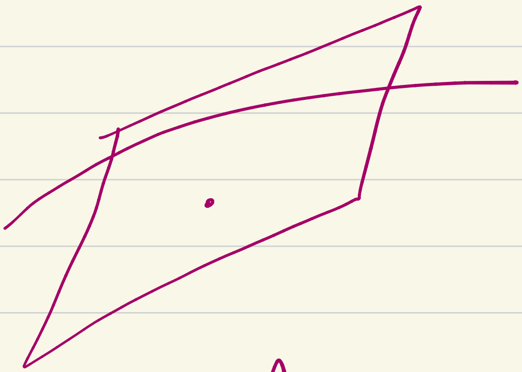
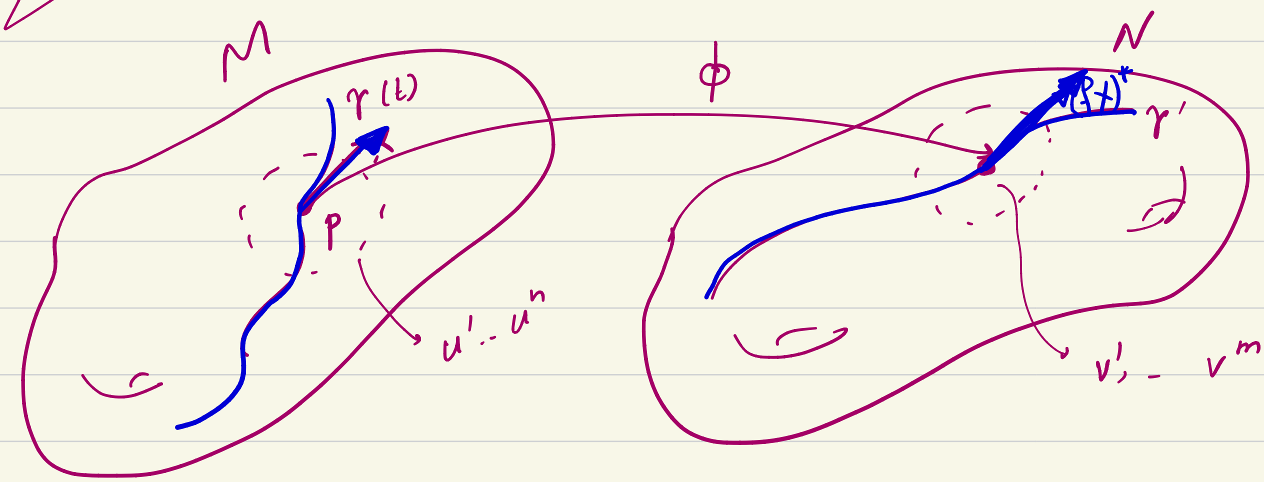
سه‌شنبه ۹,۲,۹۹ ـ جلسه بیست (۲)



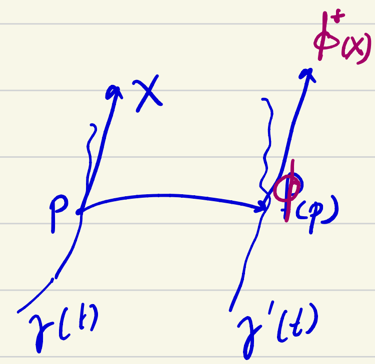
$$X = X^\mu \partial_\mu \qquad \partial_\mu = \frac{\partial}{\partial q^\mu}$$

$$\omega = \omega_\mu \, dq^\mu \qquad \langle dq^\mu, \partial_\nu \rangle = \delta^\mu_\nu \, .$$



$$\gamma(t) = (u^1(t), u^2(t), \dots u^n(t)) \qquad \gamma'(t) = (v^1(t), v^2(t), \dots v^m(t))$$

