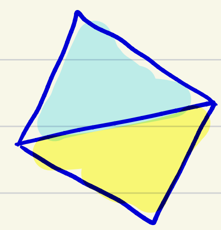
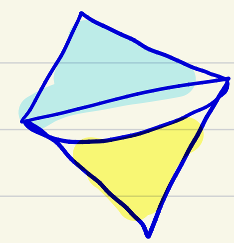
Δ_1 & Δ_3 $\qquad \Delta_1 \cap \Delta_3 = \{p\} \cup \{q\}$

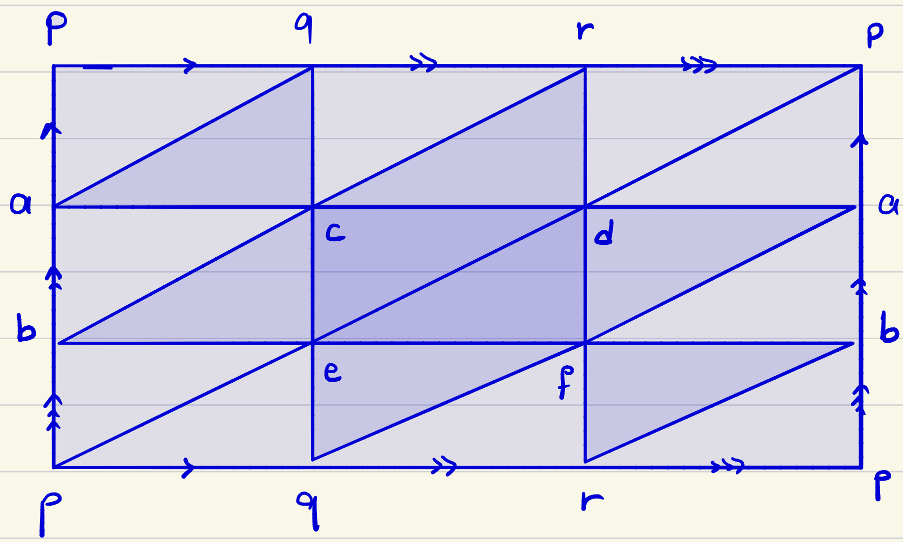


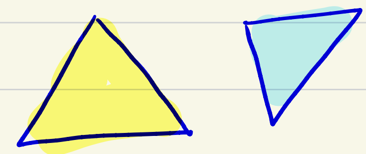
مجاز

Δ_2 & Δ_7 $\qquad \Delta_2 \cap \Delta_7 = \{q\} \cup \{r\} \cup \{s\}$

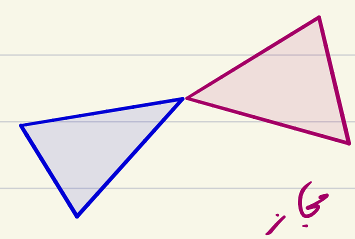


غیرمجاز



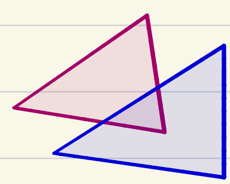


مجاز



مجاز

غیرمجاز



تمرین: Calculate $H_r(\Sigma_1)$ for $r = 0, 1, 2$.

$H_0(\Sigma_1) = \mathbb{Z}$ Since Σ_1 is path connected.

$$\begin{cases} H_1(\Sigma_1) = ? \\ H_2(\Sigma_1) = ? \end{cases}$$

this is your exercise.

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Künneth Formula یا theorem.

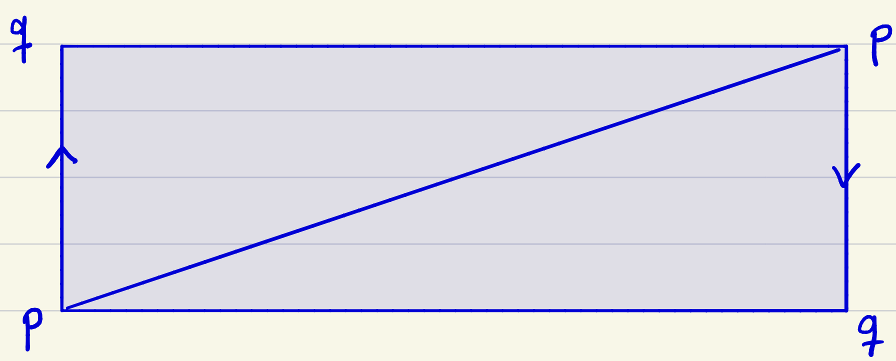
$$H_r(X \times Y) = \bigoplus_{i+j=r} H_i(X) \otimes H_j(Y)$$

