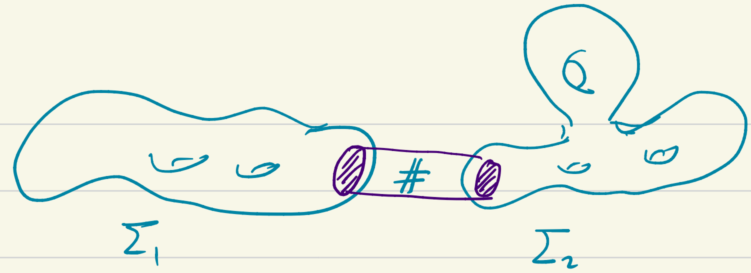
③ یک نقطه در وسط وجه قرار دهید و یک ضلع به یک رأس وصل کنید.

$V \to V+1$

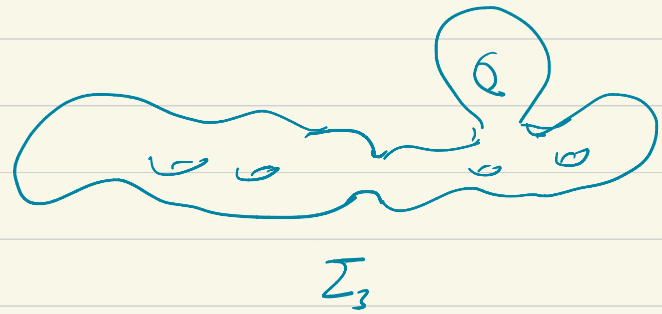
$E \to E+1$

$\chi \to \chi$

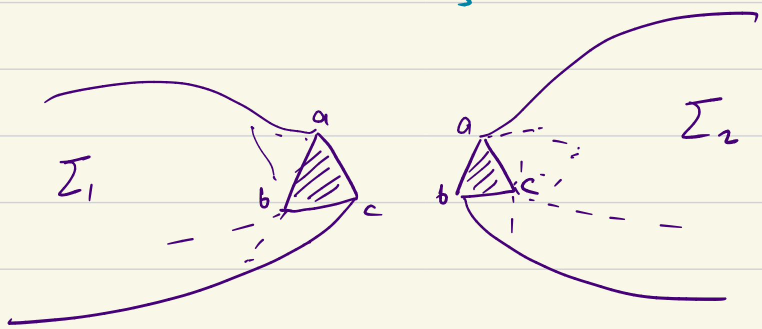
theorem:



$\Sigma_3 = \Sigma_1 \# \Sigma_2$



$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$



$V \longrightarrow V-3$

$E \longrightarrow E-3$

$F \longrightarrow F-2$

$V = V_1 + V_2 - 3$

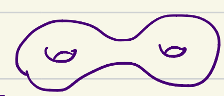
$E = E_1 + E_2 - 3$

$F = F_1 + F_2 - 2$

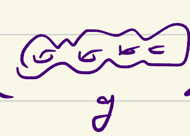
$\chi = V - E + F = \chi_1 + \chi_2 - 3 + 3 - 2$

$\boxed{\chi = \chi_1 + \chi_2 - 2}$

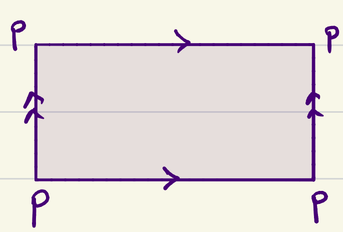
$\chi(O) = 2$, $\chi($  $) = 0$

$\chi($  $) = 0 + 0 - 2 = -2$

$\downarrow$

$\chi($  $) = 2 - 2g$

چنبره =



$V = 1$

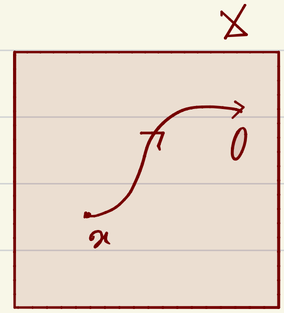
$E = 2 \quad \longrightarrow \chi = 0.$

$F = 1$

تمرین: نشان دهید که فقط پنج چندوجهی منتظم وجود دارد و مشخصات آن‌ها را پیدا کنید.

تعداد وجوه،
تعداد اضلاع
تعداد رئوس
......

Chapter 2. Homotopy Groups.



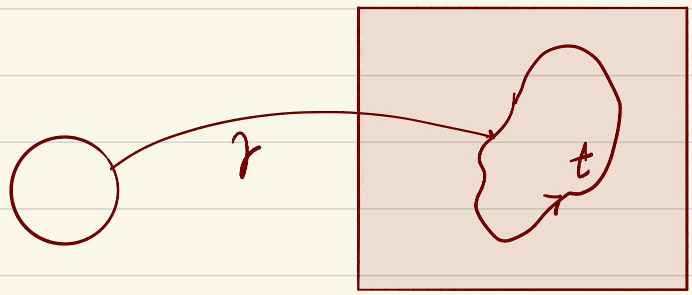
- closed $\quad \gamma : [0,1] \xrightarrow{\ \ } X$ ① $\qquad \gamma$ is Continuous.

$\gamma(0) = x \qquad \gamma(1) = y.$

γ is a loop or closed path if $\gamma(0) = \gamma(1)$ ②

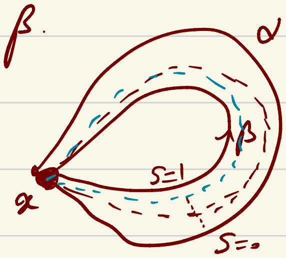
$\gamma : S^1 \longrightarrow X$

$\gamma(t)$



- Homotopy. Let $\alpha : S^1 \longrightarrow X$
$\beta : S^1 \longrightarrow X$

$\alpha \sim \beta$ if α can be continuously deformed to β.



$\exists\, F : S^1 \times [0,1] \longrightarrow X \qquad F(t,s)$

همان پارامتر سابق

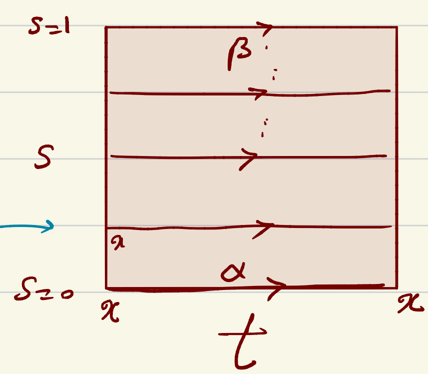
پارامتر حرکتی

deformation.

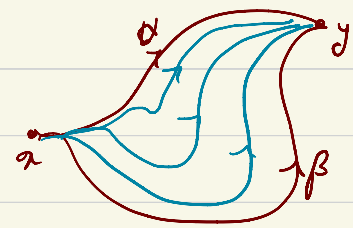
F نسبت به t , s پیوسته است.

$F(t,0) = \alpha(t)$ & $F(t,1) = \beta(t)$

$F(1,s) = F(0,s) = x$



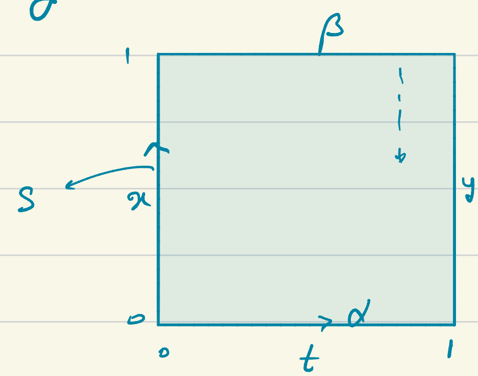
$\alpha : [0,1] \longrightarrow X \qquad \alpha(0) = x \qquad \alpha(1) = y$



$\beta : [0,1] \longrightarrow X \qquad \beta(0) = x \qquad \beta(1) = y$

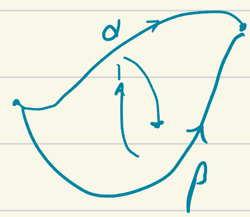
$F : [0,1] \times [0,1] \longrightarrow X \qquad \begin{cases} F(t,0) = \alpha(t) \\ F(t,1) = \beta(t) \end{cases}$

$F(0,s) = x \quad \forall s \qquad F(1,s) = y \quad \forall y$



Def. $\alpha \sim \beta$ if $\exists F$ with the above properties.

theorem: هموتوپی یک رابطه هم‌ارزی است.



proof: i) $\alpha \sim \alpha$ obvious.

$F(t,s) = \alpha(t) \quad \forall s.$

ii) $\alpha \sim \beta \rightarrow \exists F(t,s) \mid$ ادعا $\rightarrow \beta \sim \alpha$

$F(t,0) = \alpha(t) \qquad F(t,1) = \beta(t) \qquad F(0,s) = F(1,s) = x$

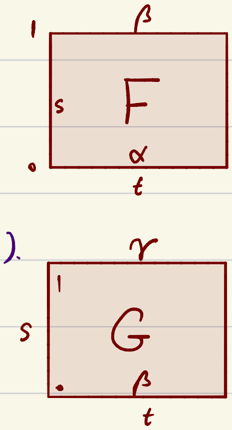
$F'(t,s) = F(t, 1-s) \qquad \begin{cases} F'(t,0) = F(t,1) = \beta(t) \\ F'(t,1) = F(t,0) = \alpha(t) \end{cases}$

$F'(0,s) = F'(1,s) = x$

iii) $\alpha \sim \beta$ & $\beta \sim \gamma \implies \alpha \sim \gamma$

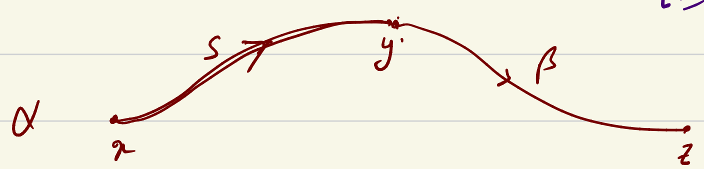
Proof: $\alpha \sim \beta \rightarrow \exists\, F(t,s) \mid F(t,0) = \alpha(t),\ F(t,1) = \beta(t)$

$\beta \sim \gamma \rightarrow \exists\, G(t,s) \mid G(t,0) = \beta(t),\ G(t,1) = \gamma(t).$



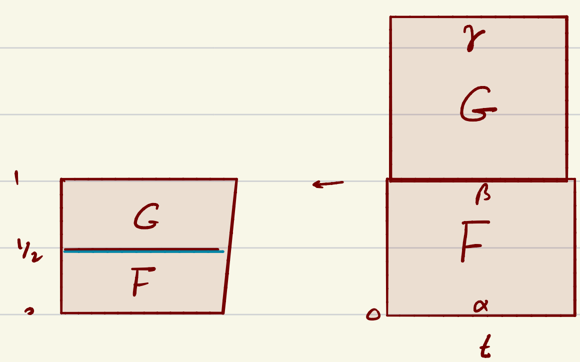
می‌خواهیم $H(t,s) \mid \quad H(t,0) = \alpha(t), \quad H(t,1) = \gamma(t)$

یک جور ترفند: خوب کردن [دنبال هم گذاشتن دو مسیر]



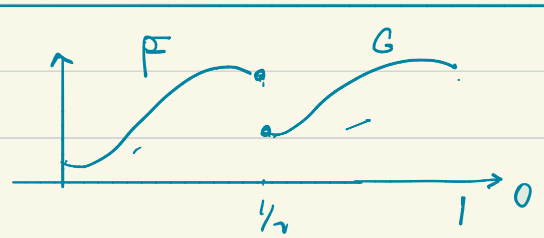
$$\begin{cases} \alpha(0) = x & \alpha(1) = y \\ \beta(0) = y & \beta(1) = z \end{cases} \qquad \gamma(s) := \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

$$H(t,s) = \begin{cases} F(t,2s) & 0 \le s \le 1/2 \\ G(t,2s-1) & 1/2 \le s \le 1. \end{cases}$$

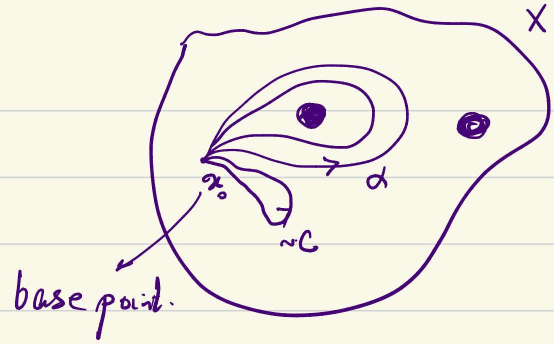


آیا $F(t,1) \overset{?}{=} G(t,0)$
$\quad \beta(t)$

$\Rightarrow$ Homotopy is an equivalence relation



Homotopy Groups:



$$\Pi_1(X, x_0) = \{ [\alpha] \}$$

کلاس‌های هم‌ارزی منحنی‌های بسته در X با نقطهٔ ثابت x_0

$e = [c]$ $\qquad$ c = Constant loop $\qquad$ $c(t) = x_0 \quad \forall t$

$$[\alpha][\beta] := [\alpha\beta] \qquad (\alpha\beta)(t) = \begin{cases} \alpha(2t) & 0 < t < 1/2 \\ \beta(2t-1) & 1/2 < t < 1. \end{cases}$$

$$[\alpha]^{-1} := [\alpha^{-1}]$$

$$\alpha^{-1}(t) = \alpha(1-t)$$

we should prove:

i) well defined-ness of multiplication.

ii) " " " inverse.

iii) $[c][\alpha] = [\alpha]$, $[\alpha][c] = [\alpha]$

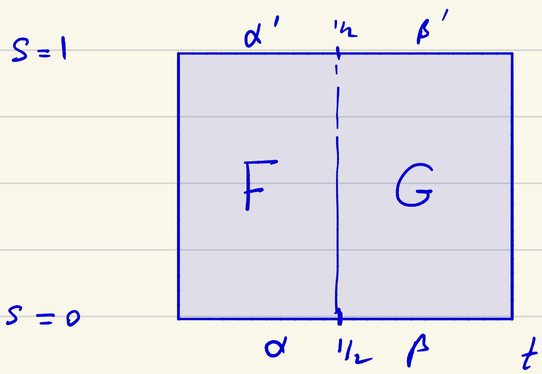
iv) $([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma])$.

---

i) $\alpha \sim \alpha' \longrightarrow \exists F \mid F(t,0) = \alpha(t) \quad F(t,1) = \alpha'(t).$ ✓

$\beta \sim \beta' \longrightarrow \exists G \mid G(t,0) = \beta(t) \quad G(t,1) = \beta'(t)$ ✓ $\qquad \alpha\beta \sim \alpha'\beta'$

$$\alpha\beta(t) = \begin{cases} \alpha(2t) & 0 < t < 1/2 \\ \beta(2t-1) & 1/2 < t < 1 \end{cases} \qquad \alpha'\beta'(t) = \begin{cases} \alpha'(2t) & 0 < t < 1/2 \\ \beta'(2t-1) & 1/2 < t < 1 \end{cases}$$



$$H(t,s) = \begin{cases} F(2t,s) & 0<t<1/2 \\ G(2t-1,s) & 1/2<t<1. \end{cases}$$

→ $\alpha\beta \sim \alpha'\beta'$ i) is proved.

ii) $[\alpha]^{-1}$ is well defined:

we want to prove that if $\alpha \sim \alpha' \longrightarrow \alpha^{-1} \sim \alpha'^{-1}$

$F(t,s) \mid \begin{cases} F(t,0) = \alpha(t) \\ F(t,1) = \alpha'(t) \end{cases}$ $\qquad F'(t,s) := F(1-t,s)$

---

iii) $[c][\alpha] = [\alpha]$, $[\alpha][c] = [\alpha]$.

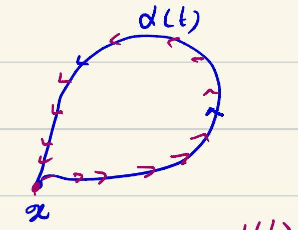
↕

$[c\alpha] = [\alpha]$

↕

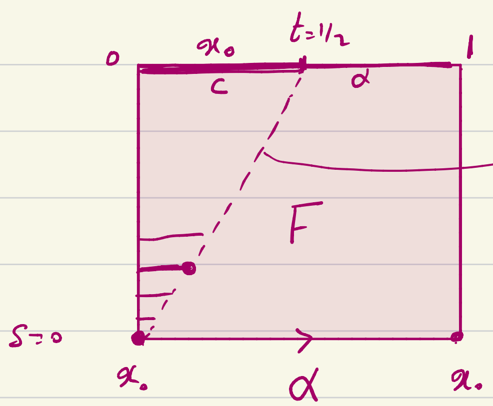
$c\alpha \sim \alpha$

$\alpha(0) = x \quad \alpha(1) = x \quad \alpha(t)$



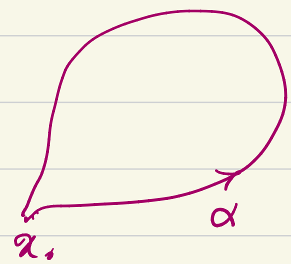
$(c\alpha)(t) \neq \alpha(t)$

$$(c\alpha)(t) = \begin{cases} c(2t) & 0<t<1/2 \\ \alpha(2t-1) & 1/2<t<1 \end{cases} = \begin{cases} x_0 & 0<t<1/2 \\ \alpha(2t-1) & 1/2\le t\le 1 \end{cases}$$



$t = As + B$

$s = 0 \to t = 0$
$s = 1 \to t = 1/2$ $\quad t = s/2$



$$F(t,s) = \begin{cases} x_0 & 0 \le t \le \frac{s}{2} \\ \alpha(2t) & \frac{s}{2} \le t \le 1 \end{cases}$$

$[c][\alpha] = [\alpha]$.