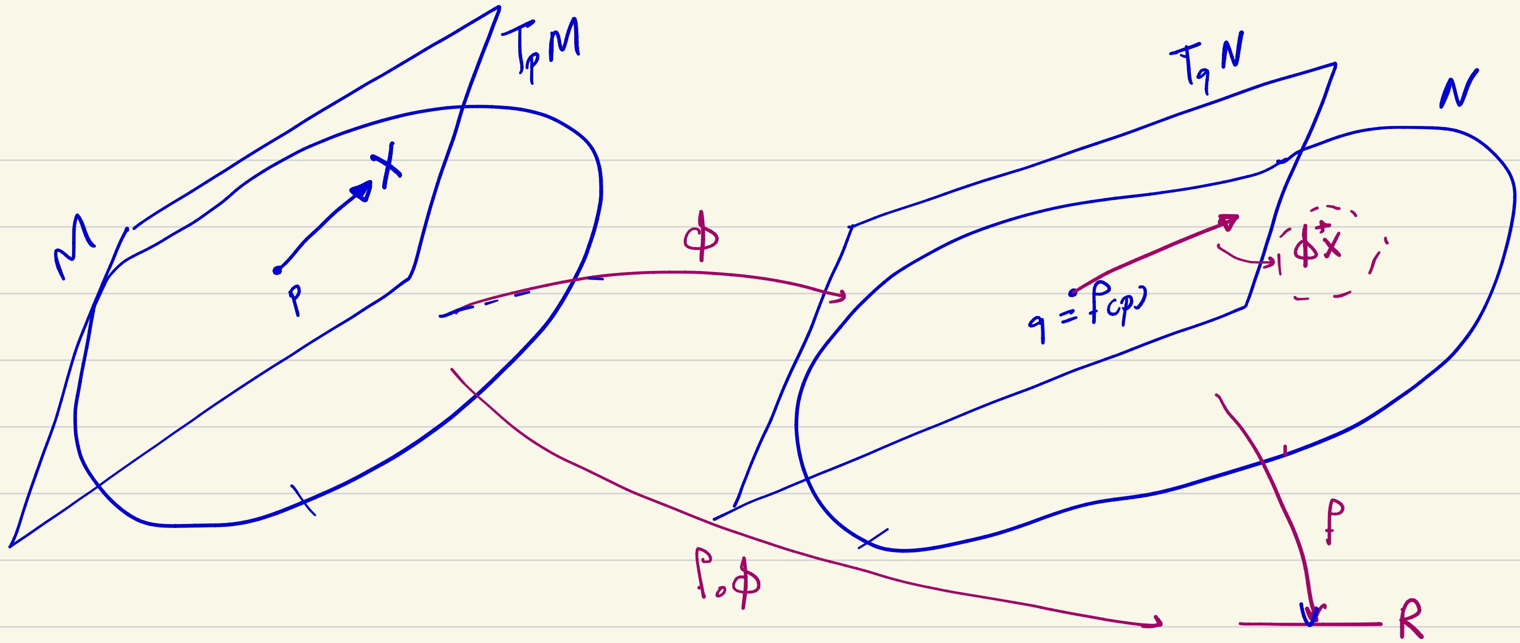
$$\gamma(0) = p \qquad \gamma'(0) = \phi(p)$$



$$\left.\frac{d\gamma}{dt}\right|_{t=0} = \frac{\partial \gamma}{\partial u^i}\left(\frac{du^i}{dt}\right) \qquad \boxed{\frac{du^i}{dt} = X^i}$$

$$\left.\frac{d\gamma'}{dt}\right|_{t=0} = \frac{\partial \gamma'}{\partial v^j}\left(\frac{\partial v^j}{\partial t}\right) \qquad \left(\frac{\partial v^j}{\partial t}\right)_{t=0} = (\phi^* X)^j = \frac{\partial v^j}{\partial u^i}\frac{\partial u^i}{\partial t}$$

$$\boxed{(\phi^* X)^j = \left(\frac{\partial v^j}{\partial u^i}\right) X^i}$$



$\phi^* X$ = push forward of X under ϕ.

