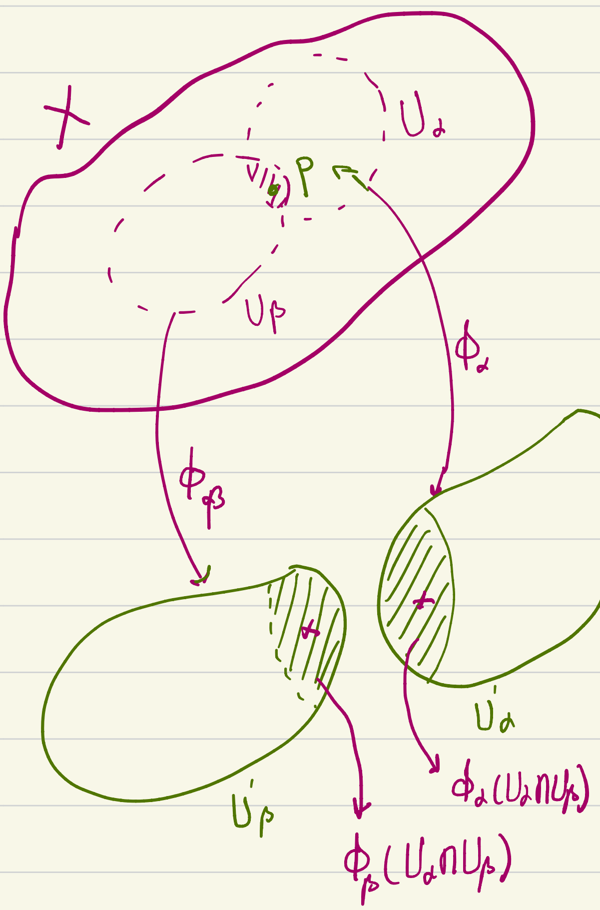
جلسه بیست و یکم، ۹۹/۱/۳۱

مروری بر مطالب جلسه گذشته.



1) $\bigcup U_\alpha \supset X$

2) $\phi_\alpha : U_\alpha \longrightarrow U'_\alpha \underset{\text{open}}{\subset} R^n$ تعریف

ϕ_α is a homeomorphism between U_α & U'_α

3) In $U'_\alpha \cap U'_\beta$: $\phi_\beta \circ \phi_\alpha^{-1}$ should be differentiable.

$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$

$$p : \xrightarrow{\phi_\alpha} \underbrace{(x^1, x^2, \ldots x^n)}$$
$$ \xrightarrow{\phi_\beta} \underbrace{(y^1, y^2, \ldots y^n)}$$
$$\phi_\beta \circ \phi_\alpha^{-1}$$

all $\left(\frac{\partial y^i}{\partial x^j}\right)$ exists. $\rightarrow$ X is a differentiable manifold.

X is a C^k manifold if all the functions $\phi_\alpha \circ \phi_\beta^{-1}$ are k-differentiable.

X is a C^∞ " " " " ∞ - " .

X is a smooth " " " " analytic; have taylor exp

$f(x) = e^{-\frac{1}{x}}$ $\quad f(0) = 0 \quad f'(0) = \frac{1}{x^2} e^{-1/x} \Big|_{x=0} = 0 \quad \dots \quad f^{(n)}(0) = 0.$

f is infinitely differentiable but not analytic:

[illegible]: (U_α, ϕ_α) = chart. $\quad \{(U_\alpha, \phi_\alpha)\}$ = Atlas.

$\text{Atlas}_1 = \{(U_\alpha, \phi_\alpha)\}_{\alpha=1 \dots N} \qquad \text{Atlas}_2 = \{(V_\alpha, \psi_\alpha)\}_{1 \dots M}$

Atlas_1 is compatible with Atlas_2 if $\text{Atlas}_1 \cup \text{Atlas}_2$ is itself an atlas

theorem: Compatibility is an equivalence relation.

[Atlas] = Differential structure.

Milnor: 1986 $\longrightarrow$ S^7 allows 28 different differential structures.

Donaldson: 200... $\longrightarrow$ R^4 " ∞ " " " .