تمرین: show that $[\alpha][c] = [\alpha]$.

iv) we want to show that $([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma])$.

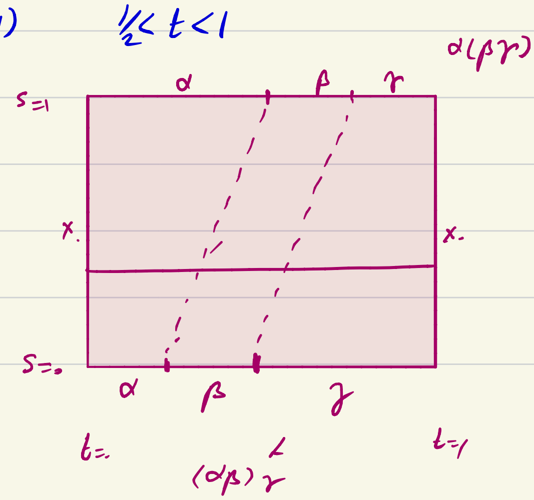
تمرین: جزئیات این بخش را کامل کنید.

$[\alpha\beta][\gamma] = [\alpha][\beta\gamma]$

$[(\alpha\beta)\gamma] = [\alpha(\beta\gamma)] \longrightarrow (\alpha\beta)\gamma \sim \alpha(\beta\gamma)$.

Let us define $(\alpha\beta)\gamma$;
$$(\alpha\beta)\gamma\,(t) = \begin{cases} (\alpha\beta)(2t) & 0 < t < 1/2 \\ \gamma(2t-1) & 1/2 < t < 1 \end{cases} \qquad (\alpha\beta)(2t) = \begin{cases} \alpha(4t) & 0 < t < 1/4 \\ \beta(4t-1) & 1/4 < t < 1/2 \end{cases}$$

$$\underline{(\alpha\beta)\gamma\,(t)} = \begin{cases} \alpha(4t) & 0 \le t \le 1/4 \\ \beta(4t-1) & 1/4 \le t \le 1/2 \\ \gamma(2t-1) & 1/2 \le t \le 1 \end{cases}$$



تمرین: نشان دهید، $[\alpha][\alpha^{-1}] = [c] = [\alpha^{-1}][\alpha]$.

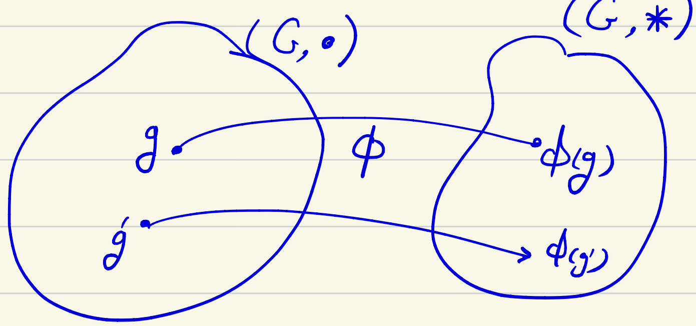
- $\Pi_1(X, a_0)$ = First homotopy Group ≡ Fundamental Group.

theorem: if X is path connected → $\Pi_1(X,a_0) \equiv \Pi_1(X,a_1)$

↑ isomorphic

Def: G & G' are homomorphic:

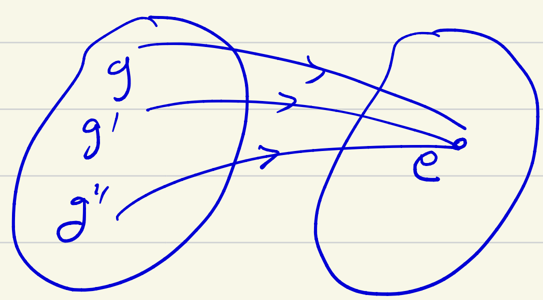
if $\phi(g) * \phi(g') = \phi(g \cdot g')$



A trivial Homomorphism:

$$G \xrightarrow{\phi} \{e\} \qquad \phi(g) = e$$

$$\underset{e}{\phi(g)} * \underset{e}{\phi(g')} = \underset{e}{\phi(gg')}$$



Isomorphism: ϕ is one to one & onto.

ϕ invertible. $\boxed{Z_2 = \{0, 1\}} \xleftrightarrow{\phi} \{1, -1\}$

$1+1=0$ $\qquad (-1)\cdot(-1)=1$.

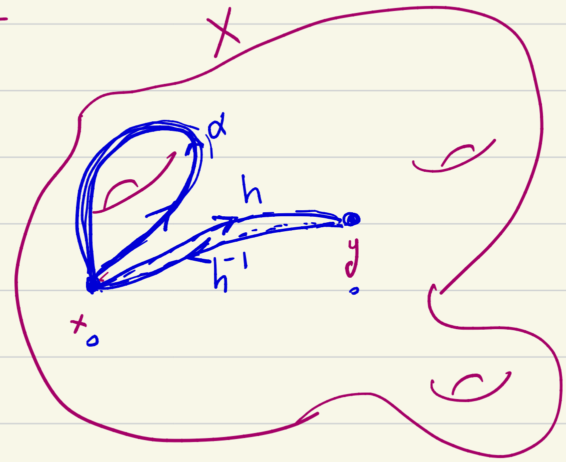
$$\begin{cases} \phi(0) = 1 \\ \phi(1) = -1 \end{cases}$$

$\mathbb{Z}_3 = \{0,1,2\}$ $a+b \mod 3.$ $1+2=0$ $2+2=1$ etc.

$$G = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} = \{I, a, a^2\}.$$

$G(X, x_0) = \{$ مجموعه‌ی همه‌ی منحنی‌های بسته که نقطه‌ی پایه‌شان x_0 است. $\}$

$G(X, y_0) = \{$ مجموعه‌ی " " " نقطه‌ی پایه‌شان y_0 است. $\}$



$\alpha \in G(X, x_0)$ $h^{-1}\alpha h \in G(X, y_0)$

$[\alpha] \in \Pi_1(X, x_0)$ $[h^{-1}\alpha h] \in \Pi_1(X, y_0)$

$$\phi([\alpha]) := [h^{-1}\alpha h]. \quad ①$$

$\phi([\alpha][\beta]) = \phi([\alpha])\phi([\beta])$ باید ثابت کنیم ✓

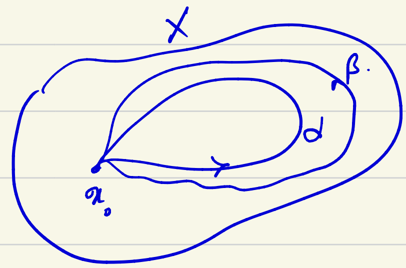
$\phi([\alpha\beta]) = \phi([\alpha])\phi([\beta])$ باید ثابت کنیم.

① $[h^{-1}\alpha\beta h] = \underbrace{\phi([\alpha])\phi([\beta])}$

$[h^{-1}\alpha\beta h] = [h^{-1}\alpha h][h^{-1}\beta h] \quad \in \Pi_1(Y, y_0)$

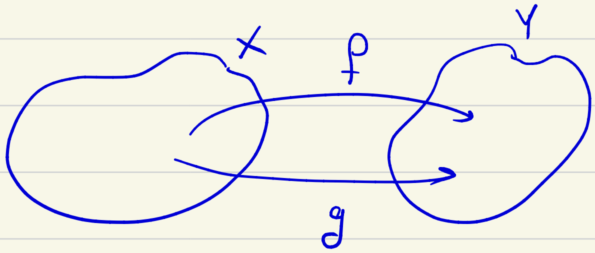
$[h^{-1}\alpha\beta h] = [h^{-1}\alpha h h^{-1}\beta h] = [h^{-1}\alpha h][h^{-1}\beta h]$

○ Def: $\alpha : S_1 \longrightarrow X$
$\beta : S_1 \longrightarrow X$



let X & Y be two topological spaces.

$f : X \longrightarrow Y$ & $g : X \to Y$



$f \sim g$ are homotopic if

$\exists\ H : X \times [0,1] \longrightarrow Y$

$H(x, 0) = f(x)$
$H(x, 1) = g(x)$
$H(x,s)$ is continuous on $X \times [0,1]$.

○ $X \simeq Y$ (homeomorphic) if $\exists\ f : X \longrightarrow Y \mid f^{-1}$ exists & f is continuous.

Exercise: show that homotopy is an equivalence relation.

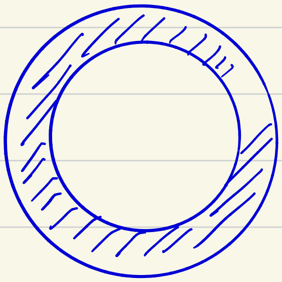
---

example of Homeomorphic spaces: $(0,1) \simeq R$ $\quad (0,1)\times(0,1) \simeq R^2$

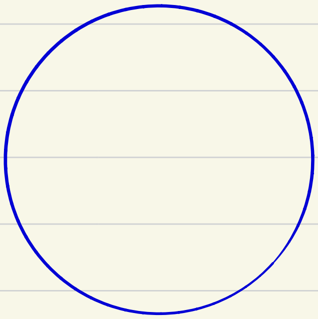
$S^2 \simeq$ elipsoid $\quad S^1 \simeq \square$

○ Def: Homotopy type < Homeomorphism.

تعریف: دو فضا از یک نوع هموتوپی خوانده می‌شوند اگر بتوان یکی را به طور پیوسته به دیگری تغییر شکل داد.



$S^1 \times (0,1) \xrightarrow{\text{Deform}}$



Def: X and Y are of the same homotopy type (denoted by $X \sim Y$)

اگر $\exists\ \underbrace{f \ \& \ g}_{\text{Continuous}}$: $f : X \longrightarrow Y$ ✓, $g : X \longleftarrow Y$ ✓

$f \circ g \neq id$ الزاماً

$g \circ f \neq id$

بلکه

$f \circ g \sim id_Y$

$g \circ f \sim id_X$

f, g نگاشت‌های هموتوپی هستند.

○ **Exercise:** نشان دهید که از یک نوع هموتوپی بودن یک رابطهٔ هم‌ارزی است.

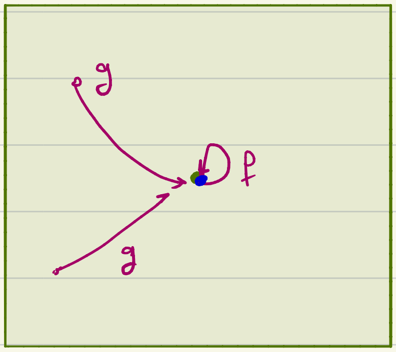
○ **Exercise:** اگر $X \simeq Y \longrightarrow X \sim Y$, $\quad X \simeq Y \;\not\longleftarrow\; X \sim Y$

○ **Example:** $R^2 \sim \{0;0\} = O$

اول: مسلماً R^2 و همریخت نیست.

$f : O \longrightarrow R^2 \qquad g : R^2 \longrightarrow O$

R^2



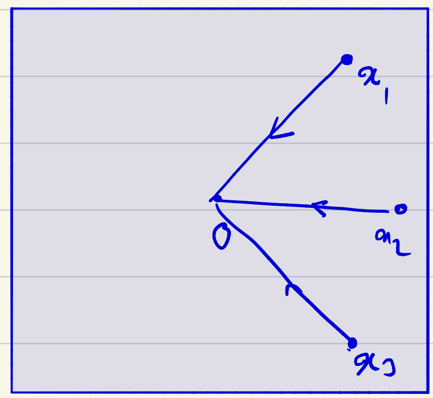
We want that: $g \circ f \simeq id_O$ $\quad f \circ g \simeq id_{\mathbb{R}^2}$.

$f(O) = 0 \in \mathbb{R}^2 \qquad g(\underset{\mathbb{R}^2}{x}) = O$

$g \circ f : O \longrightarrow O \qquad (g \circ f)(O) = O \qquad g \circ f = id_O$

$f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (f \circ g)(x) = 0 \qquad f \circ g \sim id_{\mathbb{R}^2}$

$H(\vec{x}, s) = s\vec{x}$



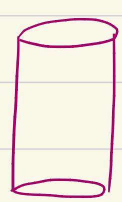
if $s = 0$ $\qquad H(\vec{x}, 0) = \bar{0} = (f \circ g)(x)$

if $s = 1$ $\qquad H(\vec{x}, 1) = \vec{x} = id_{\mathbb{R}^2}(\vec{x})$

$H(\bar{x}, s)$ is continuous in s & $\bar{x}$.

Final Result: O is not homeomorphic to $\mathbb{R}^2$ but O is of the same homotopy type as $\mathbb{R}^2$.

Exercise: $\mathbb{R} \sim O \quad \rightarrow \quad \mathbb{R}^n \sim O$

$S^1 \times \mathbb{R} =$  $\sim S^1 \sim$ Annulus.

Theorem without proof: (هم‌ارزی قضیهٔ نظریهٔ هوموتوپی)

$$\text{if } X \sim Y \longrightarrow \pi_1(X) = \pi_1(Y)$$

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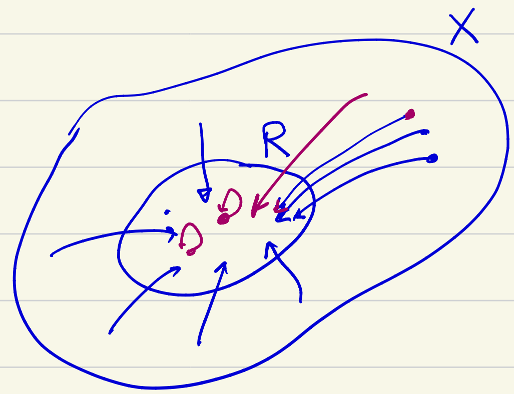
۶ فروردین ماه ۱۳۹۹، جلسه هفتم



Retract: $R \subset X$

اگر تابعی از $f: X \to R$ وجود داشته باشد

که

Retraction

① پیوسته باشد

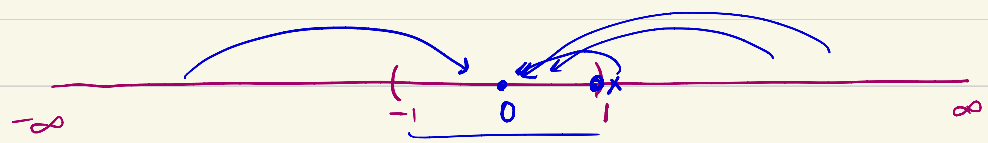
② نقاط داخل R را ثابت نگه دارد

$f|_R = id_R$

مثال:

$X = \mathbb{R}^1 = (-\infty, \infty)$.

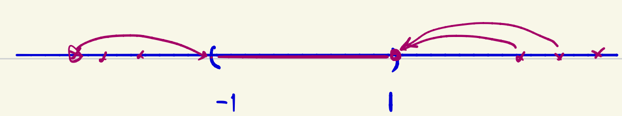
$R = (-1, 1)$

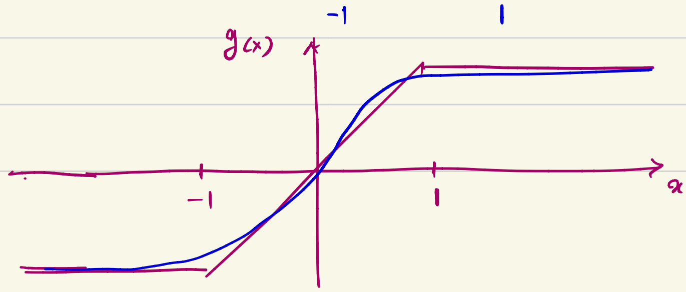


f:
$$f(x) = \begin{cases} x & x \in (-1,1) \\ 0 & x \notin (-1,1) \end{cases}$$
is not a retraction.

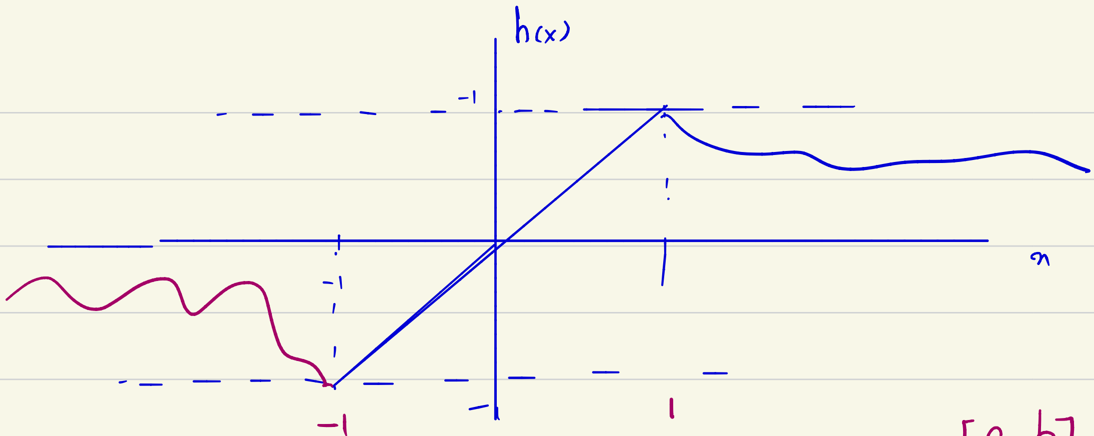
$(-1, 1)$ is a retract of $\mathbb{R}$.

$$g(x) = \begin{cases} x & x \in (-1,1) \\ 1 & \text{if } x > 1 \\ -1 & \text{if } x < -1 \end{cases}$$





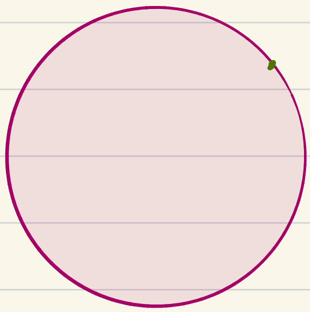
g is a retraction.