$\Sigma_1 = S_1 \times S_1$

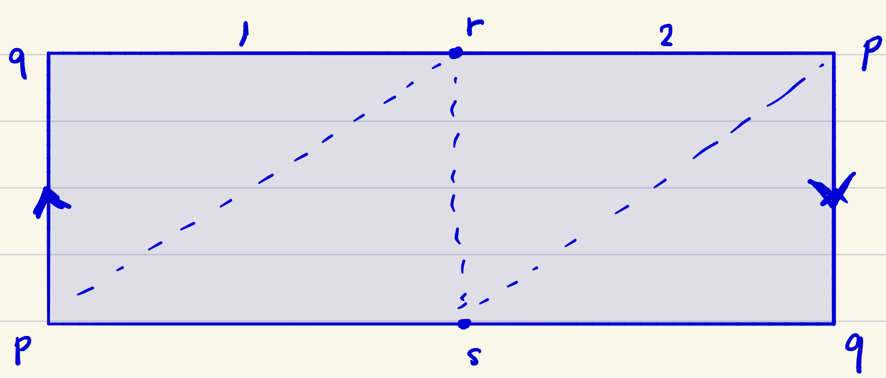
یک بار با استفاده از مثلث‌بندی این تمرین را حل کنید

یک بار هم با استفاده از فرمول Künneth.

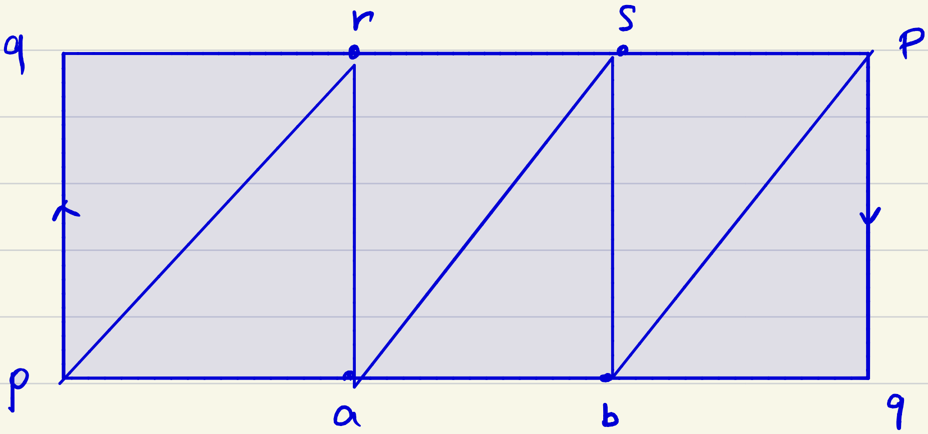
تمرین بعدی: Calculate H_r(Mobius Strip).



Simplex = $\langle P_0\, P_1 \quad P_r \rangle$ $\qquad P_0 - P_i$ independent.



مثلث بندی نوار موبیس

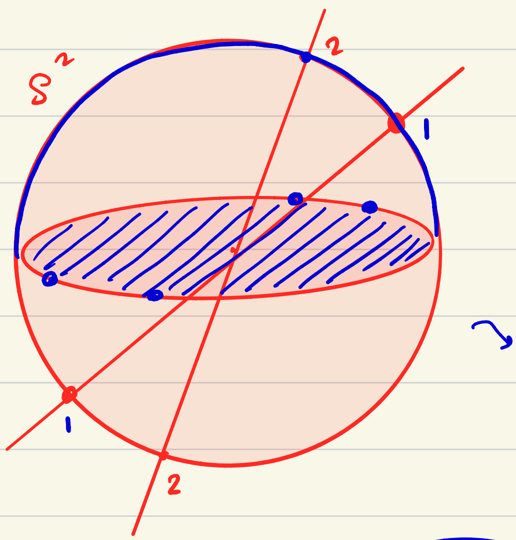


تمرین: Find)