$\phi^* X := ?$ $\qquad (\phi^* X)(f) := X(f \circ \phi)$

Is $(\phi^* X)$ a derivation? $\longrightarrow$ $\phi^*(X)(f+g) = X((f+g)\circ\phi)$

$= \cdots$

$= (\phi^* X)(f) + (\phi^* X)(g)$

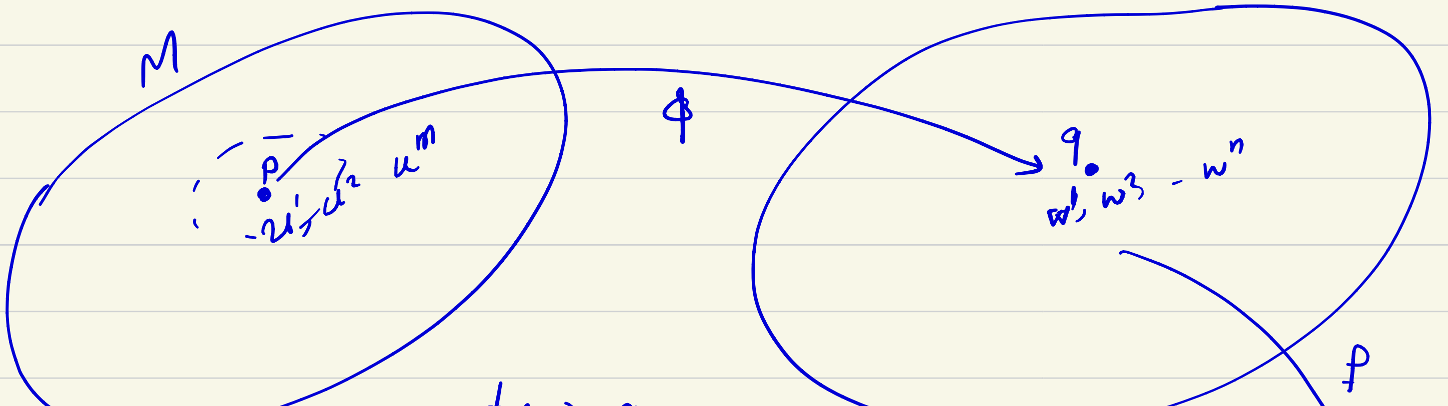
$\phi^*(X)(cf) = c\,\phi^*(X)(f)$

$\phi^*(X)(fg) = X((fg)\circ\phi) = X((f\circ\phi)(g\circ\phi)) = \cdots \longrightarrow$ Deriv.

تمرین

$(fg)\circ(\phi)(x) = (fg)(\phi(x)) = f(\phi(x))\, g(\phi(x)) = (f\circ\phi)(x)\,(g\circ\phi)(x).$

تمرین $\phi^* X$ برحسب مختصات.



$\phi(p) = q \longrightarrow \omega^i = \omega^i(u^1, u^2, \dots u^n)$

$$(\phi^* X)(f) = X(f \circ \phi)$$

$$= X^i \frac{\partial}{\partial u^i}(f \circ \phi) \circledast = X^i \frac{\partial}{\partial u^i}\underbrace{(f(\phi(u)))}_{} =$$

$$= X^i \frac{\partial f}{\partial \omega^j} \cdot \frac{\partial \omega^j}{\partial u^i} \longrightarrow = \left(X^i \frac{\partial \omega^j}{\partial u^i}\right) \frac{\partial f}{\partial \omega^j}$$

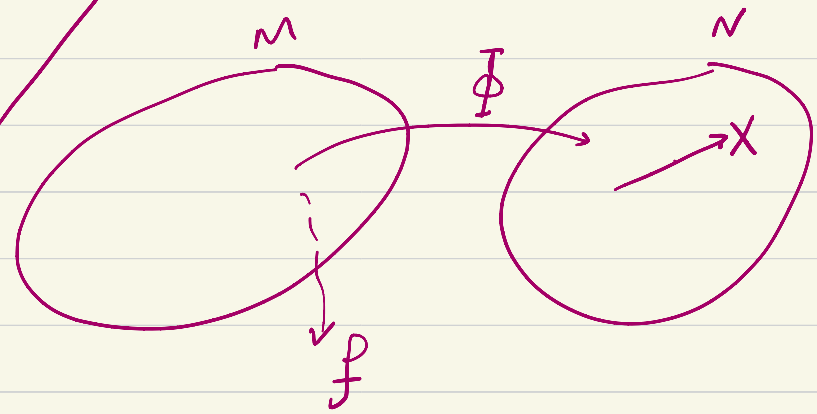
$f = f(\omega^1, \omega^2 \dots \omega^m)$