$[a, b] \subset R^1$ is a retract.

EXAMPLE:

$\cdot \vec{x}$



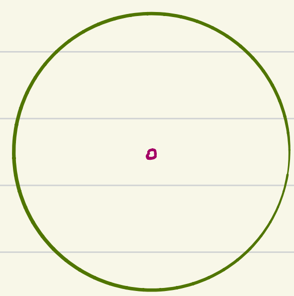
$D^2 = \{ x \in R^2 \mid |x| \leq 1 \}$

D^2 is a retract of R^2.

$$g(\vec{x}) = \begin{cases} \vec{x} & \text{if } |\vec{x}| \leq 1 \\ \dfrac{\vec{x}}{|\vec{x}|} & |\vec{x}| > 1 \end{cases}$$

EXAMPLE:

$S^1 = \{ \vec{x} \in R^2 \mid |\vec{x}| = 1 \}$



$\leftarrow S^1$

S^1 is a retract of $R^2 - \{\vec{0}\}$

$$h(\vec{x}) = \begin{cases} \vec{x} & |\vec{x}| = 1 \\ \dfrac{\vec{x}}{|x|} & |\vec{x}| \neq 1 \end{cases}$$

Deformation Retract.

0. $X \xrightarrow{f} R \subset X$
1. Conti.
2. $f|_R = id_R$.

Retraction

Def: R is a deformation retract of X if

اگر از نظر هموتوپی بتوان X را به R ، $X \supset R$ تغییر شکل داد، به طور پیوسته.

تعریف دقیق:

if $\exists \; H : X \times [0,1] \longrightarrow X$

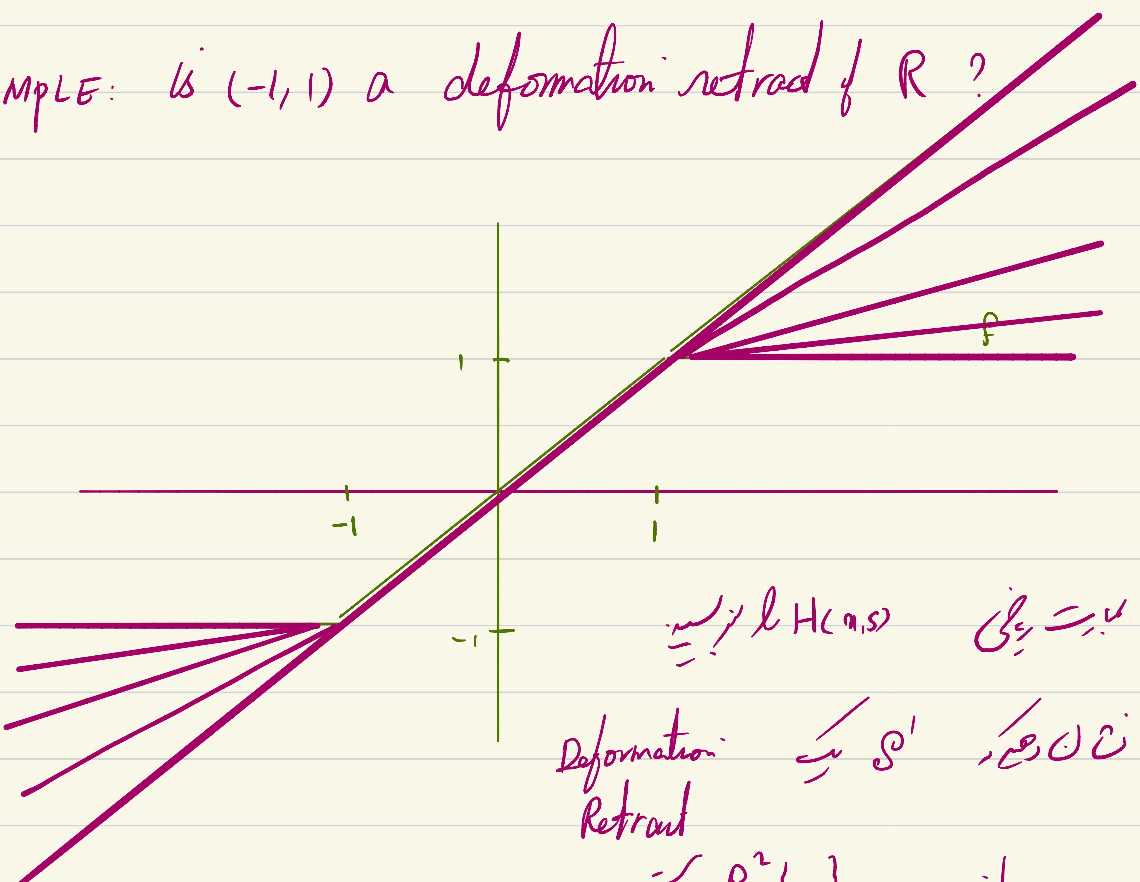
i) H is Continuous.

ii) $H(x,0) = x$

iii) $H(x,1) \in R$

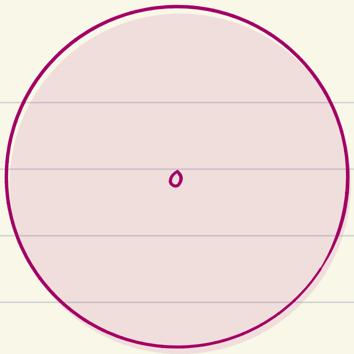
iv) $H(r,t) = r \quad \forall r \in R$

EXAMPLE: Is $(-1,1)$ a deformation retract of $\mathbb{R}$?



تمرین: عبارت دقیق $H(x,s)$ را بنویسید.

حل: نشان دهید که S^1 یک Deformation Retract از $R^2-\{0,0\}$ است.



$$f : \mathbb{R}^2 - \{(0,0)\} \longrightarrow S^1$$

$$f(\vec{x}) = \begin{cases} \vec{x} & |\vec{x}| = 1 \\ \dfrac{\vec{x}}{|\vec{x}|} & |\vec{x}| \neq 1 \end{cases}$$

$$H(\vec{x}, t) = (1-t)\vec{x} + t\, f(\vec{x}).$$ test.

i) Cont.

ii) $H(\vec{x}, 0) = \vec{x}$

iii) $H(\vec{x}, 1) = f(\vec{x})$.

iv) if $\vec{x} \in S^1$ ($|\vec{x}|=1$) $H(\vec{x}, t) = (1-t)\vec{x} + t\vec{x}$
$= \vec{x}$

تمرین: نشان دهید $S^2 \times (a,b)$ یک Deformation Retract از $\mathbb{R}^3 - \bar{0}$ است.

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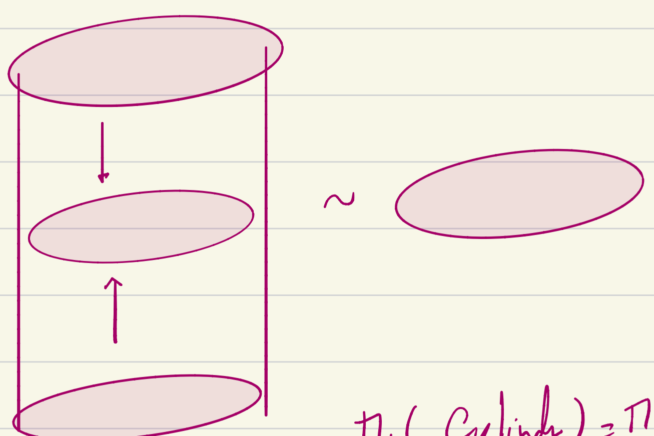
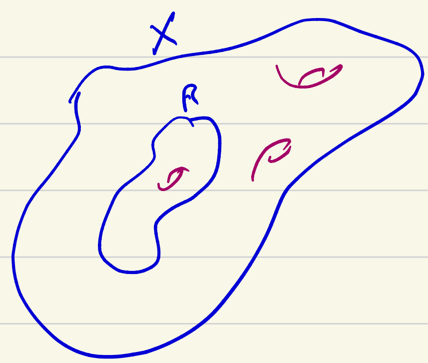
theorem: if R is a deformation retract of X, then $\pi_1(R) = \pi_1(X)$.



$\pi_1(\text{Cylinder}) = \pi_1(S^1)$.



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thm: if R is a deformation retract of X — $\pi_1(X) = \pi_1(R)$

○ G is a Group (discrete)

$\{e, a, a^2, \ \ a^{n-1}\} = G$ مثال ①:

$a^p \cdot a^q = a^{p+q}$ $\quad p+q \mod n.$

$a^n = e.$

$Z_n = \{0, 1, 2, \ \ n-1\}$ $\quad Z_n$ گروه

G is generated by a modulo a relation: $a^n = e.$

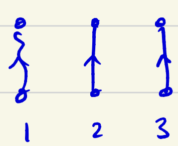
$$G = \{a^0, a^1, a^2, \ldots\ldots\}/_{a^n=e} = \langle a \rangle /_{a^n=e}.$$

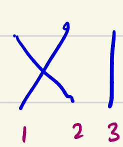
$$S_3 = \Big\{ \underset{1}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}, \underset{\sigma_1}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}, \underset{\sigma_2}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}},$$

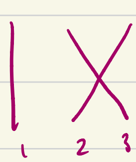
: S_3 گروه ② مثال

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} \Big\} \qquad |S_3| = 6$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow$ 

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$  $= \sigma_1$

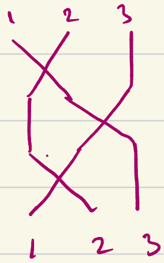
$\sigma_2 =$ 

$$S_3 = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1,$$

$$\underbrace{\sigma_1\sigma_2\sigma_1\sigma_2}_{\substack{\shortparallel \\ \sigma_2\sigma_1\sigma_2\sigma_2 = \sigma_2\sigma_1}}, \ \sigma_2\sigma_1\sigma_2\sigma_1, \ \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 \ldots\}.$$

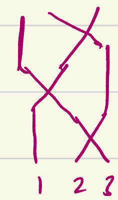
$$\sigma_1^2 = 1 \qquad \sigma_2^2 = 1 \qquad\qquad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$\sigma_1\sigma_2\sigma_1 =$



$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\sigma_2\sigma_1\sigma_2 =$



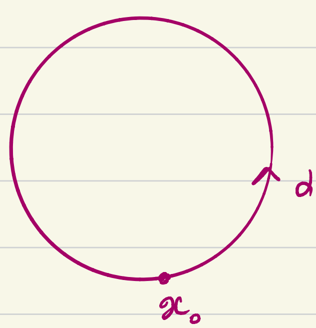
$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$$S_3 = \langle \sigma_1, \sigma_2 \rangle \Big/ \{\sigma_i^2 = I,\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}. \Rightarrow$$

$$G = \langle \text{generators} \rangle \big/ \text{relations}.$$

$$S_n = \langle \sigma_1, \sigma_2, \dots \sigma_{n-1} \rangle \Big/ \{\sigma_i^2 = 1,\ \sigma_i \cdot \sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$$

$\pi_1(S^1) = \mathbb{Z}$



$\pi_1(S^1) = \{1, \alpha, \alpha^2, - \alpha^n\}$

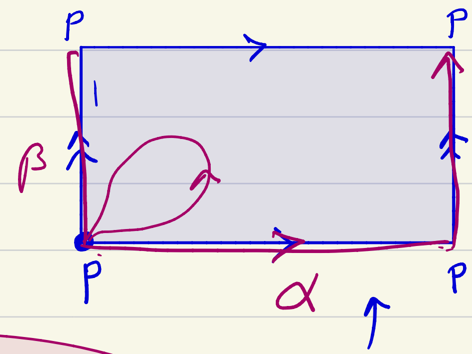
$n \in \mathbb{Z}$

$\pi_1(\text{Cylinder}) = \mathbb{Z}$

$\pi_1(S^1 \times [0,1]) = \mathbb{Z}$ $\qquad \pi_1(S^2) = \{e\}$ $\qquad \pi_1(S^n) = \{e\}.$

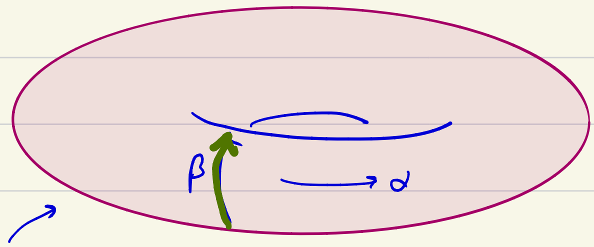
$\pi_1(\mathbb{R}^n) = \{e\}.$

$\pi_1(\Sigma_1) = \pi_1($ Torus with genus one$)$.



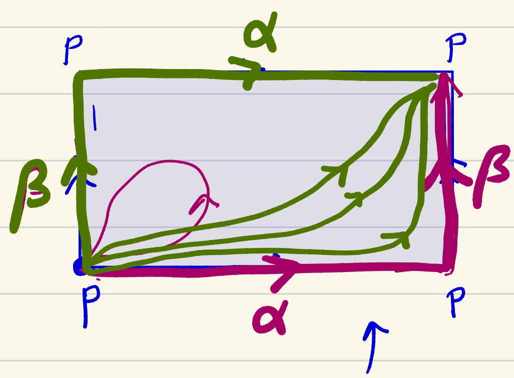
$$\pi_1(\Sigma_1) = \{ e, [\alpha], [\beta], \dots \}$$

α and β are generators of $\pi_1(\Sigma_1)$



$$\alpha\beta \sim \beta\alpha \longrightarrow [\alpha][\beta] = [\beta][\alpha]$$

[illegible]



$$\pi_1(\Sigma_1) = \langle [\alpha], [\beta] \rangle \Big/ \{[\alpha][\beta] = [\beta][\alpha]\}$$

$\rightarrow$ $[\alpha] \ni \alpha$

$[\beta] \ni \beta$.

[illegible]

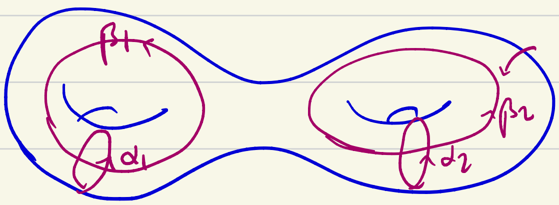
$$\pi_1(\Sigma_1) = \{ e, \alpha\beta, \alpha\beta\alpha = \alpha^2\beta, \dots \}$$

$$= \{ \alpha^m \beta^n \quad m, n \in \mathbb{Z} \}.$$

$$(\alpha^m\beta^n)(\alpha^{m'}\beta^{n'}) = \alpha^{m+m'}\beta^{n+n'} \Longrightarrow$$

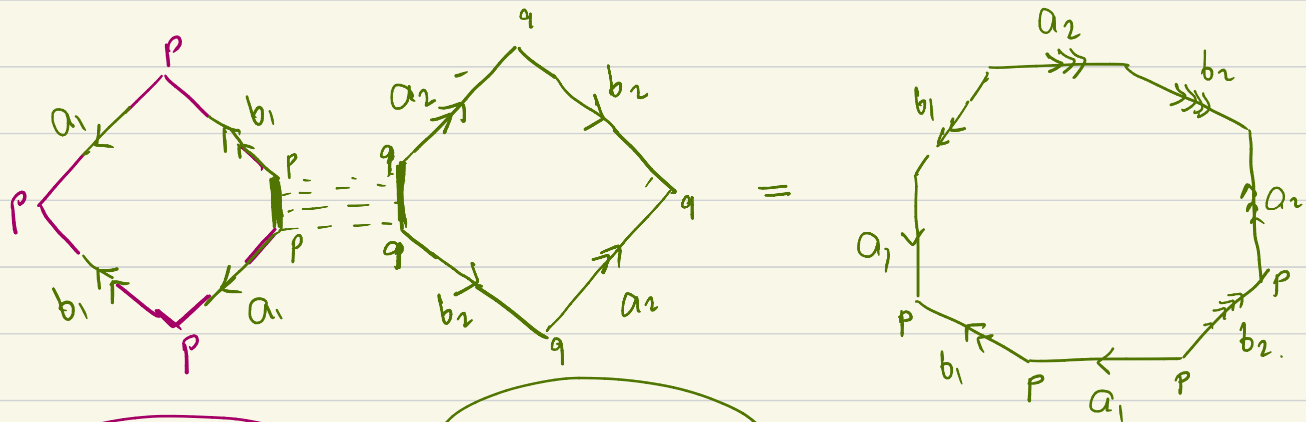
$$\pi_1(\Sigma_1) = \{ (m,n) \mid (m,n) + (m',n') = (m+m', n+n') \}$$

