جلسهٔ هجدهم: ۹۹/۱/۲۰

K $\quad C_n(K) \quad Z_n(K) \subset C_n(K)$

$$Z_n(K) = \operatorname{Ker} \partial_n \qquad B_n(K) \subset Z_n(K)$$

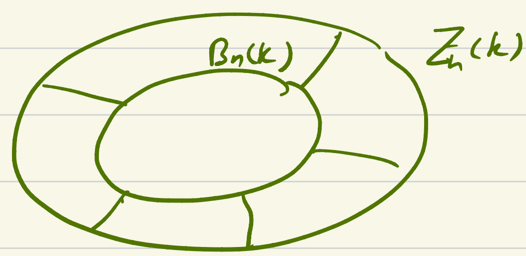
$$B_n(K) = \operatorname{Im} \partial_{n+1}$$

if $z_n \in Z_n(K) \Rightarrow \partial z_n = 0$

if $b_n \in B_n(K) \Rightarrow b_n = \partial_{n+1} c_{n+1}$

چون $\partial_n \partial_{n+1} = 0 \rightarrow$ if $b_n \in B_n(K) \rightarrow b_n \in Z_n(K) \rightarrow B_n(K) \subset Z_n(K)$

$z_n \sim z_n'$ if $z_n - z_n' \in B_n$



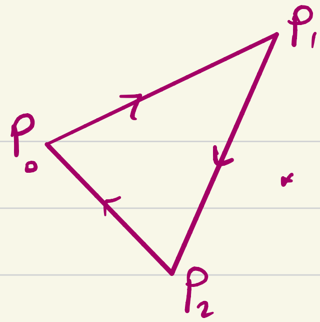
$$H_n(K) = Z_n(K) \big/ B_n(K) = \operatorname{Ker}\partial_n \big/ \operatorname{Im} \partial_{n+1}$$

تعریف: if $f : |K| \longrightarrow X$ $\quad f$: Continuous & invertible map.

K is a triangulation of X.

قضیه: $H_n(K) =: H_n(X)$.

مثال ۱)

$|K| =$ 

.یک مثلث توخالی است |K|

$$K = \{ \langle P_0 P_1 \rangle, \langle P_1 P_2 \rangle, \langle P_2 P_0 \rangle, \langle P_0 \rangle, \langle P_1 \rangle, \langle P_2 \rangle \}$$

$$C_0(K) = \{ a_0 \langle P_0 \rangle + a_1 \langle P_1 \rangle + a_2 \langle P_2 \rangle \} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

$$C_1(K) = \{ \underbrace{a_0 \langle P_0 P_1 \rangle + a_1 \langle P_1 P_2 \rangle + a_2 \langle P_2 P_0 \rangle} \} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

$$C_2(K) = 0$$

$$Z_0(K) = \{ C_0 \mid \partial = 0 \} = C_0(K)$$

$$\partial (a_0 \langle P_0 \rangle + a_1 \langle P_1 \rangle + a_2 \langle P_2 \rangle) = 0$$

$$B_0(K) = \{ C_0 \mid C_0 = \partial C_1 \}$$

$$C_0 = \underbrace{a_0 \langle P_0 \rangle + a_1 \langle P_1 \rangle + a_2 \langle P_2 \rangle} \quad \Big| \quad C_0 = \partial [b_0 \langle P_0 P_1 \rangle + b_1 \langle P_1 P_2 \rangle + \underbrace{b_2 \langle P_2 P_0 \rangle}]$$

$$\rightarrow \underbrace{a_0 \langle P_0 \rangle + a_1 \langle P_1 \rangle + a_2 \langle P_2 \rangle} = b_0 \{ \langle P_1 \rangle - \langle P_0 \rangle \} + b_1 \{ \langle P_2 \rangle - \langle P_1 \rangle \} + b_2 \{ \langle P_0 \rangle - \langle P_2 \rangle \}$$

$$\rightarrow \quad a_0 = b_2 - b_0, \quad a_1 = b_0 - b_1, \quad a_2 = b_1 - b_2$$

.یک قید بین شان وجود دارد a_0, a_1, a_2

$$a_0 + a_1 + a_2 = 0$$

$$\rightarrow \quad C_0 = a_0\langle P_0\rangle + a_1\langle P_1\rangle - (a_0 + a_1)\langle P_2\rangle$$

$$C_0 = a_0(\langle P_0\rangle - \langle P_2\rangle) + a_1(\langle P_1\rangle - \langle P_2\rangle)$$

$$\rightarrow B_0(K) = \mathbb{Z} \oplus \mathbb{Z}. \qquad Z_0(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$H_0(K) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} / \mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}.$$

