$$(f_* \omega) = x'\left(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\right) + y'\left(\frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy\right)$$

$$= x'(2x\,dx) + y'(y\,dx + x\,dy)$$

$$= (2xx' + yy')\,dx + y'x\,dy$$

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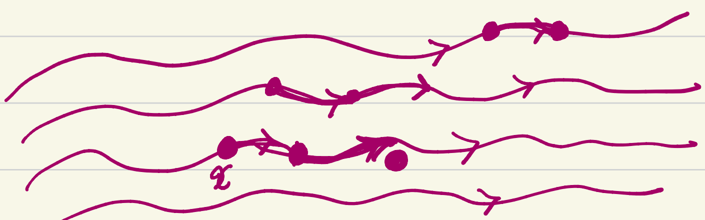
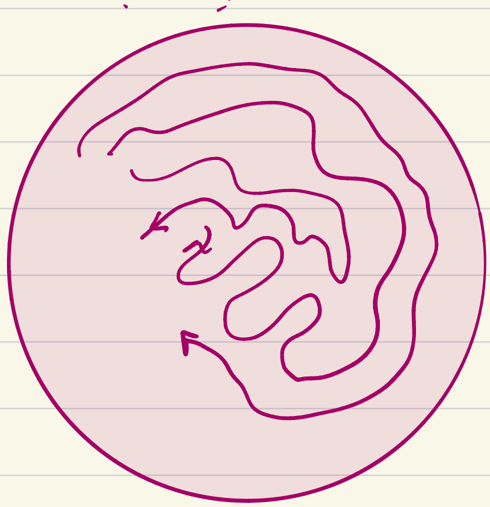
Flow : جریان



جریان آب

جریان باد



if M is a Manifold

$\alpha(t) : M \to M$ ~~یک~~ نگاشت در:

خانواده یک پارامتری پیوسته، مشتق‌پذیر و معکوس‌پذیر

1) $\alpha(0) = id_M$

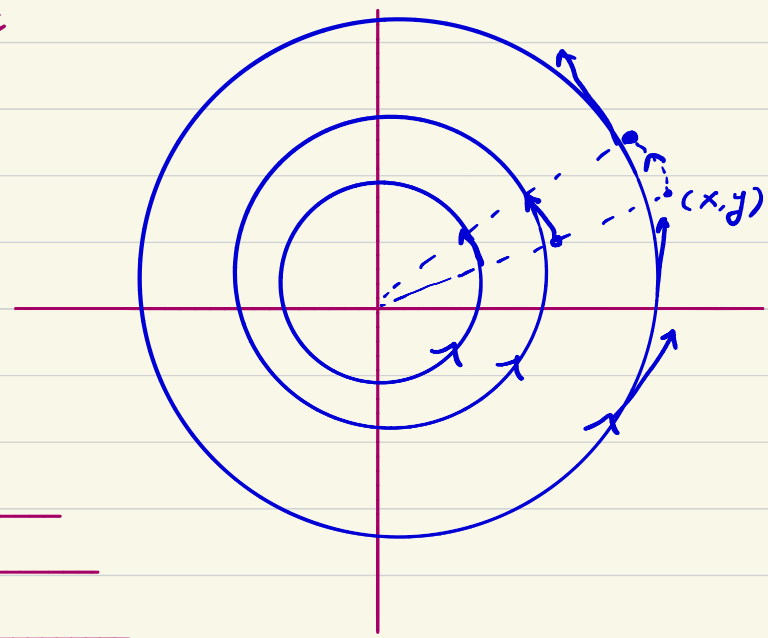
2) $\alpha(t) \circ \alpha(s) = \alpha(t+s)$ Flow : جریان

3) $\alpha^{-1}(t) = \alpha(-t)$

4) α is differentiable with respect to t.

