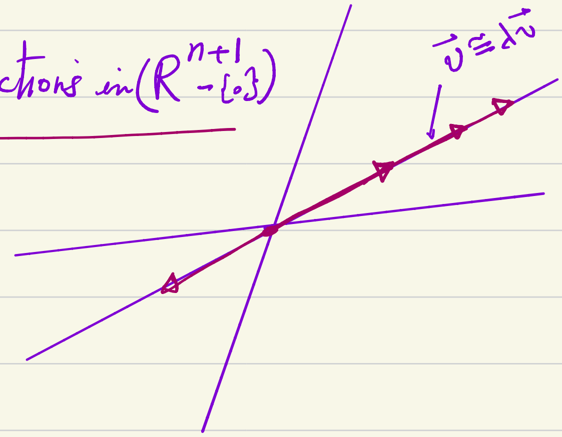
جلسه ی دوم ۹۹,۲,۲

Manifold. مشکل دار از

$\mathbb{R}P^n$ = the set of directions in $(\mathbb{R}^{n+1} - \{0\})$



$$U_1 = \{(x_1, x_2 \ldots x_{n+1}) \mid x_1 \neq 0\}$$

$$U_k = \{(x_1, x_2, \ldots x_{n+1}) \mid x_k \neq 0\}$$

in U_1: $\quad \xi^2_{(1)} = \dfrac{x_2}{x_1} \quad \xi^3_{(1)} = \dfrac{x_3}{x_1}, \ldots \quad \xi^{n+1}_{(1)} = \dfrac{x_{n+1}}{x_1}$

$\mathbb{R}P^n$ is an n-dimensional manifold.

$\mathbb{R}P^3$: $\quad U_1 = (x, y, z, t) \quad x \neq 0$

in U_1: $\xi_2 = \dfrac{y}{x} \quad \xi_3 = \dfrac{z}{x} \quad \xi_4 = \dfrac{t}{x}$

in U_2: $\eta_1 = \dfrac{x}{y} \quad \eta_3 = \dfrac{z}{y} \quad \eta_4 = \dfrac{t}{y}$

in U_3: $\zeta_1 = \dfrac{x}{z} \quad \zeta_2 = \dfrac{y}{z} \quad \zeta_4 = \dfrac{t}{z}$

in U_4: $\rho_1 = \dfrac{x}{t} \quad \rho_2 = \dfrac{y}{t} \quad \rho_3 = \dfrac{z}{t}$

in $U_1 \cap U_2 \quad x \neq 0,\ y \neq 0$

سوال: آیا $(\xi_2 \ \xi_3 \ \xi_4) \longrightarrow (\eta_1 \ \eta_3 \ \eta_4)$ بطور مشتق پذیر؟

$$\eta_1 = \frac{1}{\xi_2} \qquad \eta_3 = \frac{\xi_3}{\xi_2} \qquad \eta_4 = \frac{\xi_4}{\xi_2} \qquad (\xi_2 \neq 0, \neq \infty)$$

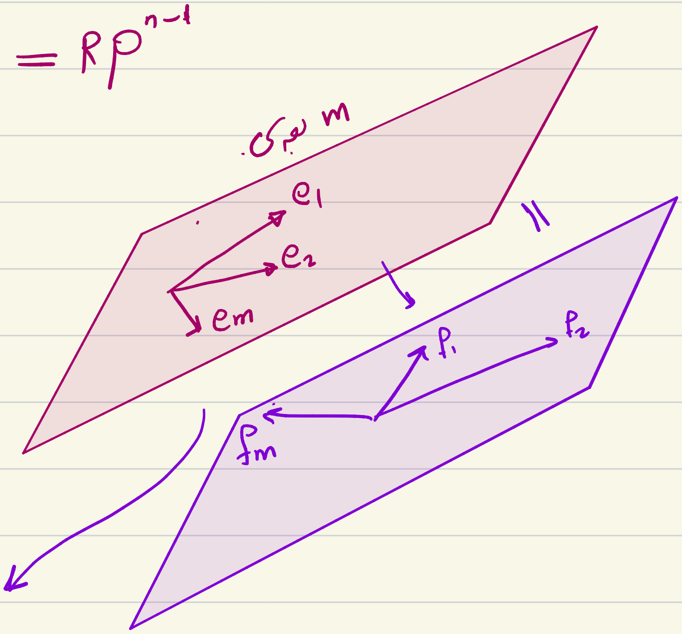
Grassmann Manifold $\xleftarrow{\text{تعمیم}}$ RP^n

