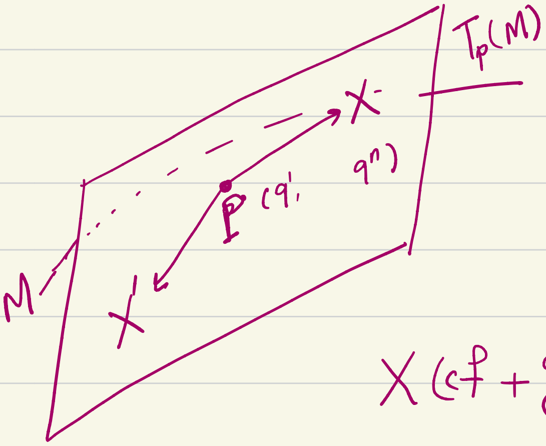
یکشنبه ۹۹٫۲٫۷ جلسه‌ی پنجم



$T_p(M) \ni X : C^1(M) \longrightarrow R$

$\rightarrow \quad X(f) = X^i \left(\frac{\partial f}{\partial q^i}\right)_p \in R. \qquad \partial_i \in T_p(M)$

$X(cf+g) = cX(f) + X(g)$

$X(fg) = f(p)\,X(g) + X(f)\,g(p)$

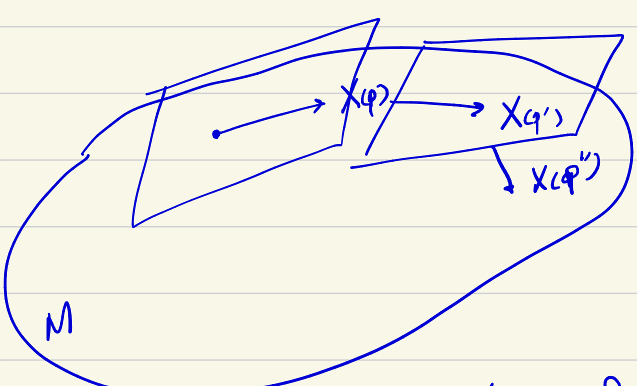
$X = X^i \partial_i \equiv X^i \frac{\partial}{\partial q^i} \rightarrow \quad X = X^i \frac{\partial}{\partial q^{i'}} \frac{\partial q^{i'}}{\partial q^i} = \left(X^i \frac{\partial q^{i'}}{\partial q^i}\right) \frac{\partial}{\partial q^{i'}}$

$\rightarrow \quad X^{i'} = X^i \frac{\partial q^{i'}}{\partial q^i}$

$(T_p M)^* : T_p(M) \longrightarrow R$

$\forall \quad \alpha \in T_p(M)^* : \qquad \alpha(X) \in R \qquad X \in T_p(M)$

$\alpha(cX+Y) = c\alpha(X) + \alpha(Y).$



$X =$ میدان برداری

$X : C^\infty(M) \longrightarrow C^\infty(M). \quad \Leftarrow$

$X_p : f \longrightarrow X_p(f) \in R \quad \leftarrow$

$X_{p'} : f \longrightarrow X_{p'}(f) \in R \quad \leftarrow$

$$X_{p''}: f \longrightarrow X_p''(f) \in \mathbb{R} \quad \leftarrow$$

$$\vdots \qquad\qquad \vdots$$

$X \in \mathfrak{J}(M)$ فضای میدان‌های برداری روی M $\qquad X : C^1(M) \longrightarrow C^1(M)$

$$\begin{cases} X(\underbrace{cf+g}_{\text{تابع}}) = \underbrace{c}_{\text{عدد}}\, \underbrace{X(f)}_{\text{تابع}} + \underbrace{X(g)}_{\text{تابع}} & ① \\ X(fg) = \underbrace{X(f)}_{\text{تابع}}\, g + f\, \underbrace{X(g)}_{\text{تابع}} & ② \end{cases}$$

$$X_p(fg) = \underbrace{f(p)}_{\text{عدد}} X(g) + X(f) \underbrace{g(p)}_{\text{عدد}}$$