$f = f(\omega^1(u^1 \dots u^m), \omega^2(u^1, u^m), \dots \omega^n(u^1 \dots u^m))$
$= (f \circ \phi)(u^1, \dots u^m)$

$$\Longrightarrow \quad (\phi^* X)(f) = \left(X^i \frac{\partial \omega^j}{\partial u^i}\right) \frac{\partial f}{\partial \omega^j}$$

$$(\phi^* X)^j = X^i \frac{\partial \omega^j}{\partial u^i}$$

$\circledast$ $(\phi^* X)(f) = X^i \frac{\partial}{\partial u^i}(f \circ \phi)$



$(\phi_* X) : f \longrightarrow \mathbb{R}$

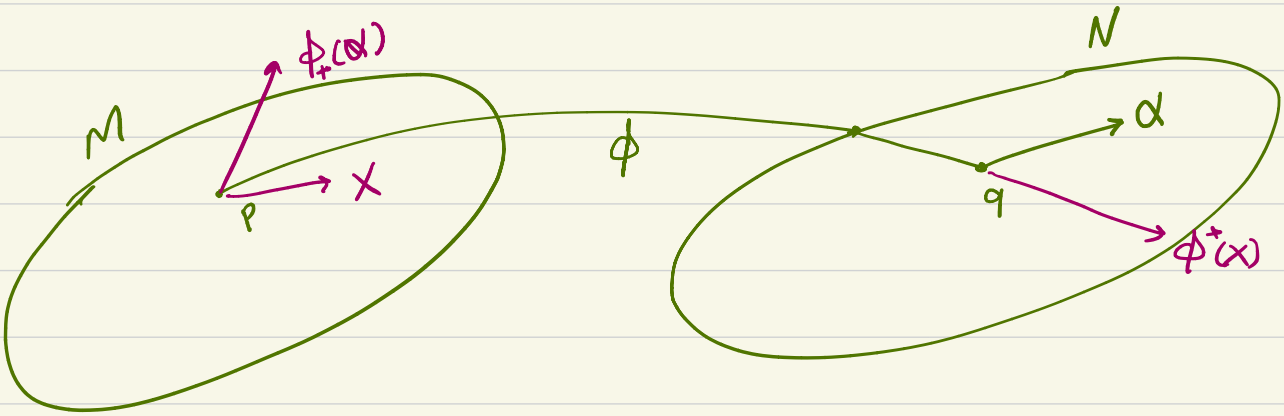
$= X(\phi \circ f)$? یعنی $\phi \circ f$ ج؟

$$X = X^i(u^1 \quad u^n)\frac{\partial}{\partial u^i} \qquad (u^1 \quad u^n) \xrightarrow{\phi} (\omega^1 - \omega^m)$$

$$(\phi^* X) = X^i(u^1 \quad u^n)\frac{\partial \omega^j}{\partial u^i}\frac{\partial}{\partial \omega^j}$$

جایگذاری $\omega^1 - \omega^m$.

pull back of forms (differential forms)



$$\langle \phi_*(\alpha), (X) \rangle := \langle \alpha, \phi^*(X) \rangle$$

Is $\phi_*(\alpha)$ a linear form? $\langle \phi_*(\alpha), X+Y \rangle = \langle \phi_*(\alpha) X \rangle + \langle \phi_*(\alpha), Y \rangle$

$$\langle \phi_*(\alpha), X+Y \rangle = \langle \alpha, \phi^*(X+Y) \rangle = \langle \alpha, \phi^*(X) \rangle + \langle \alpha, \phi^*(Y) \rangle$$
$$= \langle \phi_*(\alpha), X \rangle + \langle \phi_*(\alpha), Y \rangle.$$

$$\phi_*(\alpha+\beta) \overset{?}{=} \phi_*(\alpha) + \phi_*(\beta).$$

از نظر برداری $\phi_*(\alpha)$ که طوقه بیان کرد؟

let $\alpha = \alpha_i \, d\omega^i = \alpha_i(\omega^1, \quad \omega^n) \, d\omega^i$

$(u^1, \quad u^n)$ بطور کلی در هر فضایی : با مختصات محلی

با مختصات محلی u^i

$X = X^i \frac{\partial}{\partial u^i} \qquad \alpha = \alpha_i \, du^i$

$\langle du^i, \frac{\partial}{\partial u^j} \rangle = \delta^i_j \qquad \langle \alpha, X \rangle = \alpha_i X^i$

