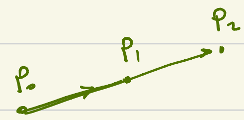
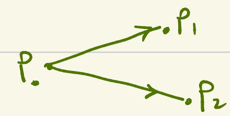
Lecture 13 ; Tuesday 18th Farvardin 1399.

$$r\text{-Simplex} = \sum_{i=0}^{r} c_i P_i \quad \Big| \quad \sum_{i=0}^{r} c_i = 1 \qquad P_i \in \mathbb{R}^n$$

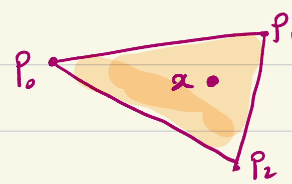
$P_0 - P_i$ linearly independent.





$$2\text{-Simplex} = \{ x = c_0 P_0 + c_1 P_1 + c_2 P_2 \mid c_0 + c_1 + c_2 = 1 , \quad c_i \in \mathbb{R}_+ \}.$$

Barycentric Coordinates of x.



$$\text{Complex } K = \{ \underline{\ \sigma_i\ } \}$$

1) if $\sigma_i \in K$, $\sigma_j < \sigma_i \rightarrow \sigma_j \in K$ (face)

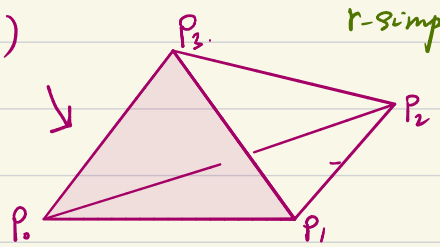
2) if $\sigma_i, \sigma_j \in K \rightarrow \sigma_i \cap \sigma_j = \phi$ & $\sigma_i \cap \sigma_j < \sigma_i, \sigma_j$

Def: Face: if $\underbrace{\sigma = (P_0, P_1, P_2 \ \ P_r)}_{r\text{-simplex}} \rightarrow \sigma_0 = (P_1 \ P_2 \ \ P_r)$ is a face of σ.

$\sigma_1 = (P_0 \ P_1 \ \ P_r)$ is another face of σ

etc. $\searrow$ $r-1$ simplex.

$(P_0 P_1 P_2 P_3)$

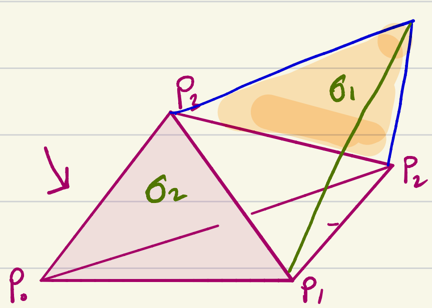


3-Simplex

$(P_1 P_2 P_3)$, $(P_0 P_2 P_3)$, $(P_0 P_1 P_3)$, $(P_0 P_1 P_2)$ 2-faces

$(P_1 P_2)$ $(P_2 P_3)$ $(P_0 P_1)$ 1-faces

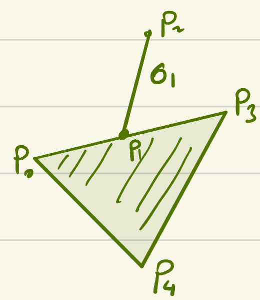
(P_0) (P_1) (P_2) (P_3) 0-face



3-Simplex

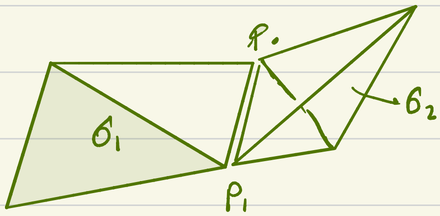
$$\sigma_1 \cap \sigma_2 = (P_2 P_3) \in K.$$

$$\sigma_1 \cap \sigma_2 = (P_1 P_2 P_3)$$

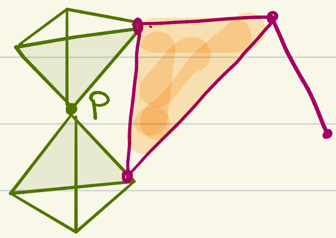


$$K = \{ \underbrace{(P_0 P_3 P_4)}_{\sigma_1}, (P_0 P_3), (P_0 P_4), (P_3 P_4), \underbrace{(P_1 P_2)}_{\sigma_2}, (P_0), (P_1), (P_2), (P_3), (P_4) \}$$

$$(P_0 P_3 P_4) \cap (P_1 P_2) = (P_1) \nless (P_0 P_3 P_4)$$



$$\sigma_1 \cap \sigma_0 = (P_0 P_1) < \sigma_1, \sigma_2$$



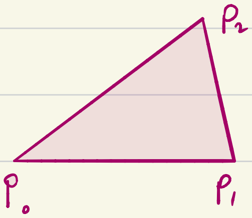
$$K = \{ \sigma_1, \sigma_2, \sigma_3 \dots \}. \qquad C_r(K) =$$