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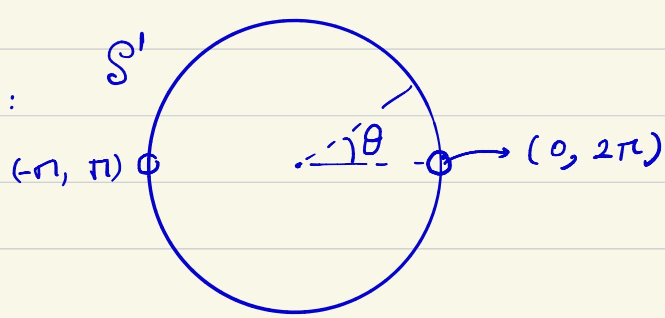
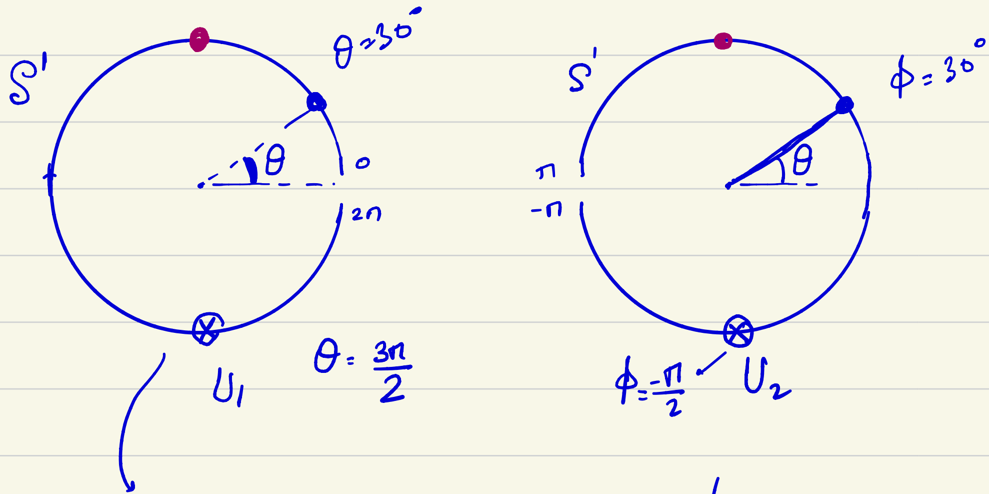
1) Example:



$U_1 = (0, 2\pi)$

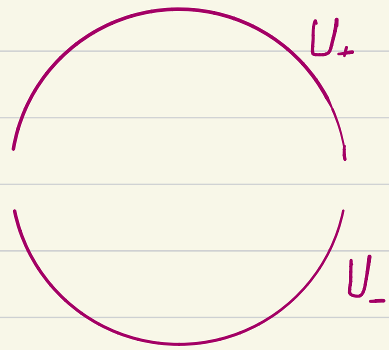
$U_2 = (-\pi, \pi).$



$\theta \in (0, 2\pi)$ $\qquad\qquad$ $\phi \in (-\pi, \pi)$

$\phi = \pi - \theta$

$U_1 \cap U_2 =$

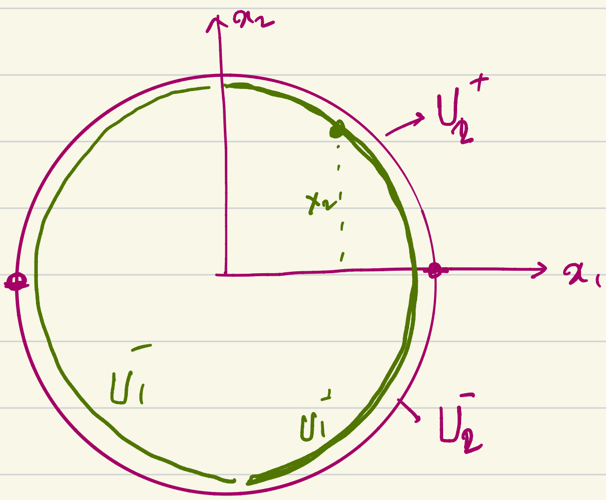


$= \underset{U_+}{(0, \pi)} \cup \underset{U_-}{(\pi, 2\pi)}$

in U_+: $\theta = \phi$ $\qquad$ in U_-: $\phi = \theta - 2\pi$.