is the set of m dimensional planes in R^n $\qquad \underline{m < n}$

$G_{n,m}$ = Grassman Manifold.

$G_{n,1} = RP^{n-1} \qquad G_{n,n-1} = RP^{n-1}$

سوال: $G_{n,m}$ چه تعبیری است؟



e_1, e_2, e_m مرتب مستقل خطی باشند

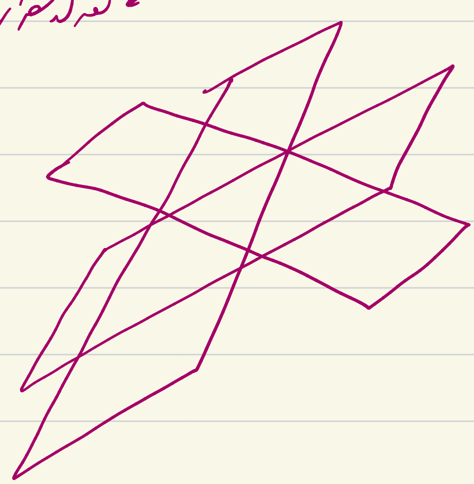
e_i is an n-dimensional vectors.

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = \underbrace{\begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ e_{m1} & e_{m2} & \cdots & e_{mn} \end{bmatrix}}_{A} \; m \times n.$$

mn coordinate?

مستقل خطی بودن بردارها

که تعداد سطرهای غیرصفر $= m$.



$$A = \begin{bmatrix} e_1 \\ e_m \end{bmatrix} \qquad A' = \begin{bmatrix} f_1 \\ f_m \end{bmatrix}$$

چه ترتیب صحیح ترکیب خطی f_i ها که بصورت ترکیب خطی

e_i ها نوشت؟

یعنی: $\quad f_i = \sum_j (G_{ij}) e_j$ $\qquad$ درایه

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} & & \\ & G & \\ & & \end{bmatrix}_{m\times m} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \qquad G \in GL(m,\mathbb{R})$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ||

گروه ماتریس‌های معکوس‌پذیر حقیقی

$G_{n,m}$ تعریف: $\longrightarrow \quad A = \begin{bmatrix} e_{11} & & e_{1n} \\ \vdots & & \\ e_{m1} & & e_{mn} \end{bmatrix}_{m\times n}$

Form the $m\times m$ minors of A: $\qquad$ همه‌ی اینا همزمان = صفر نشن.

How many minors: ? $\quad \binom{n}{m}$

U_1 = the open set where $|A_1| \neq 0 \longrightarrow A = (\underset{\#0}{A_1}, \tilde{A}_1) \longrightarrow (I, \underbrace{A_1^{-1}\tilde{A}_1}_{m\times(n-m)})$

U_2 = " " " $|A_2| \neq 0$

$\vdots$

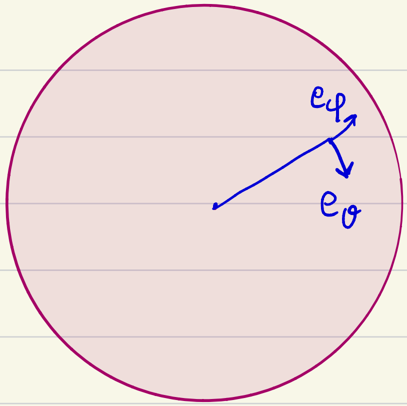
$U_{\binom{n}{m}}$ = " " $|A_{\binom{n}{m}}| \neq 0$

$$\boxed{\dim G_{n,m} = m(n-m)}$$

$$\dim G_{n,m} = \dim G_{n,n-m}$$

Optional exercise: See if you can find

Coordinate transformation between different charts.