گروه آبل

Finitely generated abelian Group.

$$C_r \in C_r(K) \qquad C_r = \sum_{i=1}^{K} c_i \sigma_r^i \qquad \sigma_r^i \in K$$

$C_r \rightarrow$ r-chain



$$K = \{ (P_0, P_1, P_2) \text{ and its faces} \}.$$

Chain Groups

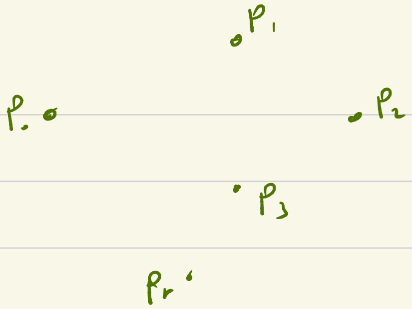
$$C_0(K) = \{ c_0 P_0 + c_1 P_1 + c_2 P_2 \mid c_i \in \mathbb{Z} \}.$$

$$C_1(K) = \{ a_0 (P_1 P_2) + a_1 (P_0 P_2) + a_2 (P_0 P_1) \mid a_i \in \mathbb{Z} \}$$

$$C_2(K) = \{ b_0 (P_1 P_2 P_3) \mid b_0 \in \mathbb{Z} \}.$$

$$C_0(K) \equiv Z \oplus Z \oplus Z \qquad C_1(K) \equiv Z \oplus Z \oplus Z \qquad C_2(K) = Z$$

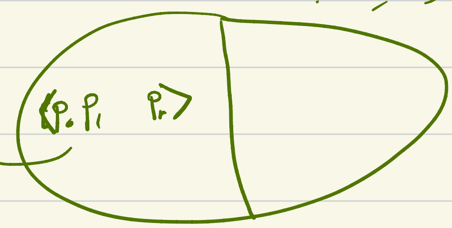
Orientation: $\underbrace{(P_0 P_1 \quad P_r)}_{} \equiv \underbrace{(P_{i_1} P_{i_2} \quad P_{i_r})}_{}$ if $(i_1 \; i_2 \quad i_r)$ is an even permutation of $(0 \; 1 \quad r-1)$



آیا این رئوس یعنی $(r+1)$ تایی $(P_0 -- P_r)$ که

ربطی به جهت‌دار هست یا نیست؟

$(P_0\, P_1 \quad P_r)$



$(P_0\, P_1 \quad P_r)$ است. کوچکترین

$C_r(K)$ گروه : $\langle P_1 P_0\rangle = -\langle P_0 P_1\rangle$ $\quad \langle P_0 P_1 P_2 \quad P_r\rangle := -\langle P_{i_0} P_{i_1} \quad P_{i_r}\rangle$

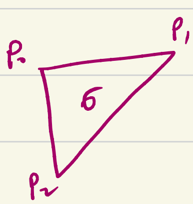
if $(i_0 \quad i_r)$ is an odd permutation of $(0\,1\,2 \quad r)$.

$\partial_r : C_r(K) \longrightarrow C_{r-1}(K)$

$\boxed{\partial_{r-1}\partial_r = 0 \qquad \forall r}$ ← in a complex of dimension n:

$$C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} C_{n-2}(K) \xrightarrow{\partial_{n-2}} \cdots C_1(K) \xrightarrow{\partial_1} C_0(K)$$

نگاشت خطی = همریختی



$$\partial_2(\sigma) = \partial\langle P_0 P_1 P_2\rangle = \langle P_1 P_2\rangle - \langle P_0 P_2\rangle + \langle P_0 P_1\rangle$$