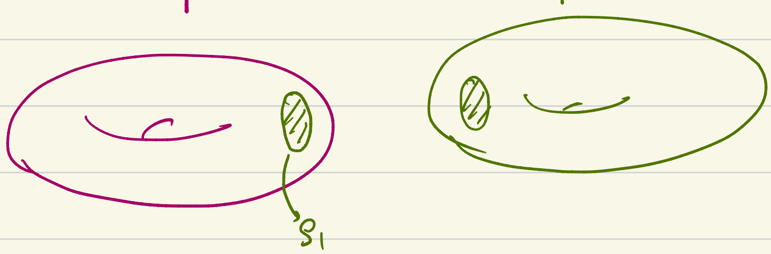
$$\Pi_1(\Sigma_1) = \mathbb{Z} \oplus \mathbb{Z}$$

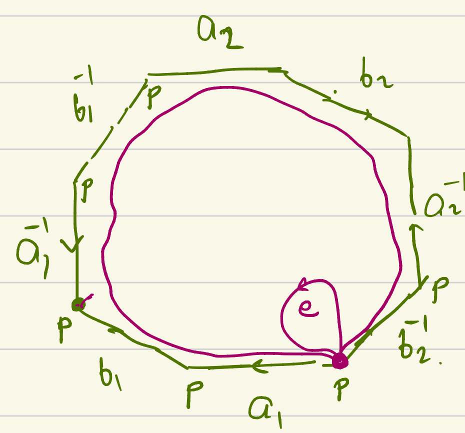


cycles.

$$\Pi_1(\Sigma_2) = ?$$







a_1, b_1, a_2, b_2

$$\Pi(\Sigma_2) = \text{free group generated by } a_1, b_1, a_2, b_2$$

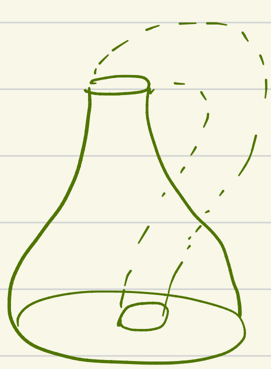
$$= \left\{ a_1^{n_1} b_1^{m_2} a_2^{m_3} b_2^{m_4} a_1^{m_1} b_1^{m_2} a_2^{m_3} b_2^{m_4} \cdots \right\} \Big/ a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = e.$$

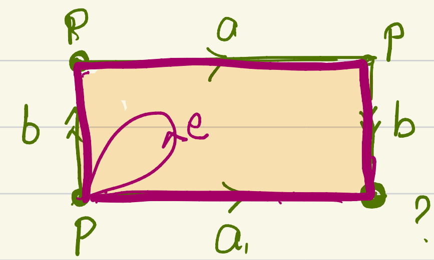
$$\Pi(\Sigma_g) = \langle a_i, b_i; \quad i = 1 \ldots g \rangle \Big/ \prod_{i=1}^{g} a_i \cdot b_i \cdot a_i^{-1} b_i^{-1} = e$$

 $=$ 12-gon $\qquad \prod_{i=1}^{3} a_i b_i a_i^{-1} b_i^{-1} = I$

$\Sigma_1 : \quad a_1 b_1 a_1^{-1} b_1^{-1} = 1 \longrightarrow a_1 b_1 = b_1 a_1 \longrightarrow \pi_1(\Sigma_1) = \mathbb{Z} \oplus \mathbb{Z}$

Klien Bottle:





$$\pi_1(K) = \langle a, b \rangle / ab^{-1}a^{-1}b^{-1} = e \quad (1)$$

$$\pi_1(K) = \langle a, b \rangle / \underline{ab^{-1}a^{-1} = b}$$

$$\pi_1(K) = \langle a, b \rangle / aba^{-1} = b^{-1}$$

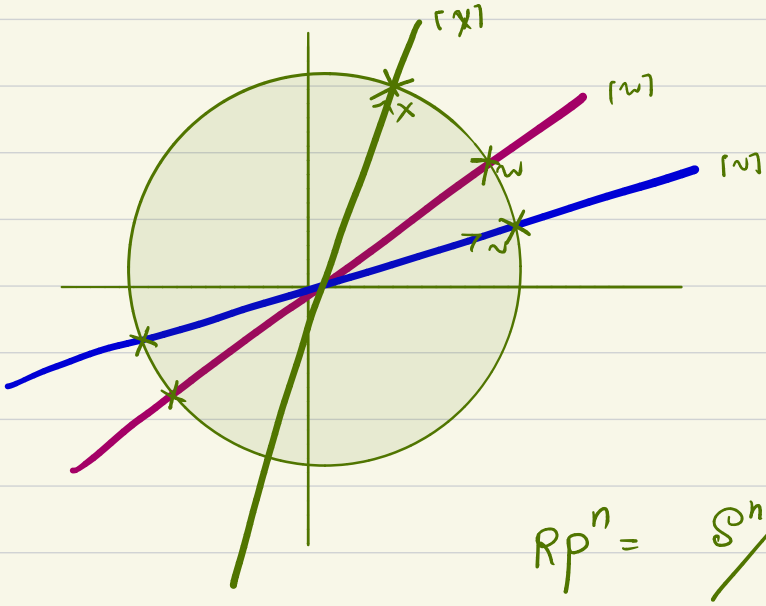
$ba b^{-1} = a \rightarrow ba = ab$ $\qquad ab^{-1} = ba \rightarrow a = bab$

تمرین: آیا می‌توانید نحوهٔ فشرده‌تری برای نوشتن الحاقی $\pi_1(K)$ پیدا کنید؟

مثال بعدی: $\mathbb{R}P^2$

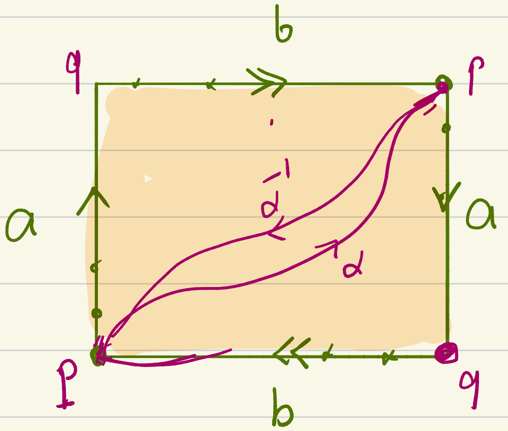
$\mathbb{R}P^2$ = Real projetive plane. $\quad \mathbb{R}^2 - \{0\}$.

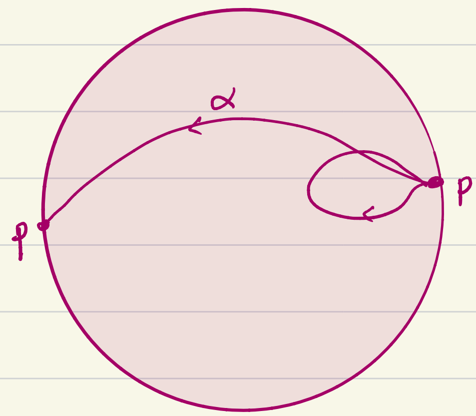
$\vec{v} \sim \vec{v}'$ if $\vec{v}' = \lambda \vec{v}$ $\rightarrow$ $\mathbb{R}P^2 = \{ [v] \mid \vec{v} \in \mathbb{R}^2, \vec{v} \neq 0 \}$



$\mathbb{R}P^2 = S^2 /$ opposite points are identified.

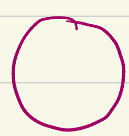
$\mathbb{R}P^n = S^n /$ opposit points are identified.





$\pi_1(\mathbb{R}P^2) = \langle \alpha \rangle / \alpha \sim \alpha^{-1}$ $\qquad \alpha^2 = 1$ $\qquad \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 = \{e, \alpha\}$

$\alpha^2 = e.$

1-D: every compact surface without boundary $\cong S^1$ 

2-D: $\begin{cases} \text{orientable} \rightarrow \Sigma_g \\ \text{non-orientable} \rightarrow (K \cong \mathbb{R}P_2) \# \Sigma_g. \end{cases}$

تمرین:

a) Find $\pi_1(\mathbb{RP}_2 \# \Sigma_1)$

b) Find $\pi_1(K \# \Sigma_1)$.

شنبه: ۱۰ فروردین ماه

$$\pi_1(SU(2)) = ?$$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad g \in U(2) \rightarrow gg^\dagger = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{cases} |a|^2 + |b|^2 = 1 \\ ac^* + bd^* = 0 \\ |c|^2 + |d|^2 = 1 \end{cases} \quad (1)$$

$$\rightarrow g \in SU(2) \rightarrow ad - bc = 1 \quad (2)$$

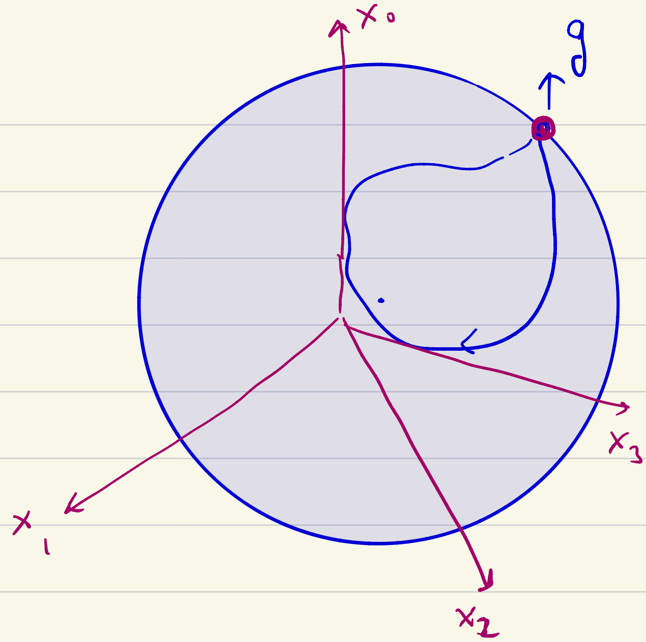
$$(1), (2) \rightarrow g = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \qquad |a|^2 + |b|^2 = 1 \quad (3)$$

$$a = x_0 + i x_1 \qquad b = x_2 + i x_3. \qquad g = \begin{bmatrix} x_0 + i x_1 & x_1 + i x_2 \\ -x_1 + i x_2 & x_0 - i x_1 \end{bmatrix}$$

$$(3) \rightarrow x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

$$x_0, x_1, x_2, x_3 \in \mathbb{R} \qquad SU(2) = \{(x_0, x_1, x_2, x_3) = \vec{x} \in \mathbb{R}^4\}$$

$$\boxed{SU(2) \cong S^3}$$ پوسته کره سه بعدی



$$\Pi_1(SU(2)) = \Pi_1(S^3) = \{e\}$$

$$\Pi_1(SO(2)) = ?$$

$$g \in SO(2) \qquad g = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad g \longleftrightarrow \theta \qquad 0 \le \theta \le 2\pi$$

$$g(\theta=0) = g(\theta=2\pi)$$

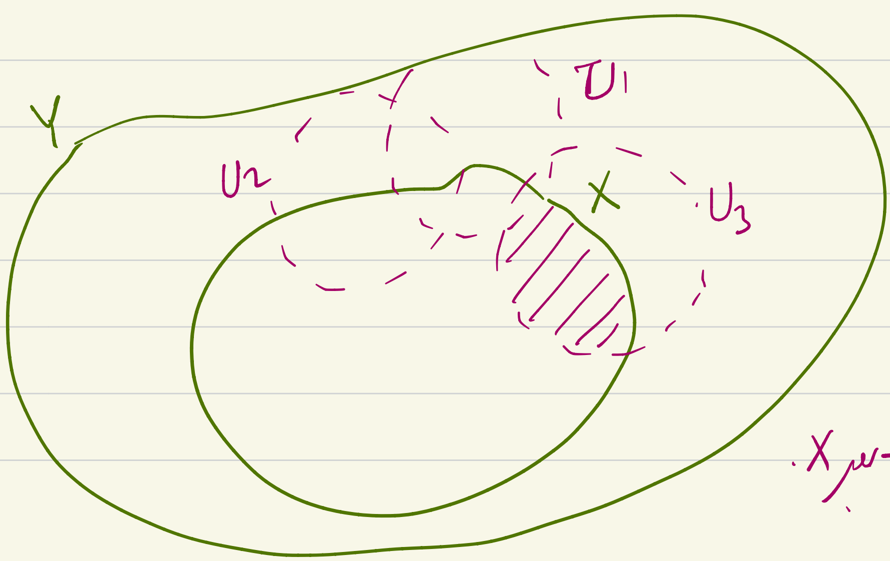
$$SO(2) = S^1 \qquad \Pi_1(SO(2)) = \mathbb{Z}$$

$$\Pi_1(SO(3)) = ? \qquad R \in SO(3) \qquad R = R_{\hat{n}}(\theta)$$

$$R_{\hat{n}}(\theta) = e^{i\hat{n}\cdot\vec{J}(\theta)}.$$

$$X \in (Y, \tau)$$

توپولوژی



$$\tau_X = \{ V_i \mid V_i = U_i \cap X ; \; U_i \in \tau_Y \}$$

تمرین: نشان دهید τ_X یک توپولوژی است بر X.

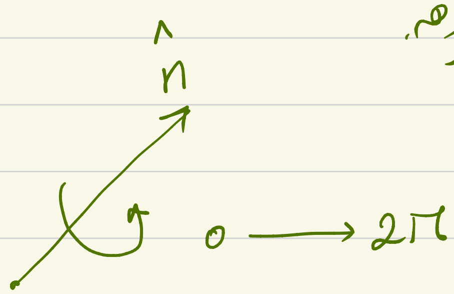
$$R_{\hat n}(\theta) \in SO(3) \longrightarrow (\hat n, \theta) \equiv \hat n \theta = \vec n$$

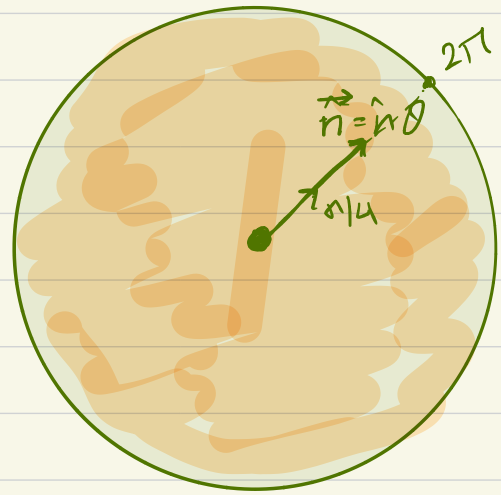
یکتا نیست.

جهت $\vec n$ جهت دوران را نشان می‌دهد. و

اندازه $\vec n$ مقدار دوران را نشان می‌دهد.

اندازه $\hat n$ از صفر است تا

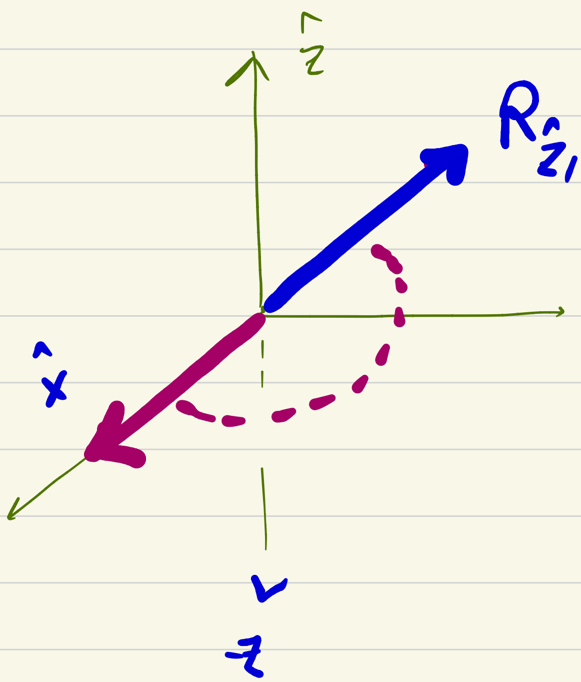




دوران حول $\hat n$ به اندازه π.

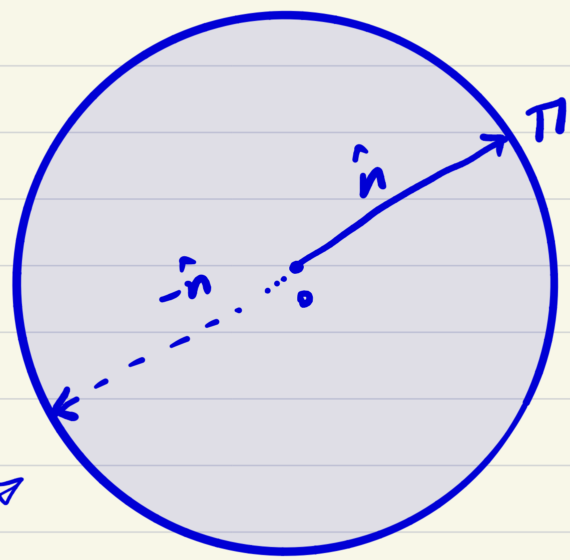
=

دوران حول $-\hat n$ به اندازه $-\pi$



$$R_{\hat z}(\pi)\,\hat x = R_{-\hat z}(\pi)\,\hat x$$

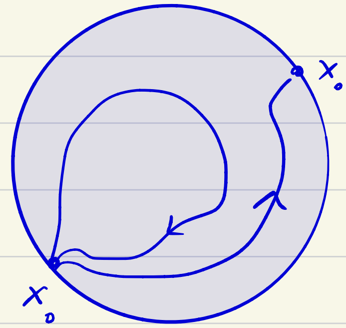
$$R_{\hat n}(\pi+\theta) = R_{-\hat n}(\pi-\theta)$$



$\Longrightarrow$ $R_{\hat n}(\pi) = R_{-\hat n}(\pi) \longrightarrow$ نقاط روبروی مرز با هم یکی‌اند

از نظر توپولوژی $SO(3) =$ گوی توپر که نقاط سرمرزی به شعاع π با هم یکی هستند.