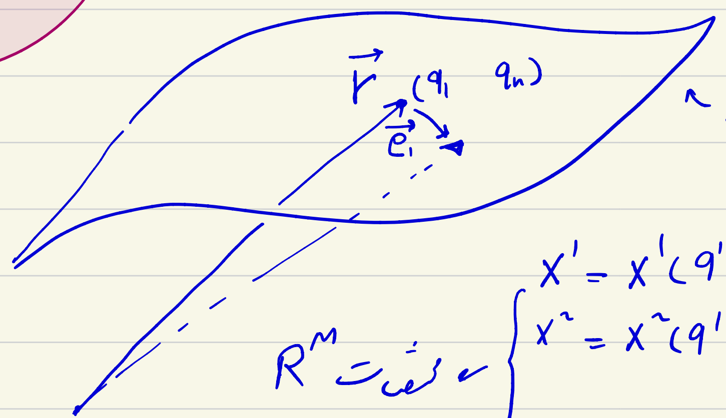
$X = \cos\theta \cos\phi$

$Y = \cos\theta \sin\phi$

$Z = \sin\theta$

$e_\theta := \dfrac{\partial \vec{r}}{\partial \theta}$

$e_\varphi = \dfrac{\partial \vec{r}}{\partial \varphi}$



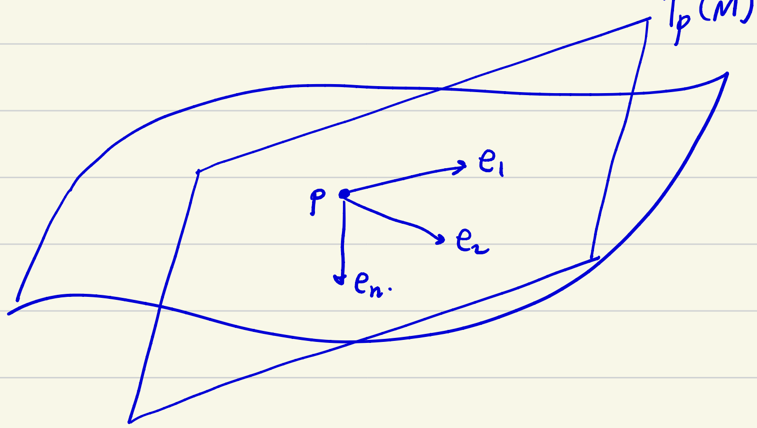
$\leftarrow$ M = Manifold

n-dim.

R^M فضای $\leftarrow$

$$\begin{cases} X^1 = X^1(q^1 - q^n) \\ X^2 = X^2(q^1 \quad q^n) \\ \\ X^M = X^M(q^1 \quad q^n) \end{cases}$$

$e_i := \dfrac{\partial \vec{r}}{\partial q^i}$ بردار مماس بر M



$T_p(M)$ = tangent space of M at p

$= \mathrm{Span}\{e_1, \quad e_n\}$

if $V \in T_p(M) \longrightarrow V = V^i e_i$

if you change the coordinates: $q^i \longrightarrow q^{i'}$ $e_i' = \dfrac{\partial \vec{r}}{\partial q^{i'}}$

$$e_i' = \frac{\partial \vec{r}}{\partial q^j} \frac{\partial q^j}{\partial q^{i'}} \Rightarrow e_i' = \frac{\partial q^j}{\partial q^{i'}} e_j$$

$$V = V^i e_i = V^{i'} e_{i'} = V^{i'} \frac{\partial q^j}{\partial q^{i'}} e_j \to$$

$$\boxed{V^j = V^{i'} \frac{\partial q^j}{\partial q^{i'}}}$$

$(T_pM)^*$ = the dual vector space of T_pM with the basis $\{e^i\}$.