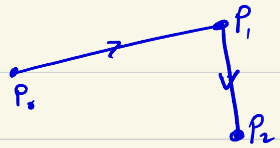
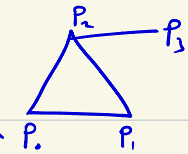
$$\partial_1\partial_2(\sigma) = \langle P_2\rangle - \langle P_1\rangle - (\langle P_2\rangle - \langle P_0\rangle) + (\langle P_1\rangle - \langle P_0\rangle) = 0$$

thm: $\partial_{r-1}\partial_r = 0$ proof: $\partial_r\langle P_0 P_1 P_2 \ldots P_r\rangle = \sum_{i=0}^{r} (-1)^i \langle P_0 P_1 \quad P_{i-1} P_{i+1} \ldots P_r\rangle$

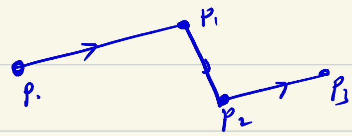
$$\partial_{r-1}\partial_r\langle P_0 P_1 \quad P_r\rangle = \sum_{i=0}^{r}(-1)^i\Big\{\sum_{j\le i-1}(-1)^j\langle P_0 \ldots P_{j-1} P_{j+1}\cdots P_{i-1} P_{i+1} \ldots P_r\rangle$$

$$+ \sum_{j\ge(i+1)}\langle P_0 P_1 \quad P_{i-1} P_{i+1} \quad P_{j-1} P_{j+1}\ldots P_r\rangle\Big\} \ldots\ldots =$$

ادامه را لطفاً کامل کنید.

Definition: $c_1 =$  $\in C_1(K)$ $\quad K = \{$  $\ldots\}$.

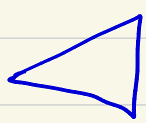
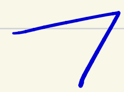
$(P_0P_1) + (P_1P_2)$

$c_1' =$  $\in C_1(K)$

$$\partial c_1 = \partial(\langle P_0P_1\rangle + \langle P_1P_2\rangle) = P_1 - P_0 + P_2 - P_1 = P_2 - P_0$$

$$\partial c_1' = P_3 - P_0$$

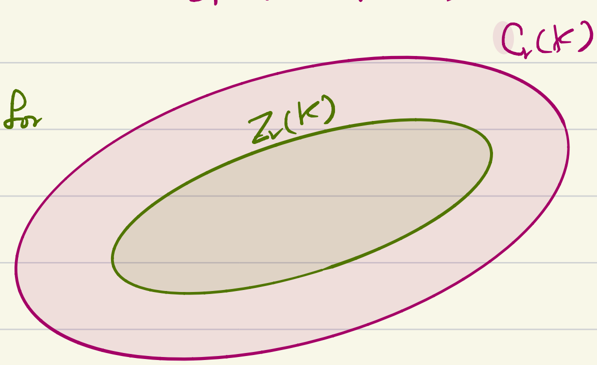
$$Z_r(K) = \operatorname{Ker} \partial_r \subset C_r(K) \qquad Z_r(K) := \{ c_r \in C_r(K) \mid \partial c_r = 0 \}.$$

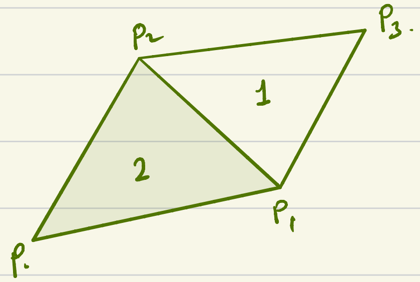
 $\in Z_1(K)$  $\notin Z_1(K)$

$Z_r(K)$ is a group. Proof: $\forall\, z_1, z_2 \in Z_r(K) \to$

$$\partial z_1 = 0 \quad \partial z_2 = 0 \longrightarrow \partial(z_1 + z_2) = 0 \longrightarrow z_1 + z_2 \in Z_r(K)$$

$$B_r(K) = \{ b \in Z_r(K) \mid b = \partial_{r+1} c \text{ for some } c \in C_{r+1}(K) \}.$$





$$\underline{b = \langle P_0P_1\rangle + \langle P_1P_2\rangle + \langle P_2P_0\rangle} \in C_1(K).$$

$$b' = \langle P_1P_3\rangle + \langle P_3P_2\rangle + \langle P_2P_1\rangle \in C_1(K).$$

$$b = \partial_2(P_0P_1P_2)$$

$$\partial(\langle P_0P_1P_2\rangle) = \langle P_1P_2\rangle - \langle P_0P_2\rangle + \langle P_0P_1\rangle = \langle P_1P_2\rangle + \langle P_2P_0\rangle + \langle P_0P_1\rangle$$

$b' \neq \partial$?