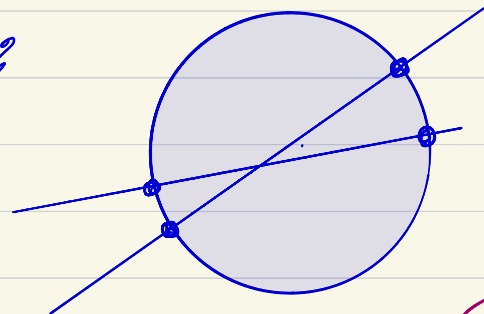
$$\pi_1(SO(3)) = \mathbb{Z}_2.$$



$RP(2) = R^3/\sim \qquad v = \lambda v$

لعبه

$RP(2) =$ مجموعه امتدادها در فضای R^3



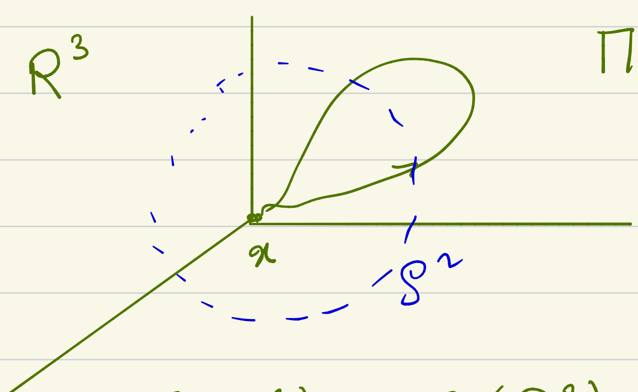
$RP(2) =$ سطح کره‌ای که دوسوی نقطه‌ها
در قطرش یکی شوند

$$\boxed{RP(n) = R^{n+1}/\sim .}$$

رابطه‌ی درست
تعریف و تا اینجا درست

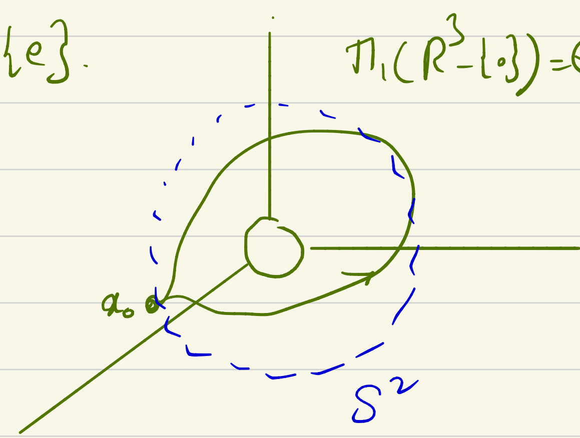
Higher Homotopy Groups π_n Motivation:

R^3



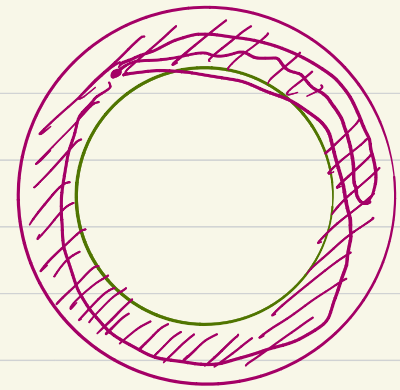
$\pi_1(R^3) = \{e\}.$

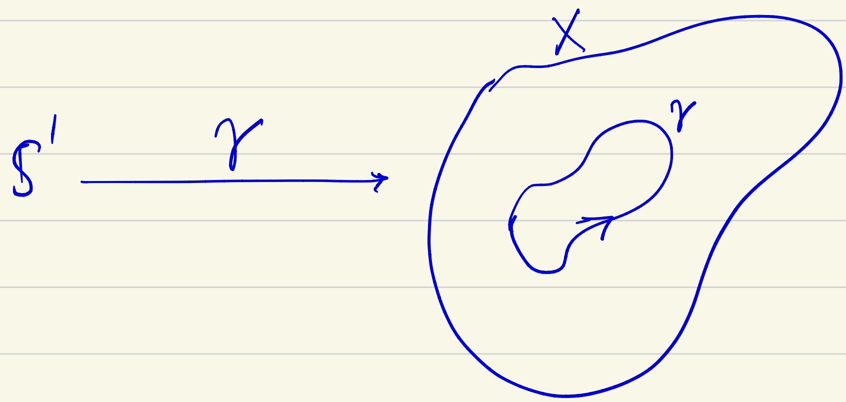
$\pi_1(R^3 - \{0\}) = e$

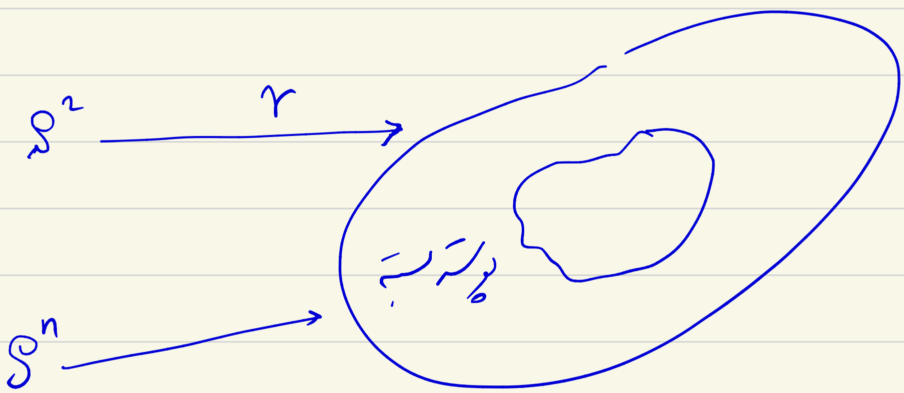


$$\pi_1(R^3) = \pi_1(S^2) = \{e\}.$$

$\pi_1(S^2 \times [0,1]) = \{e\}$







یکی: $\gamma : S_1 \longrightarrow X$

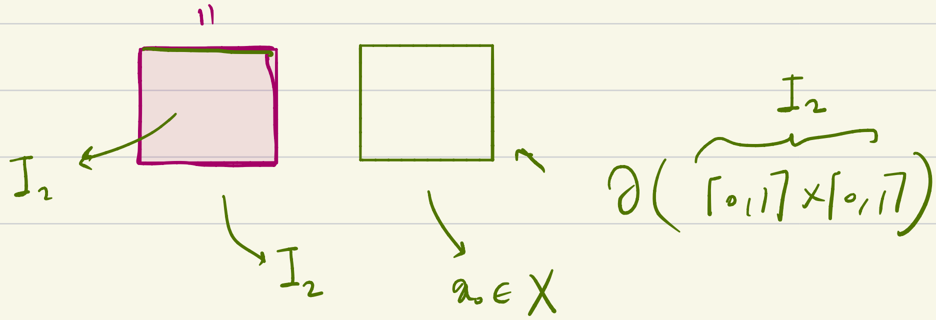
$\downarrow$

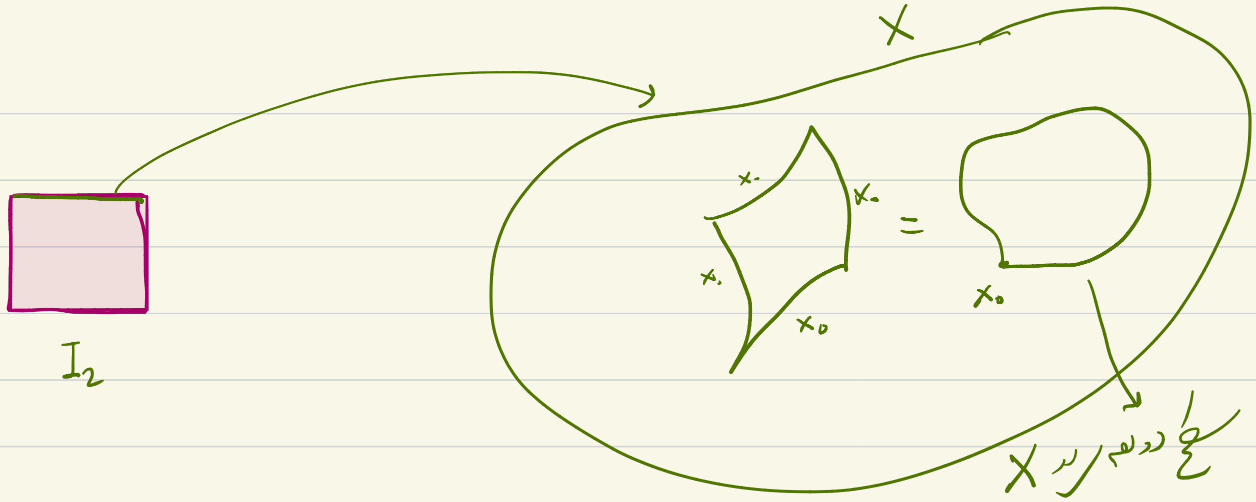
$\gamma : \underset{\| \, I_1}{[0,1]} \longrightarrow X \quad | \quad \gamma(0) = \gamma(1) = x_0$

یکی: $\gamma : S^2 \longrightarrow X$

$\gamma : \overbrace{[0,1] \times [0,1]}^{I_2} \longrightarrow X \qquad \gamma(\partial([0,1] \times [0,1])) = x_0$

$\|$





$\alpha \sim \beta$ if $\exists\ H(S^2 \times [0,1]) \longrightarrow X$

$$\begin{cases} H(\vec{s}\ \ 0) = \alpha(x) \\ H(\vec{s}, 1) = \beta(x) & \text{و} \\ H(\vec{s} \in \partial S^2, t) = x_0 & \forall\ x \end{cases}$$

$I_n = \underbrace{[0,1] \times [0,1] \times \cdots [0,1]}_{n \text{ times}}$ = مکعب n بعدی است

∂I_n = مرز I_n $\qquad \alpha$ = n-dimensional loop

$\Downarrow$

$\alpha : I_n \longrightarrow X \quad | \quad \alpha(\partial I_n) = x_0 \in X.$

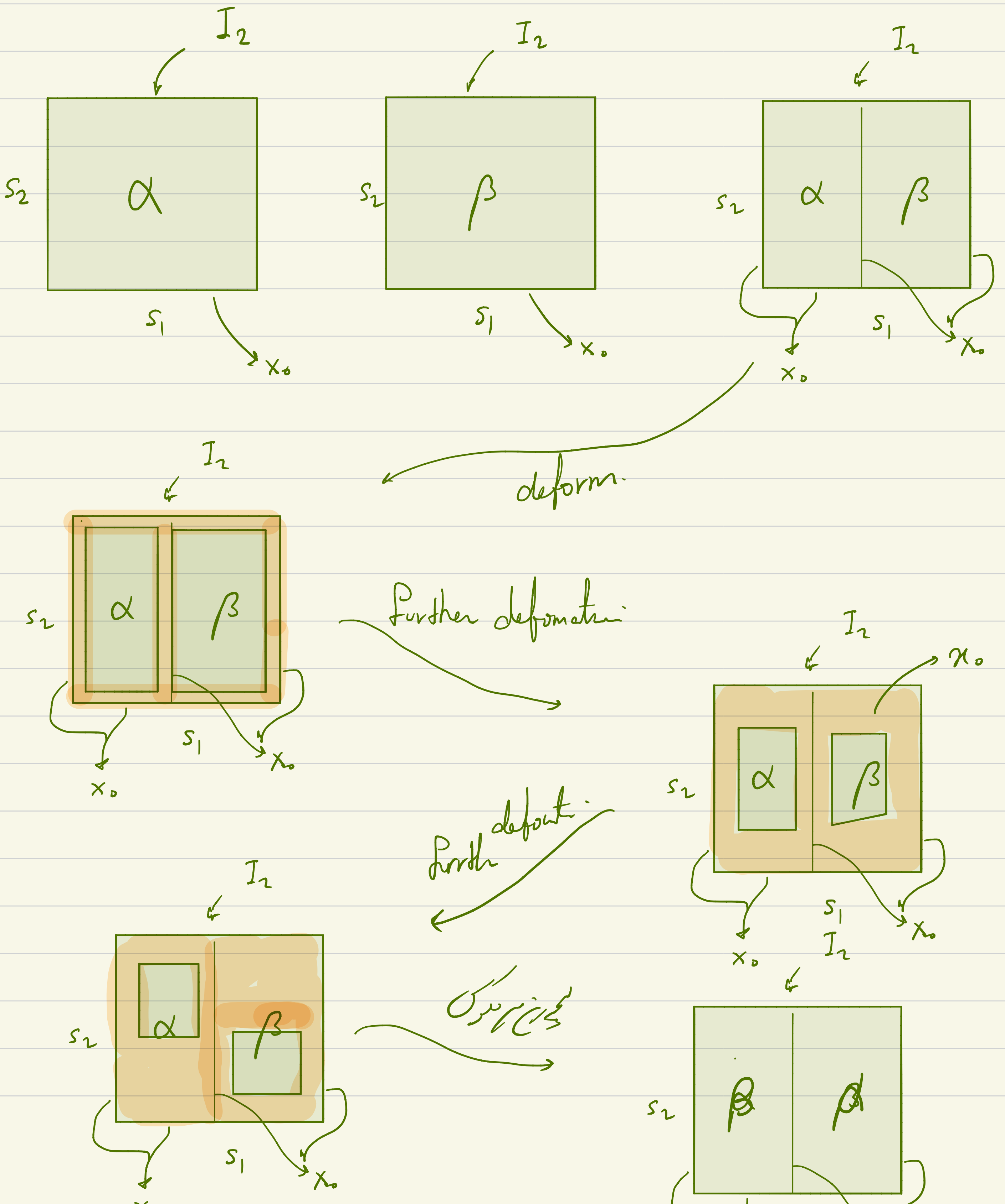
تعریف ضرب: $\quad (\alpha\beta)(s_1, s_2, \ s_n) := \begin{cases} \alpha(2s_1, s_2, \cdots s_N) & 0 \le s_1 \le 1/2 \\ \beta(2s_1 - 1, s_2, \cdots s_N) & 1/2 \le s_1 \le 1 \end{cases}$

$(\alpha^{-1})(s_1, s_2 \quad s_n) := \alpha(1 - s_1, s_2, \quad s_N)$

$\Pi_n(X) = \{ \ [\alpha] \ \} \qquad [\alpha][\beta] := [\alpha\beta] \quad [\alpha]^{-1} := [\alpha^{-1}]$

نشان دهید $\Pi_n(X)$ درواقع یک گروه آبلی است

thm: $\Pi_n(X)$ is an Abelian group $\forall X$, $n>1$.



$\pi_n(X)$ is Abelian $\forall\, n>1$ & $\forall\, X$.