$\langle e^i, e_j \rangle = \delta^i_j$ Any $\alpha \in T_pM^*$: $\alpha = \alpha_i e^i$

Ex: Find the transformation property of e^i & α_i.

$$q^i \longrightarrow q^{i'} \quad \boxed{V^j = V^{i'} \frac{\partial q^j}{\partial q^{i'}}} \xrightarrow[\text{Notation}]{\text{Dirac}} \underbrace{V^i = V^{i'} \frac{\partial q^i}{\partial q^{i'}}}$$

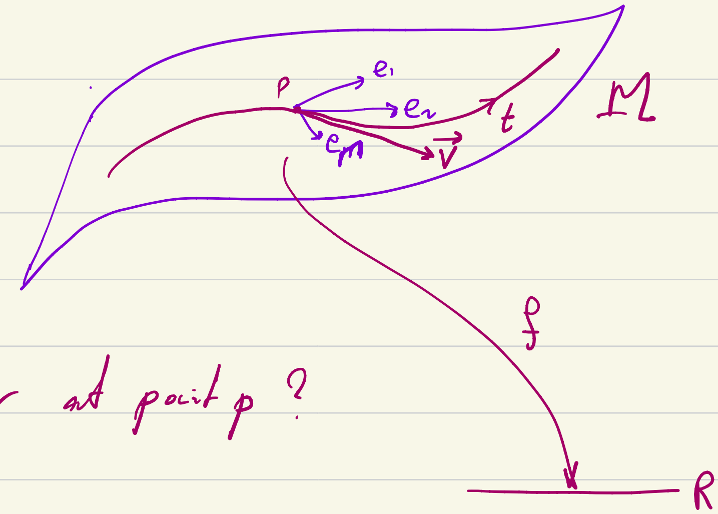
$$e_i = e_{i'} \frac{\partial q^{i'}}{\partial q^i} \qquad e^i = e^{i'} \frac{\partial q^i}{\partial q^{i'}}$$

$$e^{i'} = e^i \frac{\partial q^{i'}}{\partial q^i} \qquad \alpha_i = \alpha_{i'} \frac{\partial q^{i'}}{\partial q^i} \quad \text{etc.}$$

Tensor $\underbrace{T_pM \otimes T_pM \otimes T_pM \cdots}_{r} \otimes \underbrace{T_pM^* \otimes \cdots T_pM^*}_{s} = (T_pM)^r_s$

$\omega \in (T_pM)^r_s$: $\omega = \omega^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} e_{i_1} \otimes \cdots e_{i_r} \otimes e^{j_1} \otimes \cdots e^{j_s}$.

$\gamma: \quad \vec{r}(t) = (x^1(t), x^2(t), \quad x^n(t))$



at $t=0 \quad \vec{r}(0) \equiv P$

What is the tangent vector on γ at point p?

$$\vec{V} = \left.\frac{\partial \vec{r}}{\partial t}\right|_{t=0} = \dot{q}^i \frac{\partial \vec{r}}{\partial q^i} = \dot{q}^i e_i$$

Let $f : M \longrightarrow R$ be a differentiable function.

مشتق تابع f در امتداد منحنی γ در نقطه p چیست؟

$$\left.\frac{df}{dt}\right|_{t=0} = \frac{d}{dt} f(q^1(t), \quad q^n(t)) = \frac{\partial f}{\partial q^i} \dot{q}^i$$

$$\frac{df}{dt} = \dot{q}^i \frac{\partial f}{\partial q^i}$$ directional Derivative of f Along γ. $\quad (f \circ \phi^{-1}) \Rightarrow f$

Coordinate.

تناظر یک

$$\frac{d}{dt} = \dot{q}^i \frac{\partial}{\partial q^i} \quad (1)$$ " " of Any function.
" γ.