Question: is $\sim$ an equivalence relation?

1) $c_r \sim c_r \longrightarrow c_r - c_r = 0 = \partial 0.$

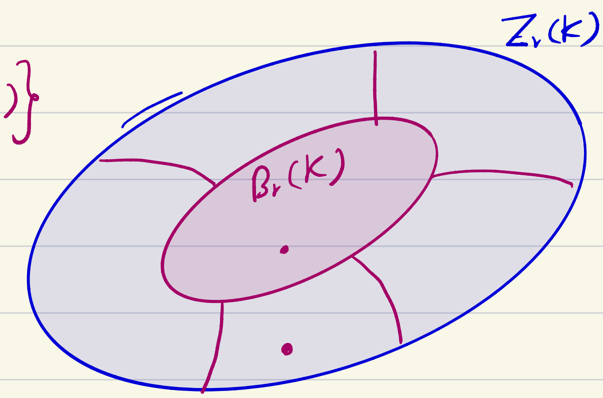
2) $c_r \sim c_r' \longrightarrow c_r - c_r' = \partial c_{r+1} \longrightarrow c_r' - c_r = \partial(-c_{r+1}) \longrightarrow c_r' \sim c_r$

3) $c_r \sim c_{r'}, \quad c_{r'} \sim c_r''$

$$c_r - c_r' = \partial c_{r+1} \qquad c_r' - c_r'' = \partial c_{r+1}'$$

$$\longrightarrow c_r - c_r'' = \partial(c_{r+1} + c_{r+1}') \longrightarrow c_r \sim c_r''$$

$$H_r(K) = Z_r(K) / B_r(K) = \{ [c_r] \in Z_r(K) \}$$

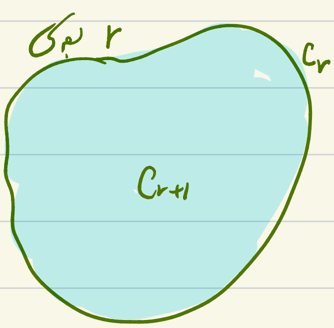


یک گروه هست؟

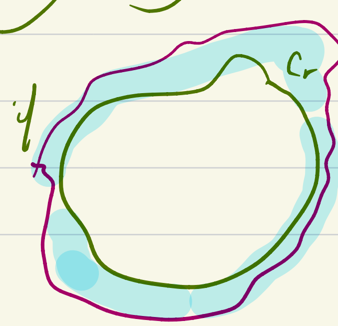
$$\begin{cases} [c_r] + [c_r'] := [c_r + c_r'] . \\ -[c_r] := [-c_r] . \end{cases}$$

$H_r(K) =$ r-th homology Group of K.

از لحاظ شهودی حفره‌های r+1 بعدی را در فضا تعیین می‌کند.



$\partial c_r = 0 \quad , \quad c_r = \partial c_{r+1}$



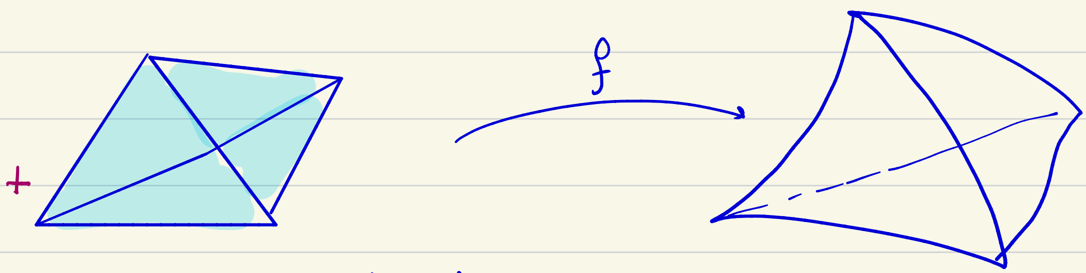
$\partial c_r = 0$

$c_r \neq \partial c_{r+1}$

Definition: if $f : K \longrightarrow X$ is a homeomorphism, then K is called a triangulation of X.

K → Complex

X → Topological space



K is a complex of dimension 2.

X