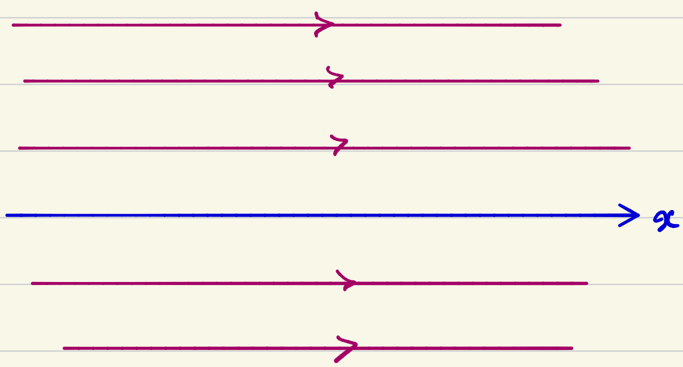
Example 1) $M = R^2$ $(x, y) \longrightarrow (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$

$\theta \Longleftrightarrow t$ زمان

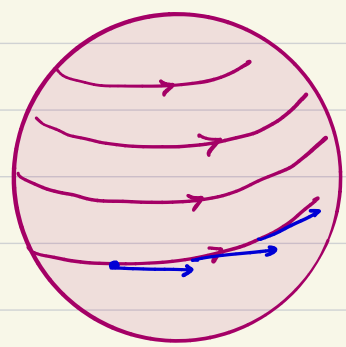


Example 2)

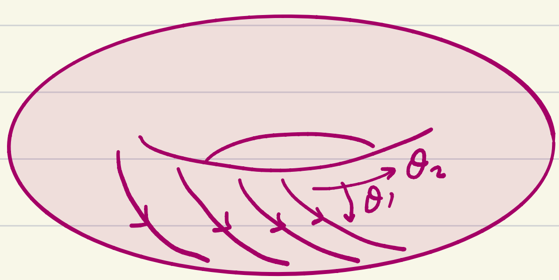
$M = R^2 : (x,y) \xrightarrow{\alpha(t)} (x+t, y)$



Example 3). $M = S^2$ $\alpha(t) : (\theta, \varphi) \longrightarrow (\theta, \varphi + t)$



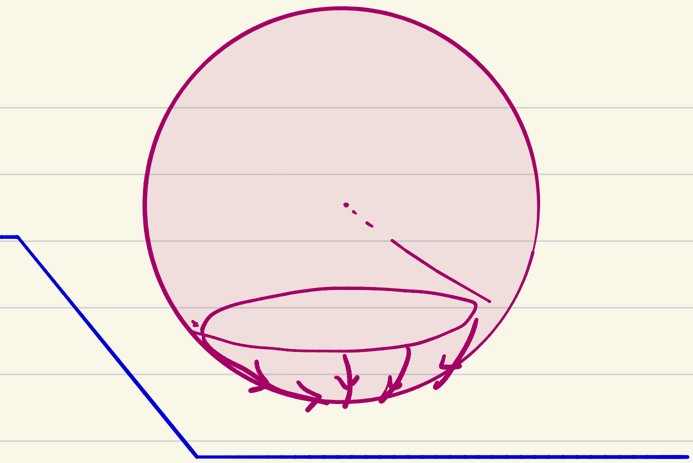
Example 4) $M = \Sigma_1 =$ چنبره‌ی جنس 1 $= S_1 \times S_1$



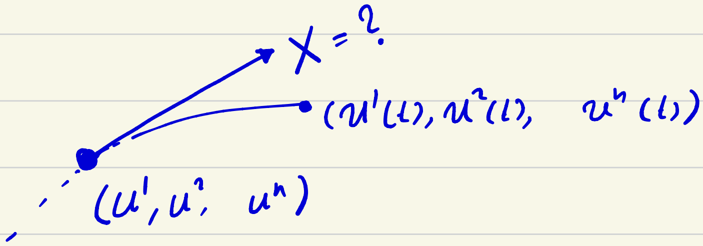
$(\theta_1, \theta_2) \longrightarrow (\theta_1 + t, \theta_2 + \alpha t)$