$$H_0(M) = \mathbb{Z} \qquad H_1(M) = ? \qquad H_2(M) = ?$$

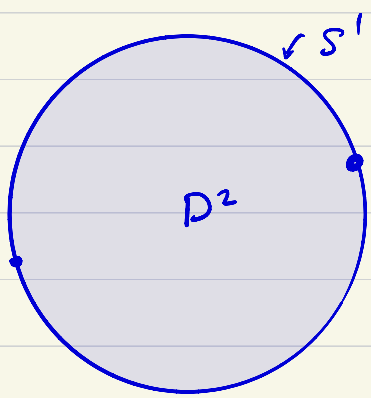
تمرین: Find $H_r(RP_2)$.

$$RP_2 = R^3 / \{\vec{x} \simeq \lambda \vec{x}\} =$$ فضای امتدادها در R^3



$$RP_2 = S^2 / (x = -x \in S^2)$$

$$RP_2 = D^2 / (x = -x,\ x \in \partial D^2)$$



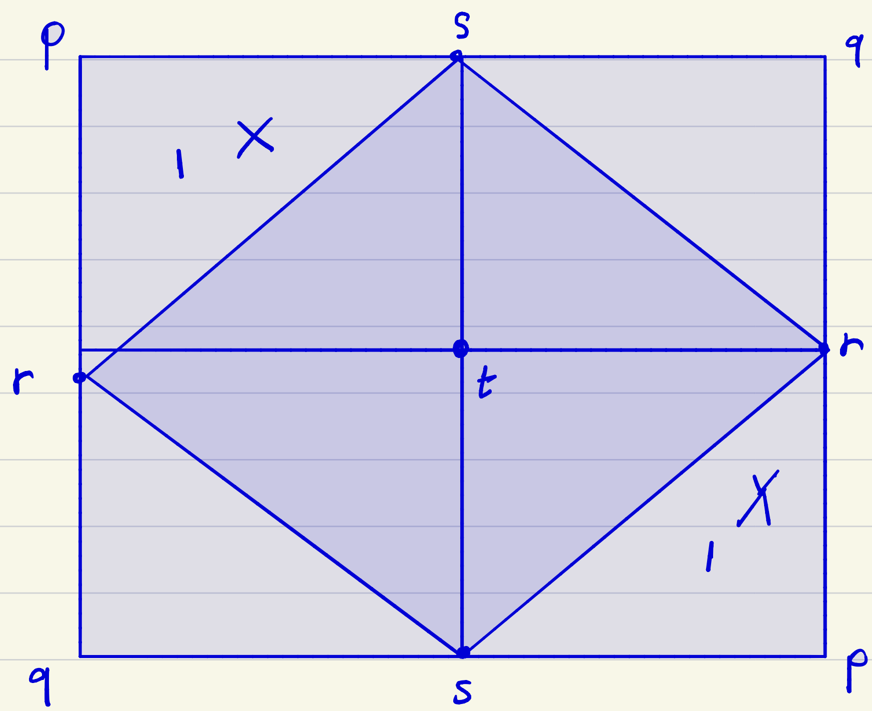
→ مثلث بندی

(1) pqr درمثلث

(2) (psr) در

با هم متفاوت اند ولی

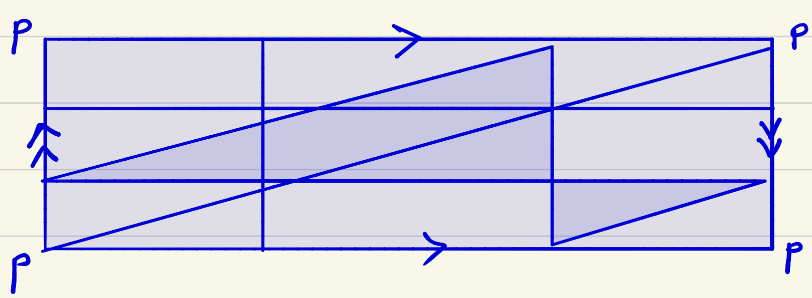
در شکل از یک وجه گرفته



Please find a good tringulation and calculat $H_r(RP^2)$.