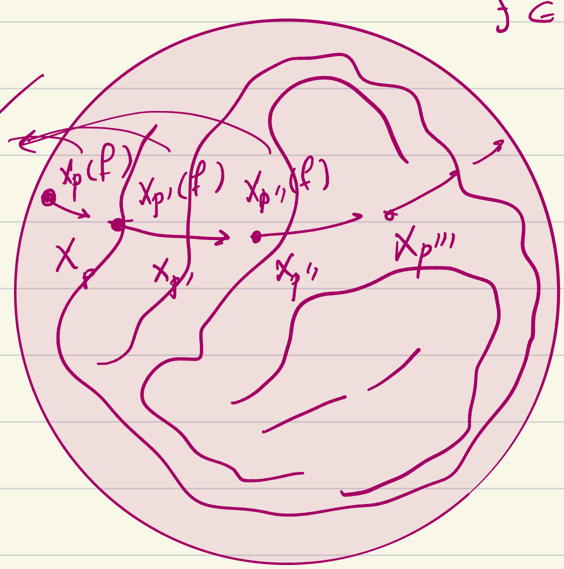
$$X = X^i \frac{\partial}{\partial q^i}$$

$$X = x^i(q^1 \ldots q^n) \frac{\partial}{\partial q^i}$$

thm: if X & Y are two vector fields — $cX+Y$ is also a vector field.

proof:
$$\begin{cases} (X+Y)(f) := X(f) + Y(f) \\ (cX)(f) := c\, X(f) \end{cases} \qquad f \in C^\infty(M)$$

یک تابع جدید



$\downarrow$

خیلی ساده‌ست که نشان بدیم

$$\begin{cases} X+Y \\ cX \end{cases}$$

هم میدان برداری هستند.

حل: $(X+Y)(fg) = X(fg) + Y(fg) = f\,X(g) + X(f)\,g + f\,Y(g) + Y(f)\,g$

$$= f\,\big(X(g)+Y(g)\big) + \big(X(f)+Y(f)\big)\,g$$

$$= f\,(X+Y)(g) + (X+Y)(f)\,g.$$

$X \in \mathcal{J}(M) \qquad Y \in \mathcal{J}(M)$

$XY(f) := X(Y(f)) \longrightarrow$ کی : ضرب

$$(XY)(f+g) = X(Y(f+g)) = X\big(Y(f)+Y(g)\big) = X(Y(f)) + X(Y(g))$$

$$= (XY)(f) + (XY)(g)$$

$$(XY)(fg) = X(Y(fg)) = X\big[\,Y(f)\,g + f\,Y(g)\,\big] =$$

$$= X(Y(f))\,g + Y(f)\,X(g) + X(f)\,Y(g) + f\,X(Y(g))$$

$$= (XY)(f)\,g + \quad '' \quad '' \quad + f\,(XY)(g)$$

$$(YX)(f) = (YX)(f)\,g + \quad '' \quad '' \quad + f\,(YX)(g)$$

$\rightarrow \boxed{[X,Y] \in \mathcal{J}(M)}$ Lie Bracket.

$\mathcal{X}(M)$ is an infinit dimensional Lie Algebra.

Def: Lie Algebra: A vector space V + a lie bracket:

$[\ \ ]: V \times V \longrightarrow V$

1) Bilinear
2) Antisymmetric.
3) Jacobi Identity.

Exampls of Lie Algebra:

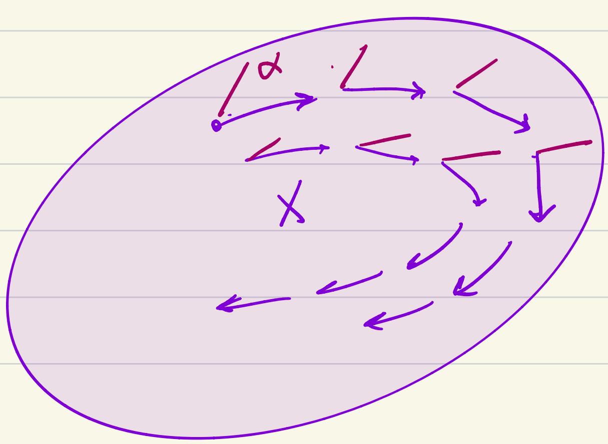
1) R^3 $\quad \vec{X} \times \vec{Y} = [\vec{X}, \vec{Y}]$.