Counter example: 5) $M = S^2$ $(\theta, \varphi) \xrightarrow{\alpha(t)} (\theta + t, \varphi)$

$$(r-t,\varphi)\xrightarrow{\alpha(t)}(r,\varphi)$$



میدان برداری وابسته به یک جریان ؟



$$u^1(t)=u(t,\ u_1,u_2,\ \ u_n)$$
$$u^2(t)=u(t,\ u_1,u_2,\ \ u_n)$$
$$u^n(t)=u(t,\ u_1,u_2\ \ \ u_n)$$

$$X=X^r\frac{\partial}{\partial u^r}$$

$$u^i(0)=u^i$$

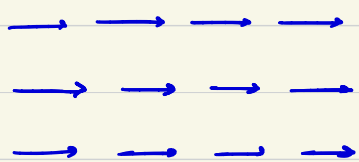
$$=X^r(u^1,\ \ u^n)\frac{\partial}{\partial u^r}$$

$$X:=\left.\frac{du^r(t)}{dt}\right|_{t=0}\frac{\partial}{\partial u^r}\qquad \frac{d}{dt}f(t)=\frac{du^r}{dt}\frac{\partial f}{\partial u^r}=X^r\frac{\partial}{\partial u^r}f=X(f)$$

Ex: R^2: $\alpha(t):(x,y)\longrightarrow(x+t,y)\ \Rightarrow\ X=X^1\frac{\partial}{\partial x}+X^2\frac{\partial}{\partial y}$

$$X^1=\frac{d}{dt}(x(t))=\frac{d}{dt}(x+t)=1\qquad X^2=\frac{d}{dt}x^2(t)=\frac{d}{dt}y=0$$

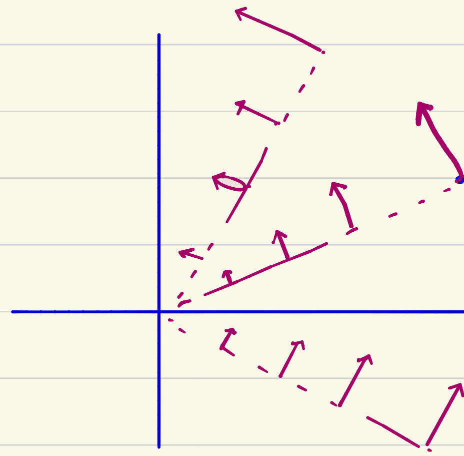
$$X=\frac{\partial}{\partial x}$$



Ex: R^2: $\alpha(t):(x,y)\longrightarrow(x\cos t+y\sin t,\ -x\sin t+y\cos t)$

$$X = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = (-x\sin t + y\cos t)\frac{\partial}{\partial x} + (-x\cos t + y\sin t)\frac{\partial}{\partial y}\Big|_{t=0}$$

$$\underbrace{X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}}$$



دوران: $(x,y,z) \longrightarrow (x\cos t + y\sin t,\ -y\sin t + x\cos t,\ z)$.

مثال: دوران حول z

$$J_z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \longrightarrow J_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \qquad J_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

تا الان فهمیدیم: Flow $\longrightarrow$ Vector Field.

Flow $\longleftarrow$ Vector Field

Let X be a vector field on a manifold: $X = X^\mu \frac{\partial}{\partial u^\mu}$

$u^\mu \longrightarrow X = X^\mu \frac{\partial}{\partial u^\mu}$

فلسفه: برای به اندازه‌ی کافی کوچک ϵ در امتداد X جلو می‌رویم.

$\downarrow$

$$\alpha(\epsilon):\ u^\mu \longrightarrow u^\mu + \epsilon X^\mu \quad (1) \quad \longleftarrow \quad X^\mu = \frac{du^\mu(\epsilon)}{d\epsilon}$$

So we defined $\alpha(\epsilon)$. How we define $\alpha(t)$?

$\downarrow t = $ finite.

$$\alpha(2\epsilon) = \alpha(\epsilon)\,\alpha(\epsilon) \qquad \alpha(N\epsilon) = [\alpha(\epsilon)]^N$$

$$\boxed{\alpha(t) := \lim [\alpha(\tfrac{t}{N})]^N}$$

Sol. let $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ we want to find the flow of X.

$$① \rightarrow \quad (x,y) \xrightarrow{\epsilon} (x + \epsilon X^1,\ y + \epsilon X^2)$$

$$(x,y) \longrightarrow (x + \epsilon(-y),\ y + \epsilon(x)) \equiv (x - \epsilon y,\ y + \epsilon x)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x - \epsilon y \\ y + \epsilon x \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

for finit t: $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{t}{N} \\ \frac{t}{N} & 1 \end{pmatrix}^N \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$N \longrightarrow \infty$$