EXAMPLE 2:

$$S^1 = \{ (x^1, x^2) \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 = 1 \}.$$



$$U_2^+ = \{ (x^1, x^2) \in S^1 \mid x^2 > 0 \}$$

$$U_2^- = \{ (x^1, x^2) \in S^1 \mid x^2 < 0 \}.$$

$$U_1^+ = \{ (x^1, x^2) \in S^1 \mid x^1 > 0 \}$$

$$U_1^- = \{ (x^1, x^2) \in S^1 \mid x^1 < 0 \}$$

$$U_1^- \cup U_1^+ \cup U_2^- \cup U_2^+ \supset S^1$$

in $U_1^+ : \quad (x^1, x^2) \xrightarrow{\phi_1^+} x_2 \qquad\qquad U_2^+ : (x^1, x^2) \longrightarrow x_1$

$U_1^- : \quad (x^1, x^2) \xrightarrow{\phi_1^-} x_2 \qquad\qquad U_2^- : (x^1, x^2) \longrightarrow x_1$

$$\begin{tikzcd}[row sep=small] & & x^2 \arrow[dd, bend left, "?"] \\ U_1^+ \cap U_2^+ : (x^1, x^2) \arrow[urr, "\phi_1^+"] \arrow[drr, "\phi_2^+"'] & & \\ & & x^1 \end{tikzcd}$$

$$x^1 = \sqrt{1-(x^2)^2}$$

در تمام $U_1^+ \cap U_2^-$ مشتق‌پذیر است.

$$\frac{dx^1}{dx^2} = \frac{-2x^2}{2\sqrt{1-(x^2)^2}} \qquad x^2 \neq \pm 1$$

$U_1^- \cap U_2^+$

$$x_1 = -\sqrt{1-(x^2)^2}$$

EXAMPLE 3: S^n

$$U_i^+ = \{ (x^1, x^2, \ldots x^{n+1}) \mid x^i > 0 \}$$

$$U_i^- = \{ (x^1, x^2, \ldots x^{n+1}) \mid x^i < 0 \}.$$

$$(x^1, x^2, \ldots x^{n+1}) \in S^n \xrightarrow{\phi_i^\pm} (x^1, x^2, \ldots x^{i-1}, x^{i+1}, \ldots x^{n+1}).$$

in $U_i^+ \cap U_j^+ \in S^n$

$$\begin{tikzcd} & (x^1, x^2, \ldots x^{i-1}, x^{i+1}, \ldots x^{n+1}) \\ (x^1 \ldots x^{n+1}) \arrow[ur, "\phi_i^+"] \arrow[dr, "\phi_j^+"'] & \\ & (x^1, x^2, \ldots x^{j-1}, x^{j+1}, \ldots x^{n+1}) \end{tikzcd}$$

$x^1 = s_1,\ x^2 = s_2,\ \ldots\ x^{i-1} = s_{i-1},\ x^{i+1} = s_i,\ \ldots\ x^{n+1} = s_{n+1}$

$x^1 = \eta_1,\ x^2 = \eta_2,\ \ldots\ x^{j-1} = \eta_{j-1},\ x^{j+1} = \eta_{j+1},\ \ldots\ x^{n+1} = \eta_{n+1}$

$\underline{N=2}$.

$$(x,y,z)\in S^2 \longrightarrow (x,y,\sqrt{1-x^2-y^2}) \xrightarrow{\phi_3^+} (x,y)$$