Prove: $H_1(RP_2) = \mathbb{Z}_2$. $H_0(RP_2) = \mathbb{Z}$ $H_2(RP_2) = ?$

Problem: Find $H_r(K)$. Klein Bottle. $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$



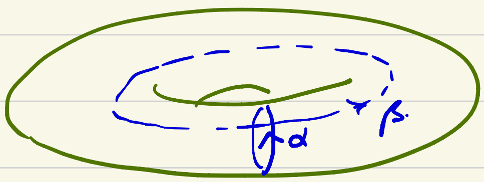
$H_2(K) = Z_2(K) / B_2(K)$ $B_2(K) = \{0\}$.

$B_2(K) = \{ z_2 \in Z_2 \mid z_2 = \partial_3 c_3 \}$ چون $C_3(K)=0$

$B_2(K) = 0 \longrightarrow H_2(K) = Z_2(K)$ از سطوح دوبعدی.

کشش = سطح دوبعدی که مرز نداشته باشد.

مثال:



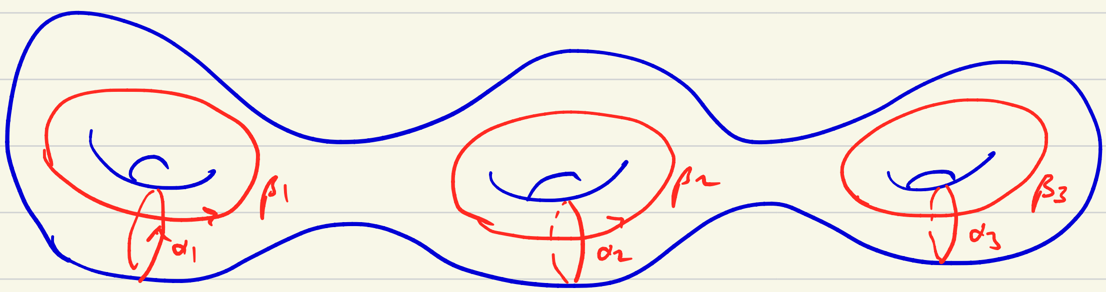
$H_1(\Sigma_1) = \underbrace{Z \oplus Z}$.

دوتا سیکل وجود دارد که مرز چیزی نیستند ← Homology Cycle

مولد H_1

سیکل = cycle = یک زنجیر است که مرزش صفر است. = یک منحنی که مرز آن صفر است.

for Σ_g



$$H_1(\Sigma_g) = \underbrace{(Z \oplus Z) \oplus (Z \oplus Z) \oplus \cdots \oplus (Z \oplus Z)}_{2g}.$$

قضیه ما بین اینها

$H_1(X)$ → Homology Group

$\pi_1(X)$ → Homology Group

$H_1(X) = \text{Abelinization of } \pi_1(X)$

$\pi_1(X)$ را آبلی می‌کند.

$$\pi_1(X) = \langle g_1, g_2, g_3 \ldots \rangle \big/ (R_1, R_2, R_3 \ldots)$$

$$\pi_1(\Sigma_1) = \langle \alpha, \beta \rangle \big/ {\alpha\beta\alpha^{-1}\beta^{-1} = 1}$$

$$\pi_1(\Sigma_g) = \langle \alpha_1, \beta_1, \ldots \alpha_g, \beta_g \rangle \Big/ \prod_i \alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1} = 1$$

$$H_1(\Sigma_g) = \langle \alpha_1, \beta_1 \quad \alpha_g, \beta_g \rangle = \mathbb{Z}^{\oplus 2g}.$$

Euler Character. $\quad \chi = V - E + F$

$b_r = \dim \underbrace{H_r(X)}_{\text{یک فضای برداری است.}}$ ═══ Betti Number.