چند تمرین ساده:

$$\mathbb{R}^2 \qquad \partial_x,\ \partial_y$$

$$\downarrow$$

مختصات دکارتی (x,y)

$$\begin{cases} L = x\,\partial_y - y\,\partial_x \\ P_x = \dfrac{\partial}{\partial x} \qquad P_y = \dfrac{\partial}{\partial z} \\ S = x\,\partial_x + y\,\partial_y \\ T = x\,\partial_x - y\,\partial_y \end{cases}$$

$$A = (x^2 + y^2)\,\partial_x - \frac{x}{y^2}\,\partial_y$$

$f: R^2 \longrightarrow R^2$ $\qquad \left(f_* \frac{\partial}{\partial x}\right) = ?$ $\quad \left(f_* \frac{\partial}{\partial y}\right) = ?$

$(x, y) \longrightarrow (r, \theta)$

$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$ $\qquad (x, y) \longrightarrow \left(r = \sqrt{x^2 + y^2},\ \phi = \mathrm{tg}^{-1}\frac{y}{x}\right)$

$$\left(f_* \frac{\partial}{\partial x}\right) = \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r} \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial \varphi}{\partial x} = \frac{-y}{r^2}$$

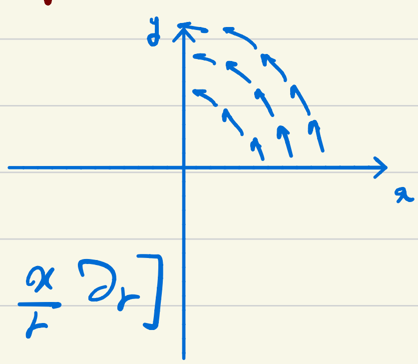
$$\left(f_* \frac{\partial}{\partial x}\right) = \frac{-y}{r^2}\frac{\partial}{\partial \varphi} + \frac{x}{r}\frac{\partial}{\partial r} = -\frac{\sin\phi}{r}\frac{\partial}{\partial \varphi} + \cos\phi\frac{\partial}{\partial r} \qquad (1)$$

$$\left(f_* \frac{\partial}{\partial y}\right) = \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r} = \frac{x}{r^2}\frac{\partial}{\partial \varphi} + \frac{y}{r}\frac{\partial}{\partial r} \quad \leftarrow$$

$$= \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi} + \sin\phi\frac{\partial}{\partial r} \qquad (2)$$

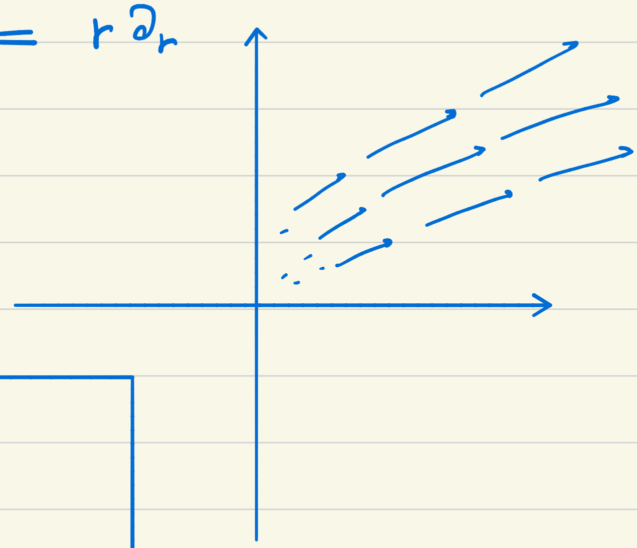
$$(f_* L) = f_*(x\,\partial_y - y\,\partial_x) = x\left[\frac{x}{r^2}\partial_\varphi + \frac{y}{r}\partial_r\right] - y\left[\frac{-y}{r^2}\partial_\varphi + \frac{x}{r}\partial_r\right]$$

$$\boxed{(f_* L) = \partial_\varphi} \qquad L = L_z$$



$$(f_* S) = f_*(x\,\partial_x + y\,\partial_y) = x\left[\frac{-y}{r^2}\partial_\varphi + \frac{x}{r}\partial_r\right]$$

$$+ y \left[\frac{x}{r^2} \partial_\varphi + \frac{y}{r} \partial_r \right] = r \partial_r$$



$$(f_* g) = r\partial_r$$

Bracket of two vector fields.