سه شنبه ۹۹,۱,۱۲

قبلاً گفته بودیم که:

$$\mathbb{R}P_n = \mathbb{R}^n/\sim \qquad v \simeq \lambda v$$

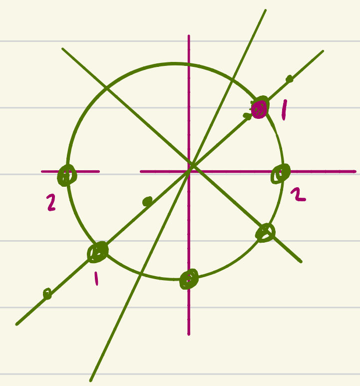
تعریف درست و متداول

$$\boxed{\mathbb{R}P_n = \mathbb{R}^{n+1}/\sim} \qquad v \simeq \lambda v$$

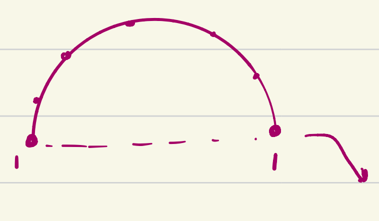
بعد فضا

مثال:

$$\mathbb{R}P_1 = \mathbb{R}^2/\sim = \text{فضای امتدادها در } \mathbb{R}^2$$

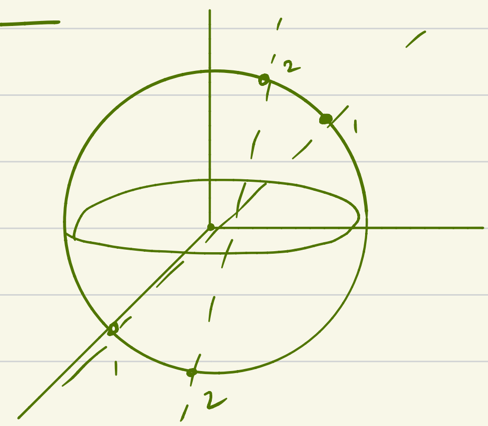


$$\mathbb{R}P_1 \equiv S^1 \longrightarrow$$ نیم دایره



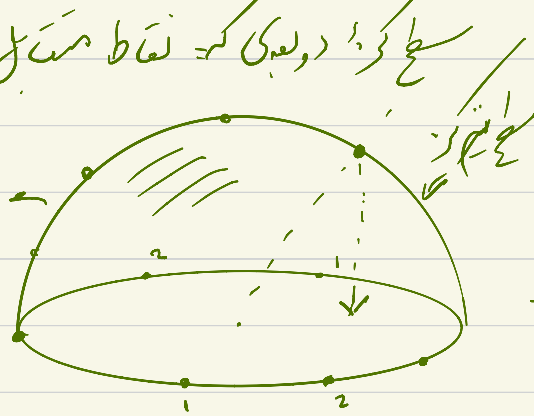
$$\boxed{\mathbb{R}P_1 = S^1}$$

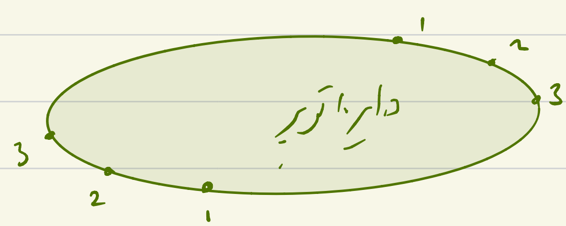
$$\mathbb{R}P_2 = \mathbb{R}^3/\sim$$



سطح کره، دولبه‌ای که نقاط متقابل آن با هم یکی هستند

$$\mathbb{R}P_2 = S^2/\sim$$





$$\mathbb{R}P_2 = D^2/\sim$$ نقاط متقاطر روی مرز یکی هستند

$RP_n = S^n$ پوسته کروی / نقاط متقاطر یکی گرفته‌اند $=$ D^n / نقاط مرزی یکی گرفته‌اند

$$\parallel$$

$$R^{n+1} / \sim = \{v = \lambda v\}$$

RP_n = real projective space of dimension n.

$$CP^n = C^{n+1} / \sim \qquad \{v \equiv \lambda v\}$$

یک تار

قبلاً دیدیم که $\Pi_n(X)$ is abelian.

$n > 1$

یک گروه آبلی $= \mathbb{Z} = \{-2, -1, 0, 1, 2, \ldots\}$ $\qquad |\mathbb{Z}| = \infty$

$\mathbb{Z}$ is generated by one single element $= \{-1-1, -1, 0, 1, 1+1, 1+1+1, \ldots\}$.

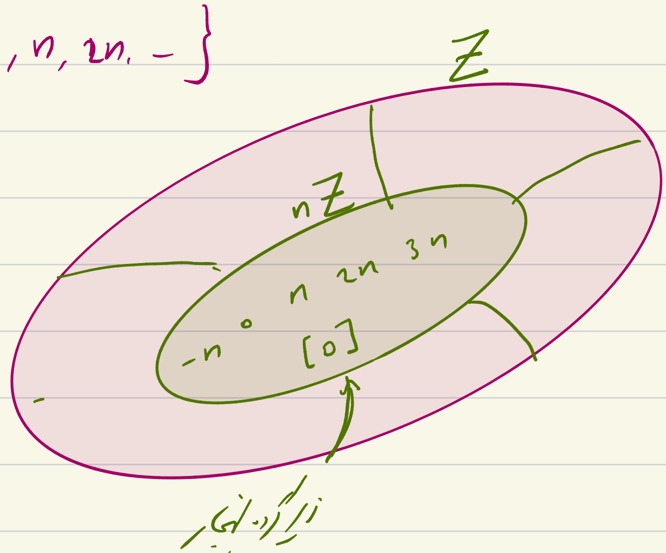
$\mathbb{Z}$ زیرگروه در $= n\mathbb{Z} = \{-2n, -n, 0, n, 2n, \ldots\}$

مضرب n

$x, y \in \mathbb{Z}$ $\qquad x \sim y$ if

$x - y \in n\mathbb{Z}$

معادل یک رابطه‌ی هم‌ارزی



$[0] = n\mathbb{Z}$

$[1] = \{ \dots -n+1, \; 1, \; n+1, \; 2n+1, \; 3n+1 \dots \} = n\mathbb{Z}+1$ $\qquad [n] \equiv [0]$

$[2] = \{ \quad -n+2, \; 2, \; n+2, \dots \quad \} = n\mathbb{Z}+2$

$[n-1] \qquad\qquad = n\mathbb{Z}+(n-1)$

$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{$Coset های مجزا$\} = \{ [0], [1], [2], \quad [n-1] \}$

$|\mathbb{Z}/n\mathbb{Z}| = n \qquad [g] + [g'] = [g+g']$

$[n-1] + [1] = [n-1+1] = [n] = [0]$

$\mathbb{Z}_n = \{0, 1, 2, \quad n-1\}$

n مرتبه

$\mathbb{Z}_n = \langle 1 \rangle = \{ \quad 1, \; 1+1, \; 1+1+1+ \dots \}$

$\underbrace{1+1+1+1}_{n} = 0$

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$G = \mathbb{Z}_n \oplus \mathbb{Z}_m = \{ (x, y) \mid x \in \mathbb{Z}_n, \; y \in \mathbb{Z}_m \}.$

$$\begin{cases} (x,y) + (x',y') := (x+x', y+y') \\ (x,y)^{-1} := (-x, -y). \qquad -x \equiv n-x \qquad -y \equiv n-y \end{cases}$$

ضرب های اکسترنال

$$G \times G' = \{ (g, g') \mid g \in G,\ g' \in G' \}$$ Pairwise

$$\mathbb{Z}_n \oplus \mathbb{Z}_m = \langle (1,0), (0,1) \rangle = \langle a, b \rangle / \{a^n = e \quad b^m = e\}$$

Finitely generated Abelian group = یک گروه آبلی که تعداد متناهی مولد داشته باشد.

thm: قضیه اصلی گروه‌های آبلی : Any FGA group is isomorphic to:

$$\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \mathbb{Z}_{p_k^{n_k}}.$$

$\underbrace{\text{اعداد اول}}$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{11}$$

$$\mathbb{Z}_6 = \{0,1,2,3,4,5\} \equiv \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

	0	1	2	5
0				
1				
2				
5				

$\mathbb{Z}_2 = \{0,1\} \qquad \mathbb{Z}_3 = \{0,1,2\}.$

تمرین: ایزومورفیسم بین $\mathbb{Z}_6$, $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ نشان دهید.

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \}$$

Pairwise operation.

$\mathbb{Z}_{10} = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \qquad \mathbb{Z}_{12} = \mathbb{Z}_3 \oplus \mathbb{Z}_4 \qquad \mathbb{Z}_4 \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$

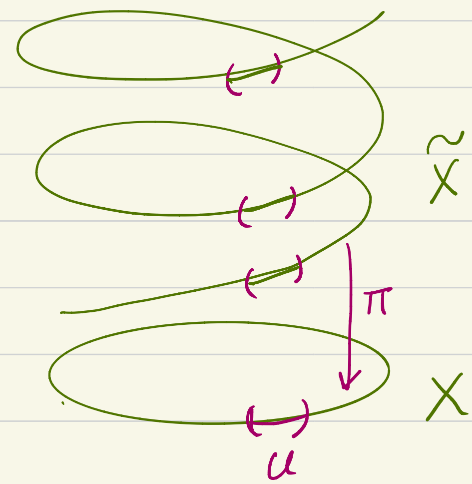
power of prime.

قضیه اصلی

theorem: Any Abelian Group $= \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \mathbb{Z}_{p_k^{n_k}} \oplus \underbrace{\mathbb{Z} \oplus \cdots \mathbb{Z}}_{m}$

نکته اساسی: هیچ راه و روش مستقیم و راحتی برای گروه‌های تورشن وجود ندارد و هر کدام را باید با یک ابزار و یک کلک و یک خصلت ویژه‌ای پیدا کرد.

Def: Covering Space.



$\pi : \tilde{X} \longrightarrow X$ پیوسته است ①

②

let $U \subset X$

یک مجموعه باز در X

$\pi^{-1}(U) =$ A union of disjoint sets in $\tilde{X}$.

$\tilde{X}$ is a Covering space of X.

if $\tilde{X}$ is path connected $\rightarrow$ $\tilde{X}$ is a universal covering space of X.

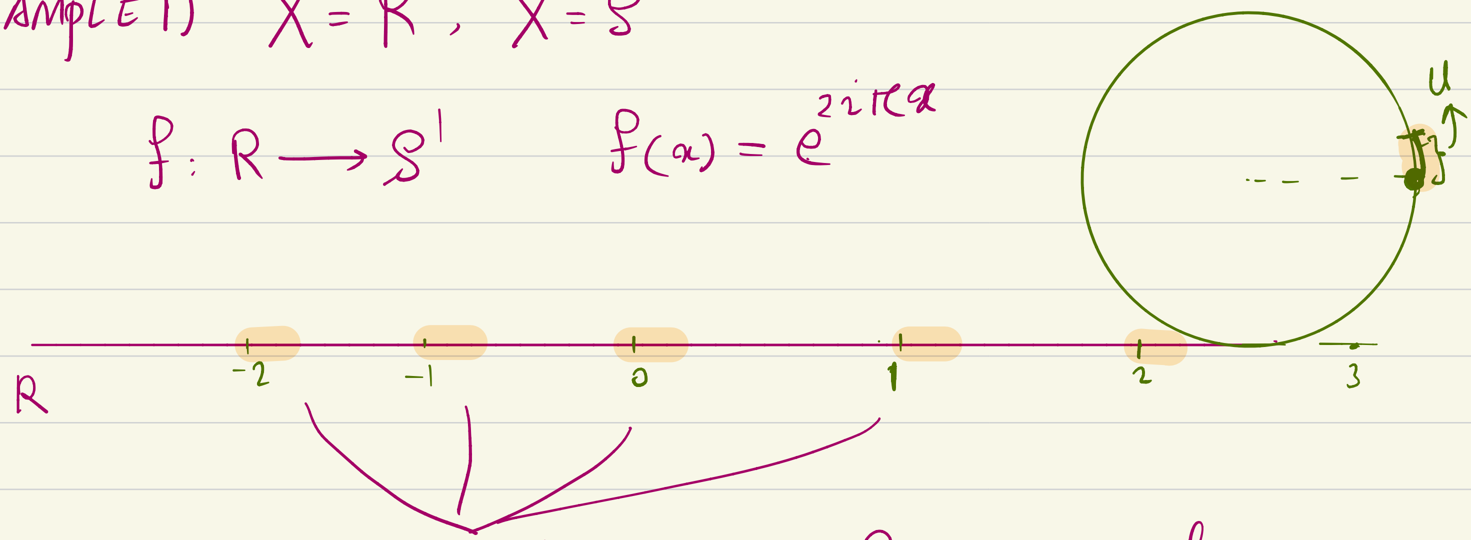
if X & $\tilde{X}$ are groups.

$$\begin{array}{cc} X & \tilde{X} \\ \downarrow & \downarrow \\ G & \tilde{G} \end{array}$$

$\tilde{G}$ is a Universal covering group of G

EXAMPLE 1) $\tilde{X} = R$, $X = S^1$

$f: R \longrightarrow S^1$ $\qquad f(x) = e^{2\pi i x}$



R is a universal covering space of S^1.

EXAMPLE 2) R^2 is a universal cover of $S^1 \times S^1 = T^2$

R^n " " $S^1 \times S^1 \times \ldots S^1 = T^n$

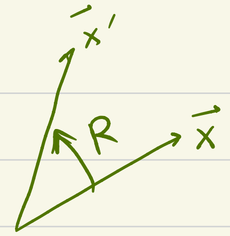
$$f: (x_1, x_2, \ldots x_n) \longrightarrow (e^{2\pi i x_1}, e^{2\pi i x_2}, \ldots e^{2\pi i x_n}).$$

EXAMPLE 3): $SU(2)$ $\qquad SO(3)$

بین $SO(3)$, $SU(2)$ یک نگاشت دو به یک هست.

if $\underbrace{R \in SO(3)}$ & $\underbrace{\vec{x} \in R^3}$ $\quad \underbrace{\vec{x} \longrightarrow \vec{x}' = R\vec{x}}$ $\longrightarrow$ دوران

$|\vec{x}'| = |\vec{x}|$



$$x \xrightarrow{R} x' \xrightarrow{R'} x'' \qquad (R'R)$$

$\vec{X} \longleftrightarrow P$

بردار ماتریس

$$P = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix} = \vec{x} \cdot \vec{\sigma} \qquad P = \begin{cases} 2\times 2 \\ P = P^\dagger \\ \operatorname{tr} P = 0 \end{cases}$$

$\det P = -|x|^2$

let $g \in SU(2)$ $\qquad P \longrightarrow P' = g P g^\dagger \quad \leadsto \quad \begin{cases} P' \quad 2\times 2 \\ P' = P'^\dagger \\ \operatorname{tr} P' = 0 \end{cases}$

$\det P' = \det P$

$|x'| = |x|$

$$P' = \begin{pmatrix} x_3' & x_1' - i x_2' \\ x_1' + i x_2' & -x_3' \end{pmatrix} \qquad \det P' = -|x'|^2$$

سوال: آیا هر دوران، $R \in SO(3)$ ایجاد میکنیم؟

با یک $g \in SU(2)$

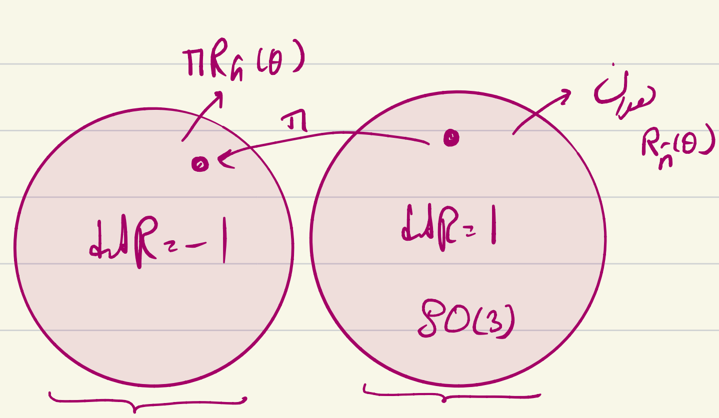
انعکاس: $\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \overset{\pi_{\parallel}}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \det R = -1$

$$\vec{x}' \cdot \sigma = g\,(x \cdot \sigma)\, g^\dagger$$

$$x_i' \sigma_i = g\,(x_j \sigma_j)\, g^\dagger$$

$$2\delta_{ij} = \operatorname{Tr}(\sigma_i \sigma_j)$$

$O(3) =$ دوران $= \{R \mid R^T R = I\}$



$$2x'_i \sigma_i \sigma_k = (g\sigma_j g^\dagger \sigma_k)\, x_j \longrightarrow \mathrm{Tr}$$

$$2x'_i \delta_{ik} = \mathrm{Tr}(g\sigma_j g^\dagger \sigma_k)\, x_j$$

$$\underset{x'_k}{\overset{\shortparallel}{}}$$

$$x'_k = \frac{1}{2}\mathrm{Tr}(g\sigma_j g^\dagger \sigma_k)\, x_j$$

$$x'_k = \frac{1}{2}\underbrace{\mathrm{Tr}(\sigma_k g \sigma_j g^\dagger)}_{R_{kj}}\, x_j$$

$$SU(2) \longrightarrow SO(3)$$

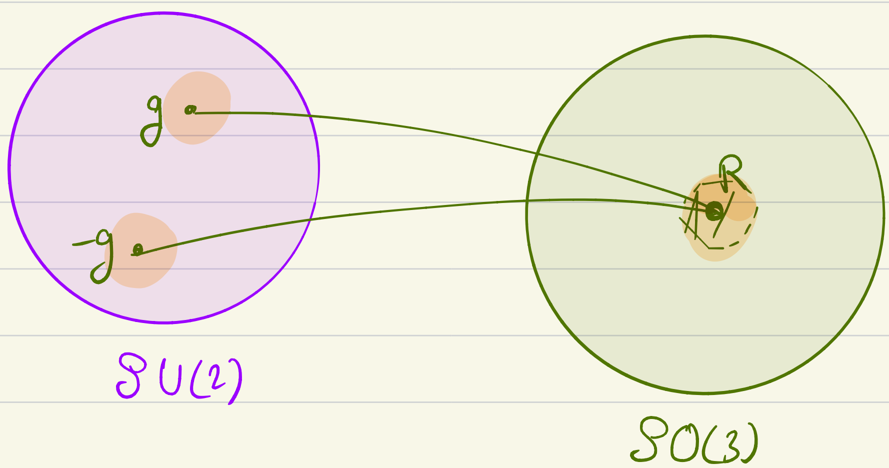
$$g \longrightarrow R$$

$$R_{jk} = \frac{1}{2}\mathrm{Tr}(\sigma_j g \sigma_k g^\dagger).$$

$$(g,-g) \longrightarrow R$$

$$SU(2)/Z_2 \equiv SO(3) \quad (*)$$

تمرین: صحت $(*)$ را بررسی کنید



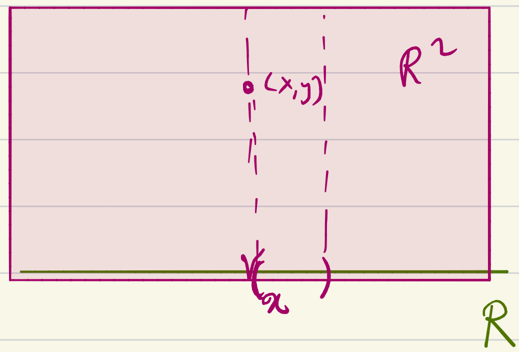
We proved that:

$$\boxed{\text{SU(2) is a universal covering group of SO(3).}}$$

Theorem: if $\tilde{X}$ is a universal covering space of X, then

$$\pi_n(\tilde{X}) = \pi_n(X) \qquad \forall\ n>1.$$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad f(x,y) = x$



$\left\{ \begin{array}{l} \text{Topological Space} \quad X \to \\ \text{Group} : \{ g \mid g \cdot g',\ \bar{g}' \} \end{array} \right.$

Topological Group: $\left\{ \begin{array}{lll} m(g,g') = gg' : & G \times G \longrightarrow G \\ i(g) = \bar{g}' & G \longrightarrow G \end{array} \right\}$

m, i should be continuous with respect to the topology of G.