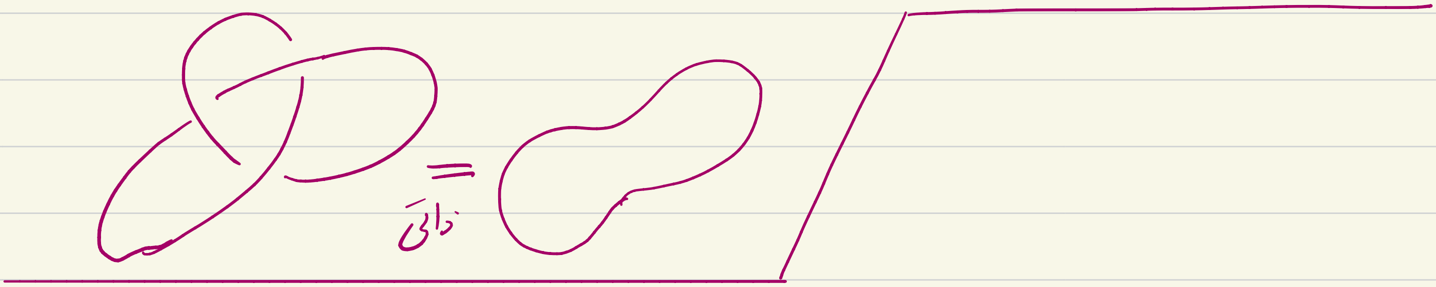
$$\vec{V} = \left.\frac{\partial \vec{r}}{\partial t}\right|_{t=0} = \dot{q}^i e_i \quad (2)$$

(1): $\quad V = V^i \dfrac{\partial}{\partial q^i} \qquad V(f) = V^i \left.\dfrac{\partial f}{\partial q^i}\right|_p \qquad$ ذاتی

(2): $\quad V = V^i e_i \qquad$ یک بردار بر حسب بردار پایه دلخواه $\qquad$ غیرذاتی

خواص ذاتی (Intrinsic) = خواصی که موجود روی خمینه حس می‌کند مستقل از نحوهٔ نشاندن آن در R^n.

خواص غیرذاتی (Extrinsic) = " " " خمینه با کمک به فضای خارجی حس می‌کند.



* Def: A tangent vector X at p is an operator $X: C^\infty(M) \longrightarrow R$

which is a derivation:

1) X is linear: $\begin{cases} X(f+g) = X(f) + X(g) \\ X(cf) = cX(f) \quad \forall\, c = \text{Const}. \end{cases}$

2) $X(fg) = f(p)\,X(g) + X(f)\,g(p)$.

thm: the set of Derivations at p is a vector space.

proof: let X & Y be derivations — $\begin{cases} \text{is } X+Y \text{ a derivation?} \\ \text{is } cX \text{ a deriv?} \end{cases}$

$$(X+Y)(f) := X(f) + Y(f).$$

i) $(X+Y)(cf+g) = X(cf+g) + Y(cf+g) =$

$= c\,X(f) + X(g) + c\,Y(f) + Y(g) =$

$= c\,\big(X(f)+Y(f)\big) + X(g)+Y(g)$

$= c\{(X+Y)(f)\} + \{(X+Y)(g)\}$

$X+Y$ is a linear operator.

ii) $(X+Y)(fg) = X(fg) + Y(fg) = X(f)\,g(p) + f(p)\,X(g) + Y(f)\,g(p) + f(p)\,Y(g)$

$= \{(X+Y)(f)\}\,g(p) + f(p)\,\{(X+Y)(g)\} \longrightarrow$ $X+Y$ is a derivation.

T_pM = the vectorspace of Derivations at p.

A Basis for T_pM. in a local chart with coordinates $(q^1 \dots q^n)$

Def ∂_i as follows: $\partial_i : f \longrightarrow \left.\dfrac{\partial f}{\partial q^i}\right|_p \equiv \dfrac{\partial f}{\partial q^i}$

ضرم

درزمان.

Q: Is ∂_i a derivation?

there are n of these derivations.

"We should prove that $\{\partial_i\}$ form a basis for T_pM.