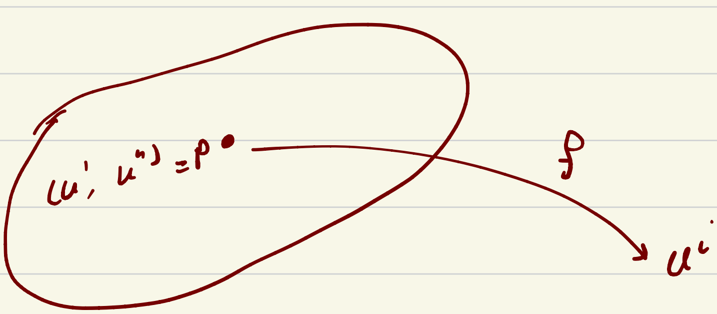
$X \,,\, Y$

XY = معمولی $\qquad (XY)(f) = X(Y(f))$ $\qquad$ این یک Der. نیست.

$$[X, Y] = XY - YX \longrightarrow [X,Y](f) := X(Y(f)) - Y(X(f))$$

$$[X,Y] = [X,Y]^i \frac{\partial}{\partial u^i} \qquad X(f) = X^i \frac{\partial f}{\partial u^i}$$

u^i is itself a function. $\quad u^i : p \longrightarrow u^i(p)$.



$$X(u^i) = X^j \frac{\partial u^i}{\partial u^j} = X^i$$

$$X^i = X(u^i)$$

$$[X,Y]^i = [X,Y](u^i) = X(Y(u^i)) - Y(X(u^i))$$

$$= X(Y^i) - Y(X^i)$$

$$= X^j \partial_j Y^i - Y^j \partial_j X^i \qquad \text{We are done}$$

مثل: on $\mathbb{R}^3$: $\quad P_i = \dfrac{\partial}{\partial u^i} \equiv \partial_i \qquad L_i = \epsilon_{ijk}\, u^j \dfrac{\partial}{\partial u^k}$

$$L_x = y\,\partial_z - z\,\partial_y \qquad L_y = z\,\partial_x - x\,\partial_z \qquad L_z = x\,\partial_y - y\,\partial_x$$

$$[L_x, L_y] = [y\,\partial_z - z\,\partial_y,\; z\,\partial_x - x\,\partial_z] = \dots\; L_z.$$

تمرین: if X is a vector field & $g : M \longrightarrow \mathbb{R}$ a function

is gX a vector field.

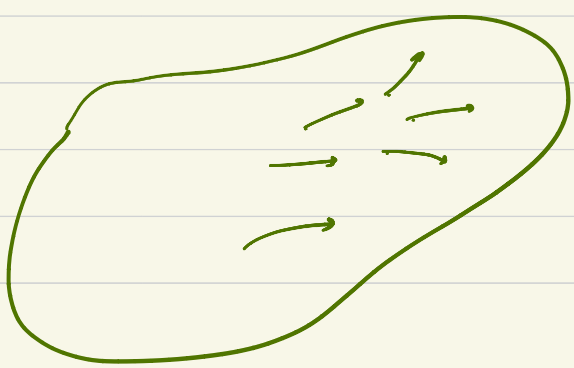
$$(gX)(f) := g\,X(f) \qquad [(gX)(f)](u) := g(u)\,[X(f)](u$$

به طور دقیق‌تر

Check:

① $(gX)(f+f') \overset{?}{=} (gX)(f) + (gX)(f')$

$$g\{X(f) + X(f')\} = g\,X(f) + g\,X(f')$$



② $(gX)(ff') \stackrel{?}{=} (gX)(f)\, f' + f\, (gX)(f')$

$$(gX)(ff') = g\{X(ff')\} = g\{\underbrace{X(f)\,f' + f\,X(f')}\}$$

$f_{(x)}\, g_{(x)} = g_{(x)} f_{(x)}$ ¿?