$$(x,y,z) \longrightarrow (x,\sqrt{1-x^2-z^2},z) \xrightarrow{\phi_2^+} (x,z).$$

$$(x,z)=(x,\sqrt{1-x^2-y^2})$$

مثلا S^3: $(x,y,z,t) \xrightarrow{U_1^+} (y,z,t)$

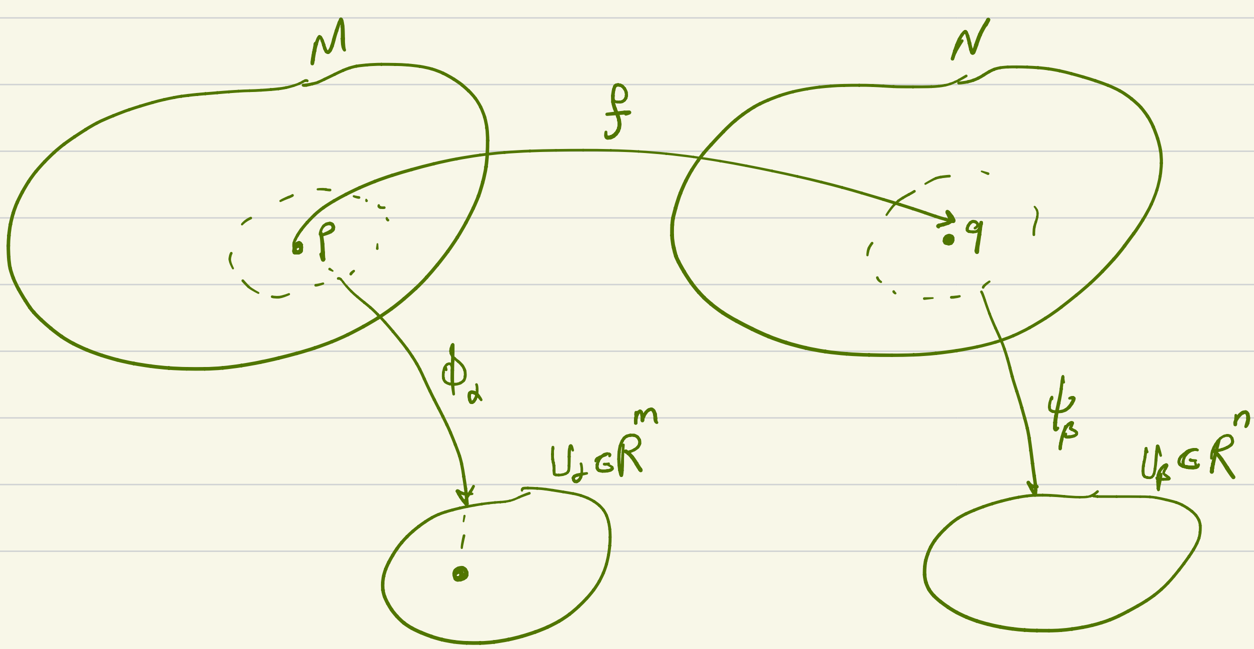
$(x,y,z,t) \xrightarrow{U_2^+} (x,z,t)$

$$(x,z,t)=(\sqrt{1-y^2-z^2-t^2},z,t)$$

مشتق‌پذیری توابع: $f: M \longrightarrow N$ یک تابع است.

به زبان ساده f مشتق‌پذیر است اگر وقتی برحسب مختصات بیان گردد یک تابع مشتق‌پذیر از R^m به R^m باشد.

$m=\dim M$ $\qquad$ $n=\dim N$.



f is diff at p. $\quad \psi_\beta \circ f \circ \phi_\alpha^{-1}$ is diff at $\phi_\alpha(p)$

f is diff on M if it is diff at all pts.

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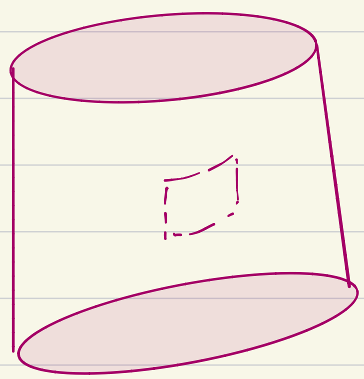
② let M is a Manifold & N is also a manifold.

Is $M \times N$ a manifold?

Let $\{(U_\alpha, \phi_\alpha)\}$ is an Atlas on M.

$\{(V_\beta, \psi_\beta)\}$ " " " N.



Define the Atlas on $M \times N$.

$M \times N = \{ (p,q) \mid p \in M, q \in N \}$.

$\{ (U_\alpha \times V_\beta), (\phi_\alpha, \psi_\beta) \}$. is an atlas on $M \times N$.