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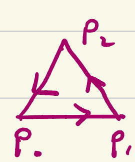
2 دن) $H_1(K)$ $\qquad C_1(K) = \{ a_0\langle P_0 P_1\rangle + a_1\langle P_1 P_2\rangle + a_2\langle P_2 P_0\rangle \}. \leftarrow$

$$Z_1(K) = \{ c_1 \mid \partial c_1 = 0 \} = ? \leftarrow$$

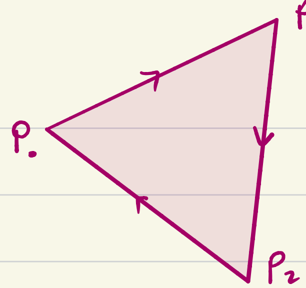
چون: $a_0(\langle P_0\rangle - \langle P_1\rangle) + a_1(\langle P_1\rangle - \langle P_2\rangle) + a_2(\langle P_2\rangle - \langle P_0\rangle) = 0$

$$\rightarrow \quad a_0 = a_2, \quad a_0 = a_1, \quad a_1 = a_2 \rightarrow a_0 = a_1 = a_2$$

$$\rightarrow Z_1(K) = \{ a_0(\langle P_0 P_1\rangle + \langle P_1 P_2\rangle + \langle P_2 P_0\rangle) \} \equiv \{ a_0$$  $$\}$$

$$B_1(K) = \{ z_1 \in Z_1(K) \mid z_1 = \partial c_2 \} = 0$$

$$H_1(K) = Z_1(K) / B_1(K) = Z_1(K) = \mathbb{Z}$$

مثال ۳) $|K| =$  ← حفره‌ی یک‌بعدی نداریم.

$$Z_1(K) = \left\{ a \, \triangle_{p_0 p_1 p_2} \right\} = \{ a(\langle p_0 p_1 \rangle + \langle p_1 p_2 \rangle + \langle p_2 p_0 \rangle) \} \cong \mathbb{Z}$$

$$\underset{z_1}{\shortparallel}$$

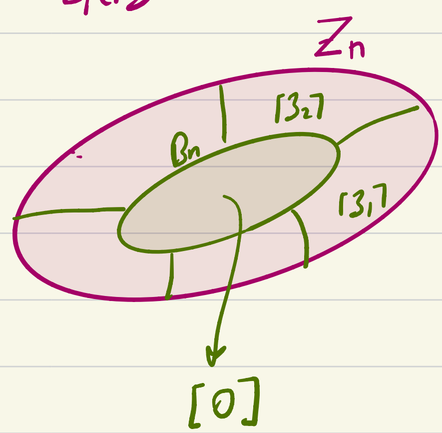
$$B_1(K) = \{ z_1 \mid z_1 = \partial c_2 \} = \ ?$$

آیا $z_1 = a\left(\langle p_0 p_1 \rangle + \langle p_1 p_2 \rangle + \langle p_2 p_0 \rangle \right) \overset{?}{=} \partial (\quad c_2 \quad)$

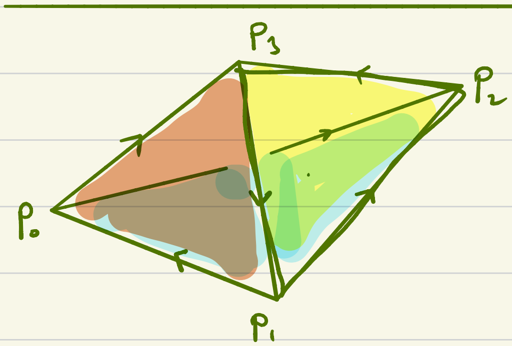
$$c_2 = \langle p_0 p_1 p_2 \rangle \qquad \partial c_2 = z_1 .$$

$$B_1(K) = Z_1(K) \longrightarrow H_1(K) = Z_1(K) / B_1(K) = 0$$

از نظر شهودی = حفره‌ی یک‌بعدی نداریم در K.