$$= (gX)(f)\, f' + f\,(gX)(f').$$

$$[X, \phi Y](f) = X[(\phi Y)(f)] - (\phi Y)(X(f)).$$

$$= X\{\underline{\phi}\ \underline{Y(f)}\} - \phi\, Y(X(f))$$

$$= X(\phi)\, Y(f) + \phi\, X(Y(f)) - \phi\, Y(X(f))$$

$$\rightarrow \boxed{[X, \phi Y] = X(\phi)\, Y + \phi\, [X, Y]}$$ ①

$$[L_x, L_y] = [y\partial_z - z\partial_y,\ z\partial_x - x\partial_z]$$

$$\underbrace{[y\partial_z, z\partial_x]} = y\{\underbrace{(\partial_z z)}_{1}\partial_x + z\,\underbrace{[\partial_z, \partial_x]}_{0}\} + \cdots$$

$$y\,\partial_x + \cdots$$

$\mathcal{H}$ = Hilbert space. $\quad \underbrace{[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k} \longrightarrow$

$$[\phi X, Y] = -[Y, \phi X] = -Y(\phi)X - \phi[Y, X]$$

$$\boxed{[\phi X, Y] = \phi[X, Y] - Y(\phi)X}$$ ②

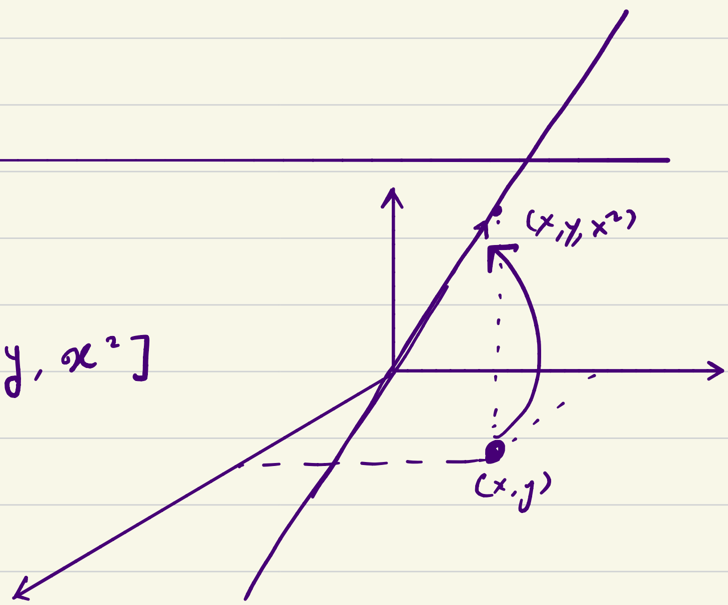
تمرین: با استفاده از روابط ①, ② این براکت‌ها را حساب کنید.

$[L_i, L_j]$, $[L_i, P_j]$

$[S, P_i]$.

مثال: $f: R^2 \longrightarrow RP^2$

$(x, y) \longmapsto [x, y, x^2]$



$\partial_x \longrightarrow ?$

$\partial_y \longrightarrow ?$

local coordinates of $RP^2 = ?$. $R^3 = (x^0, x^1, x^2)$

$[x^0, x^1, x^2] = \{\lambda x^0, \lambda x^1, \lambda x^2\}$

$U_0 : x^0 \neq 0$ $\quad \xi^1_{(0)} := \dfrac{x^1}{x^0}$ $\quad \xi^2_{(0)} = \dfrac{x^2}{x^0}$

$U_1 : x^1 \neq 0$ $\quad \xi^0_{(1)} = \dfrac{x^0}{x^1}$ $\quad \xi^2_{(1)} = \dfrac{x^2}{x^1}$

$U_2 : x^2 \neq 0$ $\quad \xi^0_{(2)} = \dfrac{x^0}{x^2}$ $\quad \xi^1_{(2)} = \dfrac{x^1}{x^2}$