قبلا دیدیم.

X, Y as topological spaces are homeomorphic if

$\exists\, f : X \to Y$, $f^{-1} : Y \to X$ and both f & f^{-1} are continuous.

Def. M, N as manifolds are diffeomorphic if

$\exists\ f: M \to N,\ f^{-1}: N \to M \mid f$ & f^{-1} are differentiable

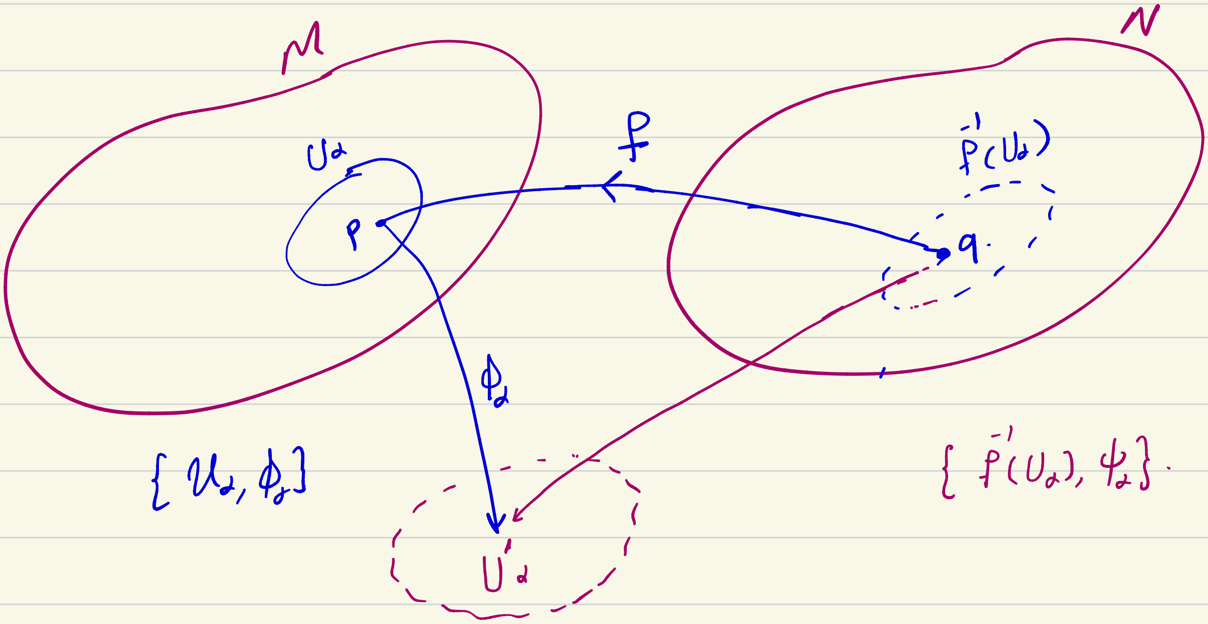
S^1 is diff. to ellipse

S^1 is not diff to △.

تمرین: آیا مثلث یک خمینه‌ی هموار است؟ به طور دقیق ثابت کنید.
[illegible]

سوال: اگر دو فضای توپولوژیک همئومورف باشند آیا اطلس اولی را می‌توان به یک اطلس روی دومی تبدیل کرد؟

تمرین:



$\{U_\alpha, \phi_\alpha\}$ $\qquad$ $\{f^{-1}(U_\alpha), \psi_\alpha\}$.

$$\underbrace{\psi_\alpha(q) := (\phi_\alpha \circ f)(q)} = \phi_\alpha(p).$$

$$\boxed{\psi_\alpha(q) := (\phi_\alpha \circ f)(q).} \qquad \psi_\beta(q) := (\phi_\beta \circ f)(q).$$

$\psi_\alpha^{-1} :== (f^{-1} \circ \phi_\alpha^{-1})$ $\quad (\psi_\beta \circ \psi_\alpha^{-1})(x) = \phi_\beta \circ f \circ (f^{-1} \circ \phi_\alpha^{-1}) \overset{?}{\equiv} \phi_\beta \circ \phi_\alpha^{-1}$