مثال ۴



$K = \{ \langle p_0 p_3 p_1 \rangle, \langle p_1 p_2 p_3 \rangle, \langle p_0 p_2 p_3 \rangle, \langle p_0 p_1 p_2 \rangle, \langle p_0 p_1 \rangle, \langle p_1 p_2 \rangle,$

$\langle p_0 p_2 \rangle, \langle p_1 p_3 \rangle, \langle p_0 p_3 \rangle, \langle p_2 p_3 \rangle, \langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle \}.$

$$C_0(K) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \qquad C_1(K) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{6} \qquad C_2(K) = \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{4}$$

$$H_1(K) = 0$$

$H_2(K) = \mathbb{Z} \rightarrow$ تمرین

$$H_2(K) = Z_2(K) / B_2(K) \qquad Z_2(K) = \{ z_2 = a_2 \langle P_0 P_3 P_1 \rangle + a_0 \langle P_1 P_2 P_3 \rangle + a_1 \langle P_0 P_2 P_3 \rangle$$

$$+ a_3 \langle P_0 P_1 P_2 \rangle \mid \partial z_2 = 0 \}$$

$$\rightarrow a_2 (P_3 P_1 - P_0 P_1 + P_0 P_3) + a_0 (P_2 P_3 - P_1 P_3 + P_1 P_2) +$$

$$a_1 (P_2 P_3 - P_0 P_3 + P_0 P_2) + a_3 (P_1 P_2 - P_0 P_2 + P_0 P_1) = 0$$

$P_3 P_1$ ضریب $\quad a_2 + a_0 = 0$

$P_0 P_1$ ضریب $\quad -a_2 + a_3 = 0$

$P_0 P_3$ ضریب $\quad a_2 - a_1 = 0$

$P_0 P_2$ ضریب $\quad a_1 - a_3 = 0$

$P_1 P_2$ ضریب $\quad a_0 + a_3 = 0$

$P_2 P_3$ ضریب $\quad a_0 + a_1 = 0$

$$a_0 = -a_2 \qquad a_3 = a_2 \qquad a_1 = a_2$$

$\rightarrow$ همه ضرایب را می‌توان

برحسب a_0 نوشت

$$\Longrightarrow$$

$$a_1 = -a_0 \qquad a_2 = -a_0 \qquad a_3 = -a_0$$

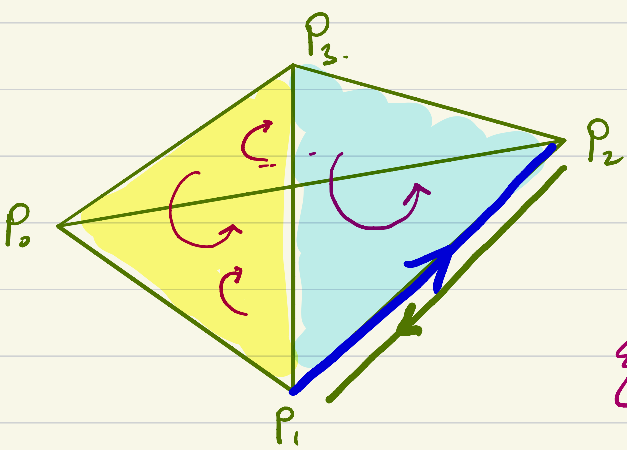
$$\downarrow$$

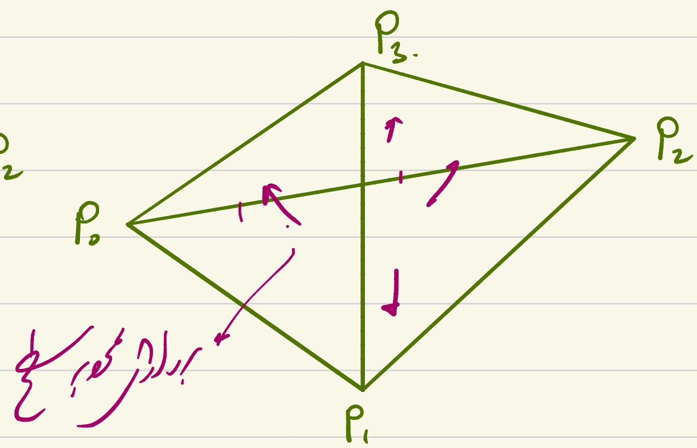
$$z_2 \in Z_1(K) \rightarrow$$

$$\partial_2 = a_0 \{ \langle P_1 P_2 P_3 \rangle - \langle P_0 P_2 P_3 \rangle - \langle P_0 P_3 P_1 \rangle - \langle P_0 P_1 P_2 \rangle \}$$

$$Z_2(K) = \mathbb{Z}$$

$$\partial_2 = a_0 \{ \underline{\langle P_1 P_2 P_3 \rangle} + \langle P_2 P_0 P_3 \rangle + \langle P_3 P_0 P_1 \rangle + \underline{\langle P_1 P_0 P_2 \rangle} \}.$$





$\partial\partial_2 = 0$

یادآوری:

$c \leftrightarrow$ chain

$\partial \leftrightarrow$ cycle

$b \leftrightarrow$ bonday

$\partial/b \longleftrightarrow$ homology cycle

$Z_2(K) = \mathbb{Z}$ $\quad B_2(K) = 0 \quad \rightarrow \quad H_2(K) = \mathbb{Z}$

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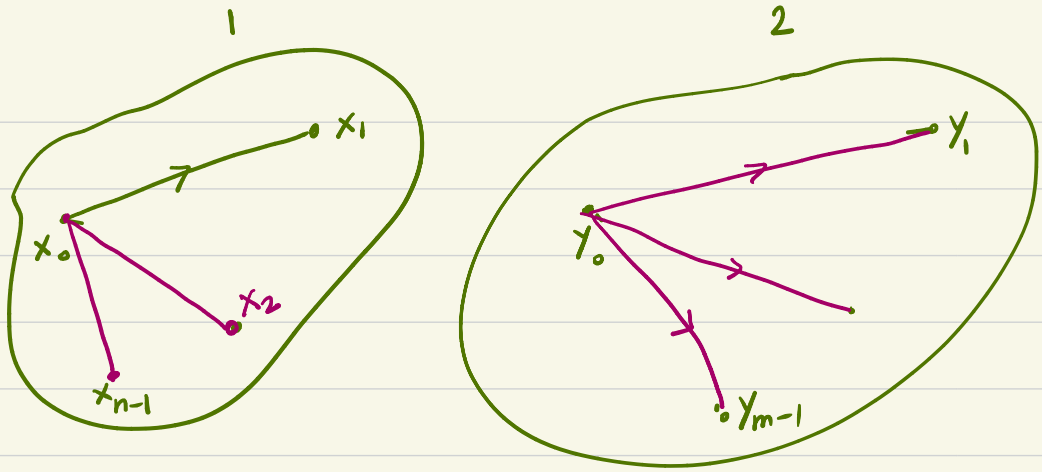
theorem: $H_0(K) = \mathbb{Z}^M$: M تعداد مولفه
path-Connected.



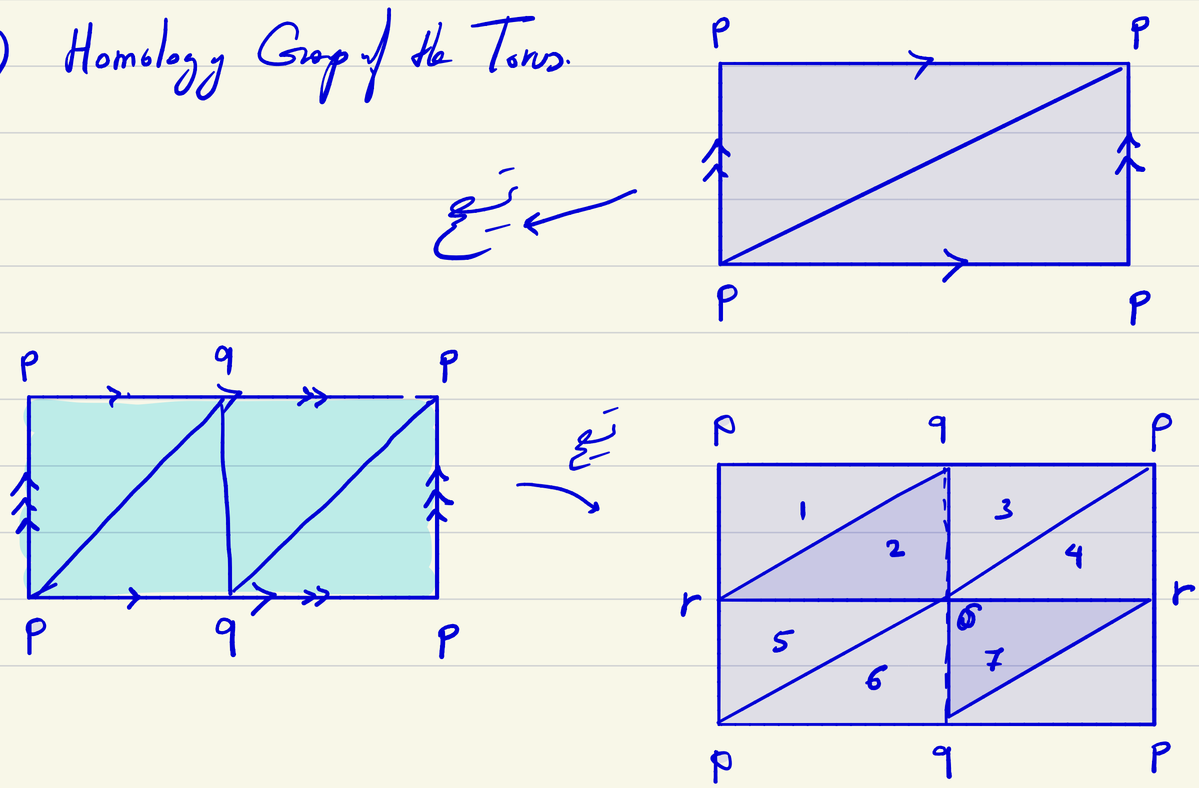
$C_0(K) = \mathbb{Z}^{m+n}$ $\qquad$ $Z_0(K) = \mathbb{Z}^{m+n}$