پنجشنبه ۱ فروردین ۹۹

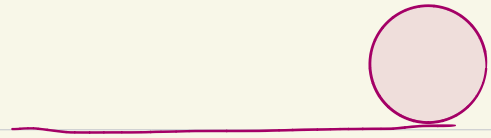
thm: (without proof): if $\tilde{X}$ is a universal covering space of X

$\updownarrow$ if $\tilde{X}$ is Simply Connected.

then $\pi_n(\tilde{X}) = \pi_n(X) \quad \forall\ n>1.$

$\tilde{G}$ " " group of G

① مثال: $f(x) : R^1 \longrightarrow S^1 \qquad f(x) = e^{i\pi x}$



$$\pi_1(S_1) = \mathbb{Z} \qquad \pi_1(R) = \{e\}$$

② مثال $\pi : SU(2) \longrightarrow SO(3) \qquad \pi^{-1}(R \in SO(3)) = \{g, -g\}$

$$R_{ij} = \frac{1}{2} Tr(\sigma_i g \sigma_j g^{+}) \qquad (g, -g) \longrightarrow R$$

$$\pi_n(SU(2)) = \pi_n(SO(3)) \qquad n > 1$$

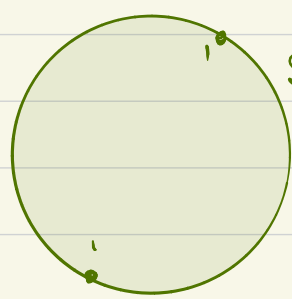
$$\parallel$$

$$\pi_n(S^3)$$

$$\pi_3(S^3) = \mathbb{Z} \longrightarrow \pi_3(SO(3)) = \pi_3(SU(2)) = \pi_3(S_3) = \mathbb{Z}$$

$$RP_3 = S^3/\sim \; = (x \equiv -x)$$

نقاط متقابل یکی‌اند.



$$\pi : S^3 \longrightarrow RP^3 \qquad RP^3 = S^3/\sim$$

$$\pi^{-1}(x \in RP^3) = \{x, -x\}.$$

S^3 is a universal cover of RP_3. $\Rightarrow \pi_n(RP^3) = \pi_n(S^3)$

$$\pi_3(RP^3) = \pi_3(S^3) = \mathbb{Z}.$$

$\pi_n(S^3)$ if $n > 3$.

Example of Covering Groups. $\qquad SU(2)/Z_2 \equiv SO(3)$

$SL(2,\mathbb{C})$, $SO(1,3)$.

Def: $SL(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C},\ ad-bc=1 \right\}$.

Special Linear group of Complex matrices.

$\det(g)=1$

$GL(2,\mathbb{C})$

general linear group.

$$\boxed{SO(1,3) = \{ \Lambda \in M_{4\times 4}(\text{Real}) \mid \Lambda^T \eta \Lambda = \eta \}}$$

$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \longrightarrow \langle x,y \rangle := x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$

ضرب داخلی مینکوسکی

$\langle x,y\rangle = x^T \eta y \qquad \eta = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ spacetime metric

Lorentz Transf. $\quad x \longrightarrow x' = \Lambda x$

$\qquad y \longrightarrow y' = \Lambda y$

$\langle x',y'\rangle = \langle x,y\rangle \longrightarrow x'^T \eta y' = x^T \eta y$

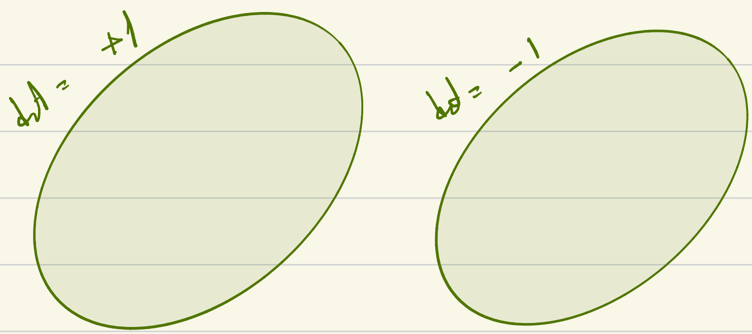
$x^T \Lambda^T \eta \Lambda y = x^T \eta y$

$\Lambda^T \Lambda = I$ متریک اقلیدسی $\qquad \Lambda^T \eta \Lambda = \eta$

$SO(p,q) = \{ \Lambda \in M_{p+q}(\mathbb{R}) \quad \Lambda^T \eta \Lambda = \eta \qquad \eta = \text{diag}(\underbrace{1,1,\dots 1}_{p}, \underbrace{-1,-1,\dots,-1}_{q})$

---

if $\Lambda \in SO(1,3) \longrightarrow \Lambda^T \eta \Lambda = \eta \longrightarrow \det(\Lambda)^2 = 1 \longrightarrow \det \Lambda = \pm 1$.



$$x \longrightarrow \Lambda x$$

$$x^i \longrightarrow x'^i = \Lambda^i{}_j\, x^j$$

$$\langle x, y\rangle = x^i \eta_{ij}\, y^j$$

Λ تمرین: ثابت کنید که در هر ترکیب $\longrightarrow$ $\Lambda^{00} \geq 1 \qquad \Lambda^{00} \leq -1$

اثبات: $\Lambda^T \eta \Lambda = \eta \quad \rightarrow \quad \eta_{ij}\, x'^i y'^j = \eta_{ij}\, x^i y^j \rightarrow \eta_{ij} \Lambda^i{}_k \Lambda^j{}_l\, x^k y^l = \eta_{kl}\, x^k y^l$

$$\eta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \eta_{kl} \longrightarrow k = l = 0$$

$$\eta_{00} = \eta_{ij} \Lambda^i{}_0 \Lambda^j{}_0$$

$$1 = \eta_{00} = \Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^1{}_0 \Lambda^1{}_0 - \Lambda^2{}_0 \Lambda^2{}_0 - \Lambda^3{}_0 \Lambda^3{}_0$$

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \longrightarrow (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \longrightarrow (\Lambda^0{}_0)^2 \geq 1$$

$\rightarrow \quad \Lambda^0{}_0 \geq 1$ یا $\Lambda^0{}_0 \leq -1$

$$\Lambda = \begin{bmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \cdots & \Lambda^0{}_3 \\ & & & \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \cdots & \Lambda^3{}_3 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$\Lambda^0{}_0 \geq 1$ — t' هم جهت t

$\Lambda^0{}_0 \leq -1$ — t' خلاف جهت t

proper Lorentz Group $\longleftarrow$ گروه لورنتز ویژه

det = 1, $\Lambda^0{}_0 \geq 1$

det = 1, $\Lambda^0{}_0 \leq -1$

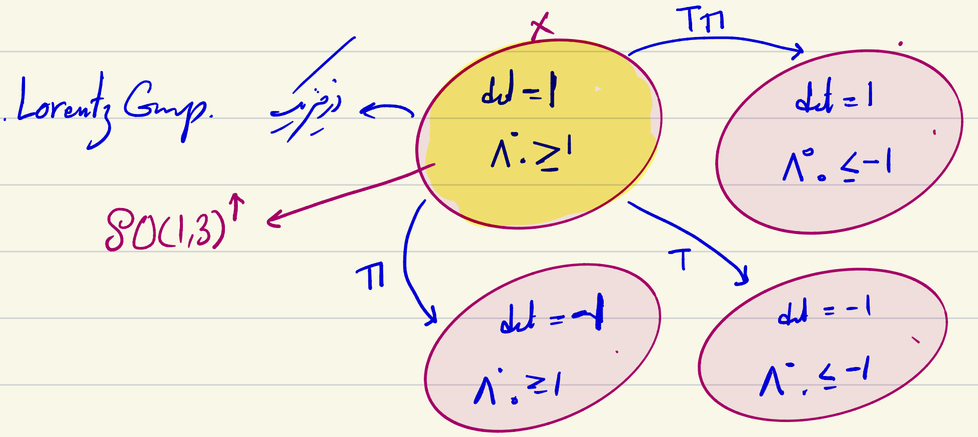
det = -1, $\Lambda^0{}_0 \geq 1$

det = -1, $\Lambda^0{}_0 \leq -1$

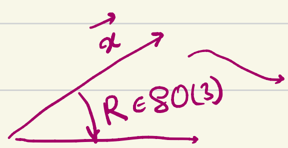
انعکاس زمان $= T = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \in SO(1,3)$? $\qquad \Lambda^{00} = -1,\ \det = -1$

انعکاس فضایی $= \Pi = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \in SO(1,3)$? $\qquad \Lambda^{00} = +1 \quad \det = -1$

$\langle x, y\rangle = x^0 y^0 - \vec{x}\cdot\vec{y} \qquad (x_0, x_1, x_2, x_3) \qquad (y_0, y_1, y_2, y_3)$



$R^3 \ni \vec{x} \longrightarrow p = \vec{x}\cdot\vec{\sigma}$ ✓ $\qquad p \to g p g^{\dagger}$, $g \in SU(2)$



$\begin{bmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{bmatrix}$

$x = (x^0, x^1, x^2, x^3) \in R^4 =$ spacetime $\qquad p = x_0 1 + \vec{x}\cdot\vec{\sigma} = \begin{bmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{bmatrix}$

خواص ماتریس p:

1) $p = p^{\dagger}$

2) $\operatorname{tr} p = 2x_0$

3) $\det p = x_0^2 - x_1^2 - x_2^2 - x_3^2 =$ نرم مینکوفسکی

$p \to \underset{p'}{g p g^{\dagger}} \qquad g \in SL(2,\mathbb{C})$

تمرین:

1) $p' = p'^{\dagger}$ ✗   2) $\mathrm{tr}\, p' \neq \mathrm{tr}\, p$ زیرا $g^\dagger g \neq 1$.

$\mathrm{Sgn}(\mathrm{tr}\, p) = \mathrm{Sgn}(\mathrm{tr}\, p')$

$\mathrm{Sgn}\, x_0' = \mathrm{Sgn}\, x_0$

3) $\det p' = \det p$

$$\det p' = x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

پس یک تبدیل $p \to p' = gpg^\dagger$ نرم مینکوسکی و رد را حفظ می‌کند

تناظر بین $SL(2,C)$ و $SO(1,3)^{\uparrow}_{+}$

پس هر یک تبدیل لورنتس: هر کار به دست $x \to x' = \Lambda x$ انجام دهیم

به دست $p \to p' = gpg^\dagger$ " .

$$\phi : SO(1,3) \longleftarrow SL(2,C)$$

$$\phi : \Lambda(g) \longleftarrow (g, -g).$$

$SL(2,C)$ is a universal covering space of $SO(1,3)$.

ادامه‌ی تمرین: : $R_{ij}(g) \equiv \frac{1}{2} Tr(\sigma_i g \sigma_j g^\dagger)$

$i, j = 1,2,3$ $R_{3\times 3}$.

$$\underset{SU(2)}{\overset{\cap}{g}} \longrightarrow R(g) \in SO(3).$$

تمرین: سوال: رابطه‌ی بین $SL(2,C)$, $SO(1,3)$ چیست؟

① $\Lambda^\mu{}_\nu$: $\mu,\nu = 0,1,2,3$.   ② μ و ν از کجا؟ بالا و پایین هستند.

$$\Lambda^\mu{}_\nu \overset{?}{=} \frac{1}{2} Tr(\sigma^\mu g \sigma_\nu g^\dagger) \quad ? \ ✓$$

$\sigma^\mu = (1, \sigma_x, \sigma_y, \sigma_z)$ $\sigma_\mu = (1, -\sigma_x, -\sigma_y, -\sigma_z)$ اوکی

$SL(2,\mathbb{C})$ is a universal covering group of $SO(1,3)$

- تمرین ، آیا کشش بالا در مورد $SO(1,3)$ درست است
  یا فقط در مورد $SO(1,3)^{\uparrow}_{+}$ درست است؟

- آیا یک g برشکننده نمایش T را باز کند.
  آیا یک g 〃 〃 〃 Π 〃 〃 .

---

① $\Pi_n(U^1) = \Pi_n(S^1) = \Pi_n(\mathbb{R}) = \{e\}$. $\quad n>1$

② $\Pi_n(RP_n) = \Pi_n(S^n) = \mathbb{Z}$

قضیه اول $\qquad \Pi_k(SO(n)) = \Pi_{k+n}(S^n)$

تعمیم: $\quad$ قضیه تناوبی Bott $\qquad$ Raul Bott

$k \leq 3$

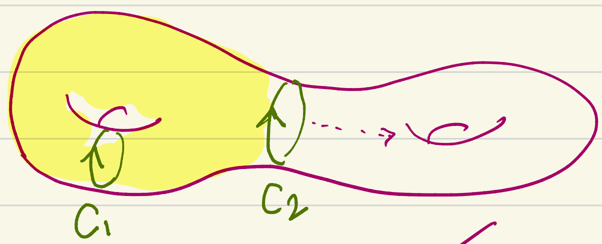
$k+1 \leq 4$

- $\underline{\Pi_k(U(n)) = \Pi_k(SU(n))} = \begin{cases} \{e\} & k = \text{even} \\ \mathbb{Z} & k = \text{odd} \end{cases}$ $\qquad \boxed{n \geq \frac{k+1}{2}}$

- $\Pi_k(O(n)) = \Pi_k(SO(n)) = \begin{cases} \{e\} & k = 2,4,5,6 \mod 8 \\ Z_2 & k = 0,1 \mod 8 \\ Z & k = 3,7 \mod 8. \end{cases}$

Lecture 12; Sunday 17th Bahman 1399.

Homology Groups.



C_1 is different from C_2.

برش در امتداد C_2 ← سطح را به دو تکه مجزا تقسیم می‌کند.

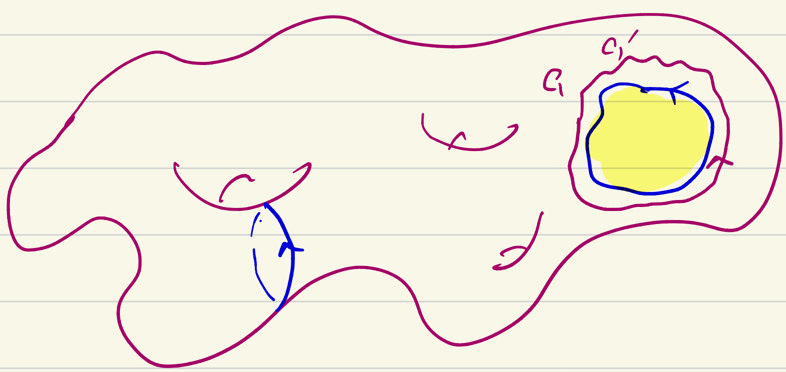
برش در امتداد C_1 ← سطح را یک تکه نگاه می‌دارد.

C_2 مرز یک جایی است.

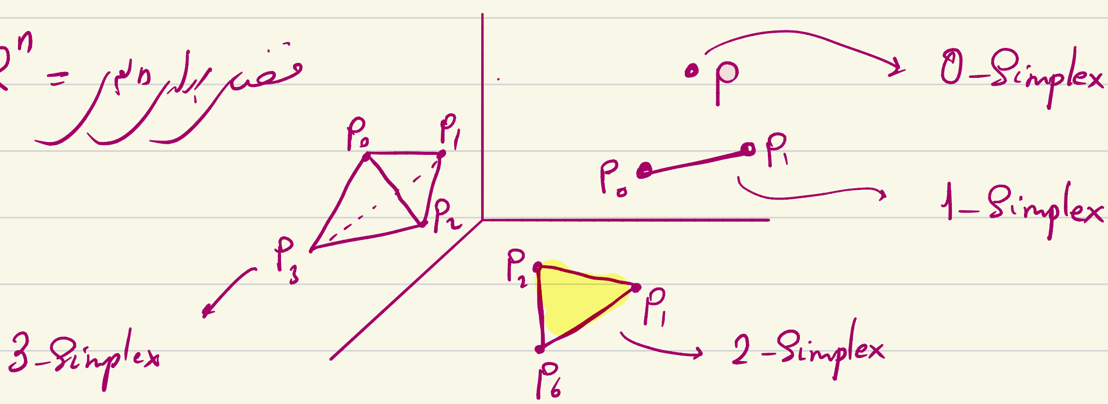
C_1 مرز جایی نیست.

Homology ابداع.

$M \to H_n(M)$ → Homology Group.