$\langle \alpha, \frac{\partial}{\partial u^i} \rangle = \alpha_i$ ① $\quad \langle du^i, X \rangle = X^i$

$\langle du^i, X^j \frac{\partial}{\partial u^j} \rangle = X^j \langle du^i, \frac{\partial}{\partial u^j} \rangle = X^i$

$[\phi_*(\alpha)]_i = ?$

$$[\phi_*(\alpha)]_i = \langle \phi_*(\alpha), \frac{\partial}{\partial u^i} \rangle = \langle \alpha, \phi^*(\frac{\partial}{\partial u^i}) \rangle =$$

① بنا بر رابطهٔ

$$= \langle \alpha, \frac{\partial}{\partial \omega^j} \frac{\partial \omega^j}{\partial u^i} \rangle = \frac{\partial \omega^j}{\partial u^i} \langle \alpha, \frac{\partial}{\partial \omega^j} \rangle$$

$$= \frac{\partial \omega^j}{\partial u^i} \alpha_j$$

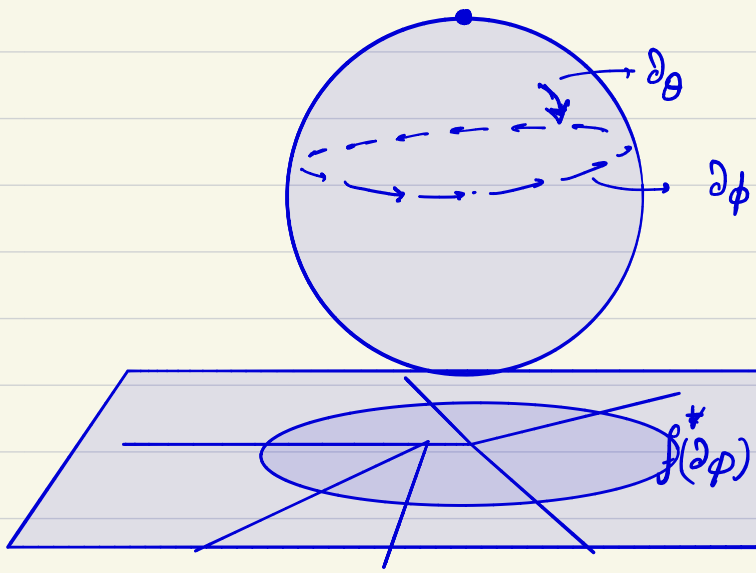
$[\phi_*(\alpha)]_i = \frac{\partial \omega^j}{\partial u^i} \alpha_j \rightsquigarrow \phi_*(\alpha) = [\phi_*(\alpha)]_i \, du^i =$

$$= \frac{\partial w^j}{\partial u^i} \alpha_j \, du^i \longrightarrow \quad \text{if } \alpha = \alpha_i \, dw^i$$

$$(\phi_+ \alpha) = \alpha_j \frac{\partial w^j}{\partial u^i} du^i$$

$$= \alpha_i \frac{\partial w^i}{\partial w^j} dw^j$$

تمرین: معادلات ژئودزیک را در نظر بگیرید.



R^2 : مختصات ریاضی (x,y)

$$f : (x,y) \longrightarrow (x^2, xy) = (x', y').$$

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$(f^* X) = x\left[\frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'}\right] - y\left[\frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x}\right]$$

$$= x\left[0 + x\frac{\partial}{\partial y'}\right] - y\left[2x\frac{\partial}{\partial x'} + y\frac{\partial}{\partial y'}\right]$$

$$= -2xy\frac{\partial}{\partial x'} + (x^2 - y^2)\frac{\partial}{\partial y'}$$

$$\boxed{(f^+ X) = -2y'\frac{\partial}{\partial x'} + \left(x' - \frac{y'^2}{x'^2}\right)\frac{\partial}{\partial y'}}$$

let $\omega = x'dx' + y'dy'$

$$(f_* \omega) = x'\left(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\right) + y'\left(\frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy\right)$$

$$= x'(2x\,dx) + y'(y\,dx + x\,dy)$$

$$= (2xx' + yy')\,dx + y'x\,dy$$