$B_0(K) = \langle x_0x_1,\ x_0x_2,\ \ldots\ x_0x_{n-1},\ y_0y_1,\ y_0y_2,\ \ldots,\ y_0y_{m-1} \rangle$

why $x_1x_2 \notin B_0(K)$ $\qquad$ $x_0x_2 = x_0x_1 + x_1x_2$

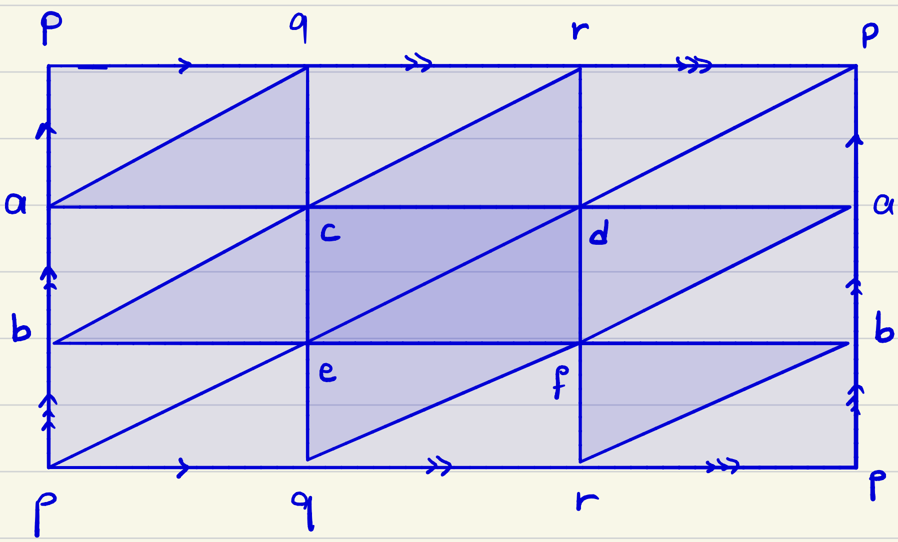
$B_0(K) = \mathbb{Z}^{m-1+n-1} \rightarrow Z_0(K)/B_0(K) = \mathbb{Z}^{m+n}/\mathbb{Z}^{m+n-2} = \mathbb{Z}^2$

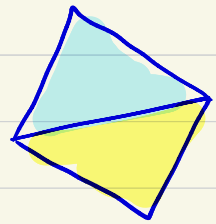
Eg 2) Homology Group of the Torus.



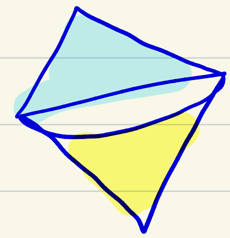
Δ_1 & Δ_3 $\quad \Delta_1 \cap \Delta_3 = \{p\} \cup \{q\}$

Δ_2 & Δ_7 $\quad \Delta_2 \cap \Delta_7 = \{q\} \cup \{r\} \cup \{s\}$

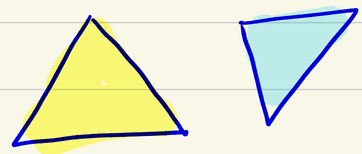




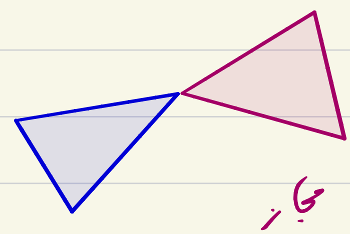
مجاز



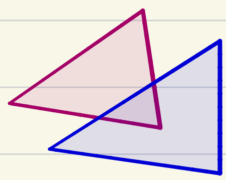
غیرمجاز



مجاز



مجاز



غیرمجاز