2) $M_n(R) \subseteq M_n(C)$ $\quad [A,B] = AB - BA$

3) $C^\infty(M_{2n})$ $\quad \{f,g\} := \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$

Differential forms: $\alpha \in T_p^*M$, $X \in T_p(M)$ $\quad \langle \alpha, X \rangle \in R$

$\alpha \in \mathcal{X}^*(M)$ $\quad X \in \mathcal{X}(M)$



$\langle \alpha, X \rangle$ is a function $\in C^\infty(M)$.

form.

let f be a function to define the following form:

تعریف $\longrightarrow$ $\langle \tilde{f}, X \rangle := X(f) \in C^\infty(M)$.

$c \in \mathbb{R}$ $\quad \langle \widetilde{cf}, X \rangle = X(cf) = c\,X(f)$

$$\langle \tilde{f}, X+Y \rangle = (X+Y)(f) = X(f) + Y(f) = \langle \tilde{f}, X \rangle + \langle \tilde{f}, Y \rangle.$$

$$\langle \tilde{f}, cX \rangle = (cX)(f) = c\,(X(f)) = c\langle \tilde{f}, X \rangle \overset{?}{=} c \langle \tilde{f}, X \rangle$$

$$\langle \widetilde{f+g}, X \rangle = \qquad \overset{?}{=} \langle \tilde{f}, X \rangle + \langle \tilde{g}, X \rangle$$

$$\langle \widetilde{fg}, X \rangle = X(fg) = X(f)\,g + f\,X(g)$$

$$= \langle \tilde{f}, X \rangle g + f \langle \tilde{g}, X \rangle = \langle \tilde{f} g + f \tilde{g}, X \rangle$$

$$\boxed{\widetilde{fg} = \tilde{f} g + f \tilde{g}}$$

$\tilde{f} \Longrightarrow df$ $\qquad d(fg) = (df)g + f(dg)$.

$\forall$ function $f \in C^\infty(M)$ define $df \in \mathcal{T}^*(M)$

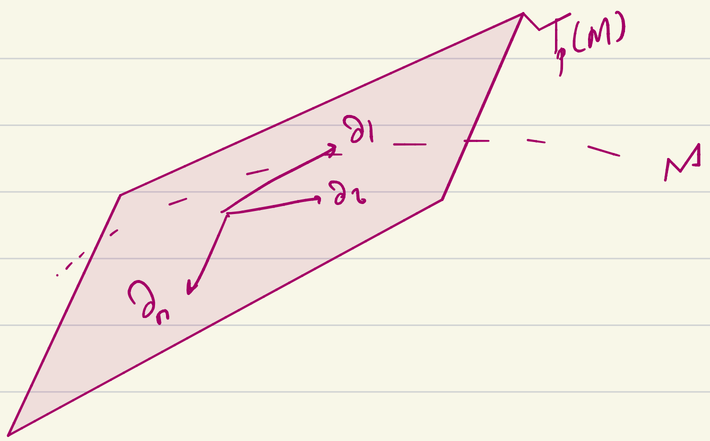
$$\langle df, X\rangle = X(f).$$

differential form $\longrightarrow$ Basis

$$\langle dq^i, \frac{\partial}{\partial q^j}\rangle = \frac{\partial}{\partial q^j}(q^i) = \delta^i_j$$

Every form $\quad \omega = \omega_i\, dq^i \qquad \omega = \omega_i(q^1 \cdots q^n)\, dq^i$

$$\omega = \omega_i\, dq^i = \omega_i \frac{\partial q^i}{\partial q^{i'}} dq^{i'} \longrightarrow \omega_{i'} = \omega_i \frac{\partial q^i}{\partial q^{i'}}$$



$T_p^*(M)$

دل نگرانی !!

$dq^1 \qquad dq^n$ $\quad$ چرا؟

$$\langle dq^i, \frac{\partial}{\partial q^j}\rangle = \delta^i_j$$

$$\| $$

$$\frac{\partial}{\partial q^j}(q^i) = \delta^i_j$$

$$\omega = \omega_{ij}\, dq^i \otimes dq^j$$

$$\alpha = \omega_{ij}{}^{k}\, dq^i \otimes dq^j \otimes \partial_k .$$

تعریفِ قدیمی تانسور، آنطور که [illegible] :

$$T^{mn}{}_{p} = \frac{\partial q^m\, \partial q^n}{\partial q^{m'}\, \partial q^{n'}} \frac{\partial q^{p'}}{\partial q^{p}} T^{m'n'}{}_{p'}$$