$\boxed{\xi^j_{(i)} = \dfrac{x^j}{x^i}}$ بطور کلی

چرا: $\xi^j_{(i)} = 1/\xi^i_{(j)}$.

$$(x,y) \xrightarrow{f} [(x,y,x^2)].$$

$$(x,y) \longrightarrow \quad \text{if } x \neq 0 \quad \zeta^1_{(0)} = \frac{y}{x} \quad \zeta^2_{(0)} = \frac{x^2}{x} = x.$$

$$\text{if } y \neq 0 \quad \zeta^0_{(1)} = \frac{x}{y} \quad \zeta^2_{(1)} = \frac{x^2}{y}$$

?

$$(f_* \partial_x) = ? \xrightarrow{\text{in } U_0} (f_*\partial_x) = \frac{\partial \zeta^1_{(0)}}{\partial x}\frac{\partial}{\partial \zeta^1_{(0)}} + \frac{\partial \zeta^2_{(0)}}{\partial x}\frac{\partial}{\partial \zeta^2_{(0)}}$$

$$(f_* \partial_y) = ?$$

$$= \frac{-y}{x^2}\frac{\partial}{\partial \zeta^1_{(0)}} + \frac{\partial}{\partial \zeta^2_{(0)}}$$

$$(f_*\partial_x) = \frac{-\zeta^1_{(0)}}{\zeta^2_{(0)}}\frac{\partial}{\partial \zeta^1_{(0)}} + \frac{\partial}{\partial \zeta^2_{(0)}} \quad \text{in } U_0 \quad 1$$

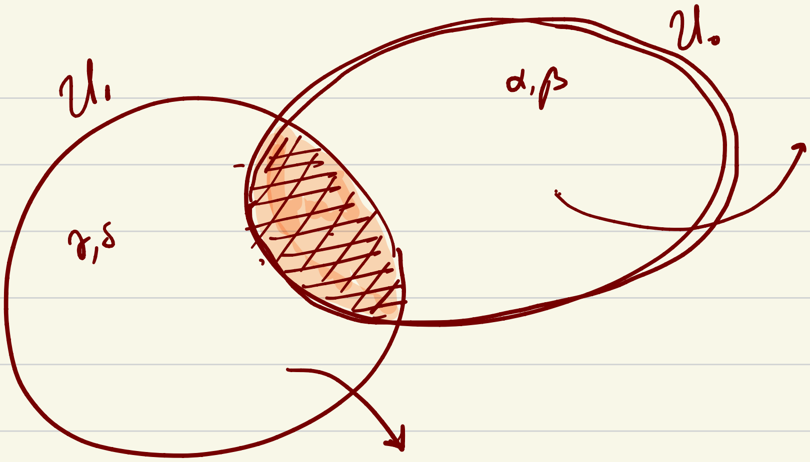
$$\text{in } U_1: \quad (f_*\partial_x) = \frac{\partial \zeta^0_{(1)}}{\partial x}\frac{\partial}{\partial \zeta^0_{(1)}} + \frac{\partial \zeta^2_{(1)}}{\partial x}\frac{\partial}{\partial \zeta^2_{(1)}}$$

2

$$= \frac{1}{y}\frac{\partial}{\partial \zeta^0_{(1)}} + \frac{2x}{y}\frac{\partial}{\partial \zeta^2_{(1)}} = \underbrace{\frac{(\zeta^2_{(1)})^2}{\zeta^0_{(1)}}\frac{\partial}{\partial \zeta^0_{(1)}} + 2\zeta^0_{(1)}\frac{\partial}{\partial \zeta^2_{(1)}}}_{\text{in } U_1.}$$

$$\text{if in } U_0: \quad \zeta^1_{(0)} = \alpha \quad \zeta^2_{(0)} = \beta$$

$$\text{\& in } U_1: \quad \zeta^0_{(1)} = \gamma \quad \zeta^2_{(1)} = \delta$$



$$(\varphi_* X) = \frac{-\alpha}{\beta}\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}.$$

$$(\varphi_* X) = \frac{\delta^2}{\gamma}\frac{\partial}{\partial \gamma} + 2\gamma\frac{\partial}{\partial \delta}$$