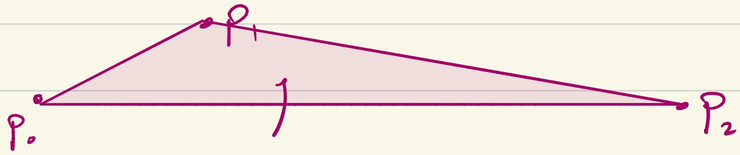
$R^n =$ فضای برداری n بعدی



$$(P_0P_1) = \{\, x \in R^n \mid x = \lambda P_0 + (1-\lambda) P_1 \quad \lambda \in [0,1] \,\}$$

$(P_0P_1P_2) = \{ x \in R^n \mid \quad x = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 , \quad \lambda_0+\lambda_1+\lambda_2=1 \}$

ثابت



2-Simplex.

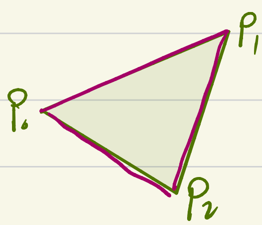
$(P_0P_1P_2 \dots P_k) = k\text{-Simplex} = \{ x \in R^n \mid \quad \vec{x} = \sum_{i=0}^{k} \lambda_i P_i \ , \ \sum_{i=0}^{k} \lambda_i = 1 \}$

Simplex تعریف

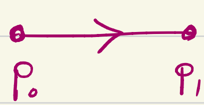
(P_0P_i) should be linearly independent.

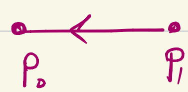
if (P_0P_1) is a 1-simplex P_0 , P_1 وجوه این سیمپلکس $P_0 < (P_0P_1)$
$P_1 < (P_0P_1)$.

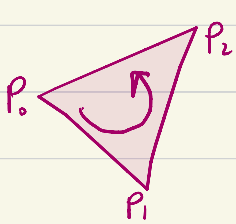
if $(P_0P_1P_2)$ is a 2-Simplex $\underbrace{(P_0P_1),(P_0P_2),(P_1P_2)}_{\text{are 1-face of } (P_0P_1P_2)} < (P_0P_1P_2)$



Oriented Simplex.

 $= \langle P_0P_1 \rangle$

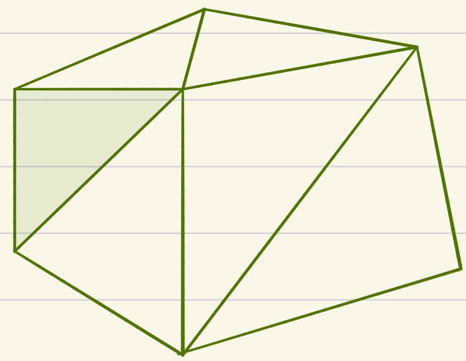
 $= \langle P_1P_0 \rangle$

two simplex :  $= \langle P_0P_1P_2 \rangle \neq \langle P_1P_0P_2 \rangle \neq \langle P_2P_1P_0 \rangle \dots$

هم ارز $\underbrace{\langle P_0P_1P_2 \dots P_k \rangle} \sim \underbrace{\langle P_{i_1}P_{i_2} \dots P_{i_k} \rangle} \mid$ if $\langle P_{i_1} \dots P_{i_k} \rangle$ is an even permutation of $\langle P_0 P_1 \ldots P_k \rangle$.

$\langle P_0 P_1 P_2 \rangle \equiv \langle P_1 P_2 P_0 \rangle \equiv \langle P_2 P_0 P_1 \rangle$.

$\langle P_1 P_2 P_0 \rangle = \langle P_2 P_0 P_1 \rangle$



K = Complex or Simplicial complex.

Simplex $= \sigma$

$K = \{ \sigma_1, \sigma_2, \ldots \sigma_N \} \to$ Such that:

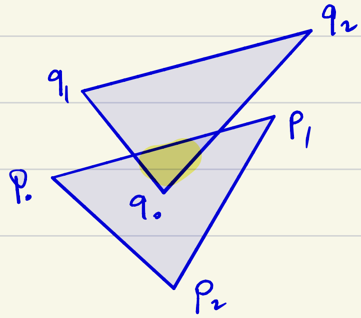
i) if $\sigma_i \in K \longrightarrow$ All faces of $\sigma_i \in K$.

ii) if $\sigma_i, \sigma_j \in K \longrightarrow \sigma_i \cap \sigma_j = \phi$ یا σ_i, σ_j در یک وجه مشترکند.

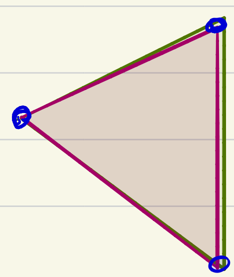
مثال ۱: $K = \{ \langle P_0 P_1 P_2 \rangle \} \xrightarrow{\text{تعمیم}} \{ \langle P_0 P_1 P_2 \rangle, \langle P_1 P_2 \rangle, \langle P_0 P_1 \rangle, \langle P_0 P_2 \rangle, \langle P_0 \rangle, \langle P_1 \rangle, \langle P_2 \rangle \}$

$\times$

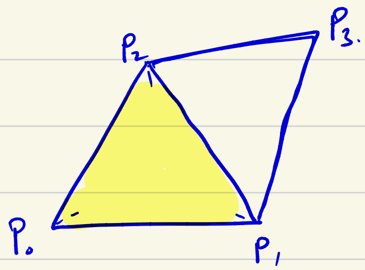
مثال ۲)



is not a complex.



3 مثال)



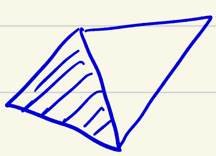
$$K = \left\{ \langle P_0 P_1 P_2 \rangle , \langle P_1 P_2 \rangle, \langle P_1 P_3 \rangle, \langle P_2 P_3 \rangle \right\}$$

$\langle P_0 P_1 \rangle, \langle P_0 P_2 \rangle$

$\langle P_1 P_2 \rangle, \langle P_1 \rangle, \langle P_2 \rangle, \langle P_0 \rangle, \langle P_3 \rangle$

$C_n(K) =$ گروه آزاد $=$ formal group $= \left\{ \sum_i c_i \sigma_n^i \mid c_i \in \mathbb{Z} ,\right.$

$\left. \sigma_n^i \text{ are } n\text{-simplexes in } K \right\}$.

if $K =$  $\underline{C_1(K) = \{ c_1 \langle P_1 P_2 \rangle + c_2 \langle P_1 P_3 \rangle + c_3 \langle P_2 P_3 \rangle \cdots \}}$

if $S = \sum_i c_i \sigma_n^i$, $S' = \sum_i c_i' \sigma_n^i$ $\quad S, S' \in C_1(K)$

$$S + S' := \sum_i (c_i + c_i') \sigma_n^i \qquad (-S') := -\sum c_i' \sigma_n^i$$

$C_n(K)$ is an Abelian group. $\equiv \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \mathbb{Z}}_{\text{تعداد } n\text{-simplex ها در } K}$

Def: Boundary operator: $\partial : C_n(K) \longrightarrow C_{n-1}(K)$

چطوری ∂ تعریف می‌شه؟

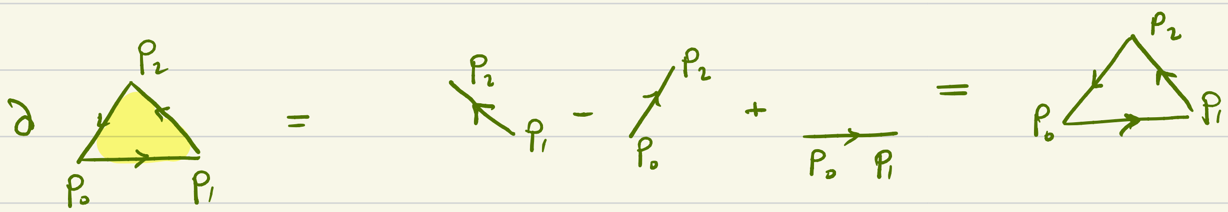
$$\partial \left(\underset{P_0}{\bullet} \longrightarrow \underset{P_1}{\bullet} \right) = P_0 + P_1$$

$$\partial\left(\underset{P_0}{\bullet} \longrightarrow \underset{P_1}{\bullet} \longrightarrow \underset{P_2}{\bullet} \right) \equiv \partial\left(\underset{P_0}{\circ} \longrightarrow \underset{P_1}{\bullet} + \underset{P_1}{\circ} \longrightarrow \underset{P_2}{\bullet} \right) = (P_0 + P_1) + (P_1 + P_2)$$

از نظر کاربردی گویا نیست، جهت‌ها در اینجا مهم است. ←

$$\partial(\langle P_0 P_1 \rangle) := \langle P_0 \rangle - \langle P_1 \rangle$$

$$\partial \langle P_0 P_1 P_2 \rangle := \langle P_1 P_2 \rangle - \langle P_0 P_2 \rangle + \langle P_0 P_1 \rangle$$



$$\partial \langle P_0 P_1 P_2 \cdots P_k \rangle := \sum_{i=0}^{k} (-1)^r \langle P_0 P_1 \cdots \hat{P}_r \cdots P_k \rangle$$

Definition: on $C_n(K)$: $\langle P_0 P_1 P_2 \rangle := - \langle P_0 P_2 P_1 \rangle$

$$\partial \langle P_0 P_1 P_2 \rangle = \langle P_1 P_2 \rangle - \langle P_0 P_2 \rangle + \langle P_0 P_1 \rangle$$

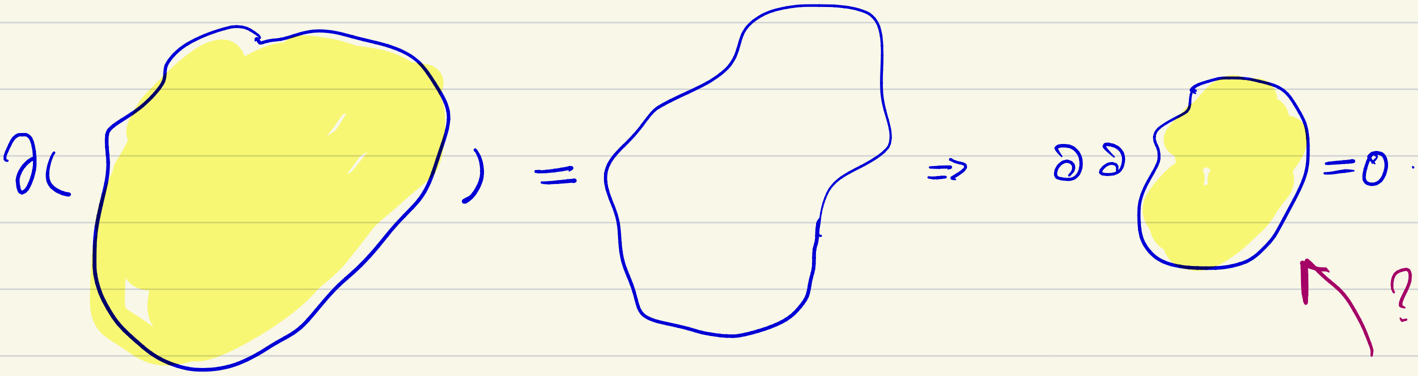
$$\parallel \qquad ?$$

$$\partial \langle P_1 P_2 P_0 \rangle = \langle P_2 P_0 \rangle - \langle P_1 P_0 \rangle + \langle P_1 P_2 \rangle$$

$$\partial : C_n(K) \longrightarrow C_{n-1}(K) \qquad \partial(\langle P_0 P_1 \cdots P_n \rangle) = \partial \sigma_n^i$$

$$\partial(c \in C_n(K)) = \partial\left(\sum c_i \sigma_n^i \right) \equiv \sum_i c_i \, \partial \sigma_n^i$$

theorem: $\partial^2 = 0$. $\qquad \partial^2 : C_n(K) \longrightarrow C_{n-2}(K)$



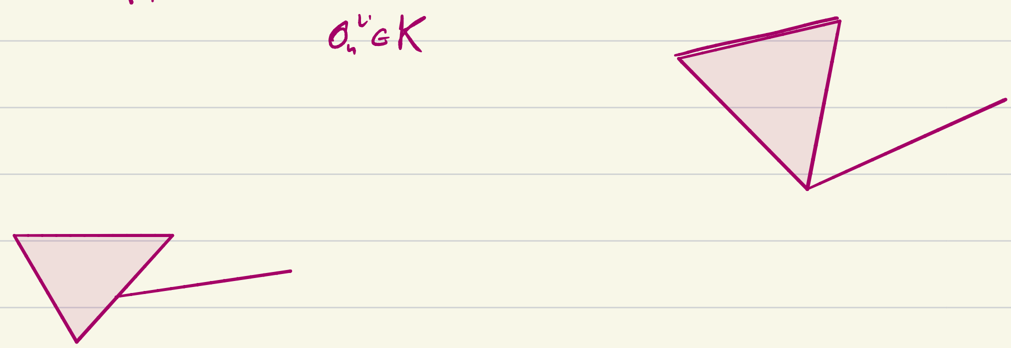
اثبات: $\quad \partial^2 \langle P_0P_1P_2\rangle = \partial\big(\langle P_1P_2\rangle - \langle P_0P_2\rangle + \langle P_0P_1\rangle \big) =$

$$= \langle P_1\rangle - \langle P_2\rangle - \big(\langle P_0\rangle - \langle P_2\rangle \big) + \langle P_0\rangle - \langle P_1\rangle = 0$$

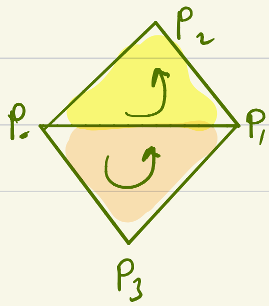
$$\partial^2\big(\langle P_0P_1\dots P_r\rangle\big) = \partial\Big\{ \sum_{i=0}^{r} (-1)^i \langle P_0P_1 \dots \hat{P_i} \dots P_r\rangle \Big\} = \cdots \parallel$$

لطفاً خودتان تکمیل اش کنید.

تعریف $\quad |K| = \bigcup_{\sigma_n^i \in K} \sigma_n^i$



Example:

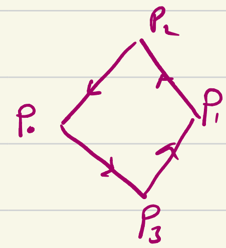


$\sigma = \langle P_0P_1P_2\rangle + \langle P_0P_3P_1\rangle$

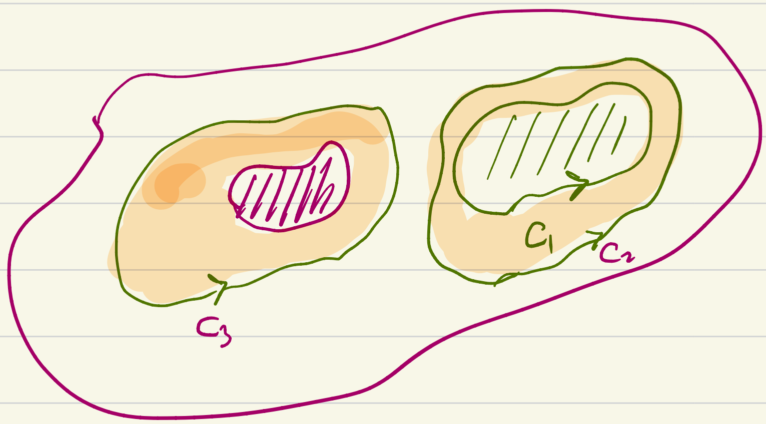
$$\partial\sigma = \partial\langle P_0P_1P_2\rangle + \partial\langle P_0P_3P_1\rangle = \langle P_1P_2\rangle - \langle P_0P_2\rangle + \cancel{\langle P_0P_1\rangle} + \langle P_3P_1\rangle - \cancel{\langle P_0P_1\rangle} + \langle P_0P_3\rangle$$

$$= \langle P_1P_2\rangle - \langle P_0P_2\rangle + \langle P_3P_1\rangle + \langle P_0P_3\rangle$$

$$= \langle P_1P_2\rangle + \langle P_2P_0\rangle + \langle P_3P_1\rangle + \langle P_0P_3\rangle =$$



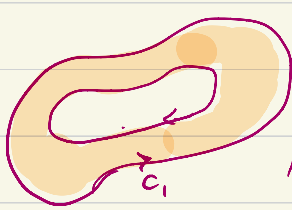
شکل



$$C_1 \sim C_2 \in C_n(K) \quad \text{iff} \quad C_1 - C_2 = \partial c \qquad c \in C_{n+1}(K)$$

Chain group of order n.

$C_1 - C_2$



$\longrightarrow$ $\sim$ is an equivalence relation.

[illegible]

$C_1 \sim C_1 \longrightarrow C_1 - C_1 = 0 \qquad C_1 \sim C_2 \longrightarrow C_2 \sim C_1$

$C_1 \sim C_2,\ C_2 \sim C_3 \longrightarrow \quad C_1 - C_2 = \partial\alpha \quad C_2 - C_3 = \partial\beta \quad \alpha \in C_{n+1}\ \beta \in C_{n+1}$

$$c_1 - c_3 = c_1 - c_2 + c_2 - c_3 = \partial\alpha + \partial\beta = \partial(\alpha+\beta) \qquad \alpha+\beta \in C_{n+1}$$

if $c_1 \sim c_2 \Rightarrow c_1 - c_2 = \partial\alpha$