$Z_r(X)$ یک فضای برداری $\qquad C_r(X) = \{ \sum c_i \cdot \sigma_i^r \}$ یک فضای برداری

$B_r(X)$ یک فضای برداری $\qquad$ بردار پایه

$H_r(X) =$ یک فضای برداری

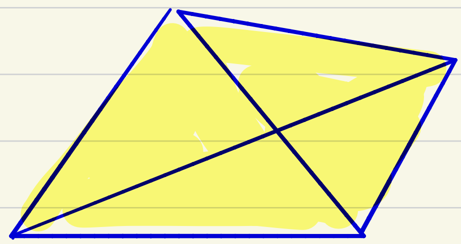
چون $H_r(X)$ یک ناوردای توپولوژیک است $\longleftarrow$ b_r هم یک ناوردای توپولوژیک است.

$$\rightarrow \quad \chi := \sum_{r=0}^{M} (-1)^r b_r = b_0 - b_1 + b_2 - b_3 \ldots \pm b_M$$

$\rightarrow \quad \chi \equiv d_0 - d_1 + d_2 - d_3 + \cdots$

d_r = Number of r-Simplexes

EXAMPLE: χ(کره‌ی توپر دو بعدی)



$d_0 = 4$ V

$d_1 = 6$ E

$d_2 = 4$ F

$d_3 = 1$

$$\chi = 4 - 6 + 4 - 1 = 1.$$

proof of Equality:

$$\chi = b_0 - b_1 + b_2 - b_3 + \cdots$$

$$b_r = \dim H_r(K) = \dim Z_r(K)/B_r(K) = \dim Z_r(K) - \dim B_r(K)$$

$$= \dim(\operatorname{Ker} \partial_r) - \dim(\operatorname{Im}(\partial_{r+1}))$$

$$\chi = \dim(\operatorname{Ker} \partial_0) - \dim(\operatorname{Im} \partial_1) - [\dim(\operatorname{Ker} \partial_1) - \dim(\operatorname{Im} \partial_2)]$$

$$+ [\dim(\operatorname{Ker} \partial_2) - \dim(\operatorname{Im} \partial_3)] - [\dim(\operatorname{Ker} \partial_3) - \dim(\operatorname{Im} \partial_4)] + \cdots$$

Remind: $T : V \longrightarrow W$ linear map

$$\dim(\mathrm{Ker}\, T) + \dim(\mathrm{Im}\, T) = \dim V$$

$$\partial_r : C_r(K) \longrightarrow C_{r-1}(K)$$

$$\dim(\mathrm{Ker}\, \partial_r) + \dim(\mathrm{Im}\, \partial_r) = \dim(C_r(K)) = d_r$$

$$\dim(\mathrm{Ker}\, \partial_0) = d_0 \qquad \partial_0 : C_0(K) \longrightarrow 0$$

$$\dim(\mathrm{Im}\, \partial_{M+1}) = d_M \qquad \partial_M : C_M(K) \longrightarrow C_{M-1}(K).$$

$$\chi = \dim(\mathrm{Ker}\, \partial_0) - \overbrace{\dim(\mathrm{Im}\, \partial_1) - [\dim(\mathrm{Ker}\, \partial_1)}^{1} - \underbrace{\dim(\mathrm{Im}\, \partial_2)}_{\times}]$$

$$+ [\underbrace{\dim(\mathrm{Ker}\, \partial_2)}_{\times\ 2} - \underbrace{\dim(\mathrm{Im}\, \partial_3)] - [\dim(\mathrm{Ker}\, \partial_3)}_{3} - \dim(\mathrm{Im}\, \partial_4)] + \cdots$$

$$\boxed{\chi = d_0 - d_1 + d_2 - d_3 + \cdots \pm (-1)^M d_M} \pm (\underbrace{\dim \mathrm{Im}(\partial_{M+1})}_{0})$$

نتیجه:

مثلاً: $M=3$.

$$\chi = b_0 - b_1 + b_2 - b_3 = \overset{d_0}{\dim(\mathrm{Ker}\, \partial_0)} - \overset{d_1}{\dim(\mathrm{Im}\, \partial_1)}$$

$$-[\dim(\mathrm{Ker}\, \partial_1) - \dim(\mathrm{Im}\, \partial_2)] + d_2$$

$$+[\dim(\mathrm{Ker}\, \partial_2) - \dim(\mathrm{Im}\, \partial_3)] - d_3$$

$$-[\dim(\mathrm{Ker}\, \partial_3) - \underbrace{\dim(\mathrm{Im}\, \partial_4)}_{0}]$$