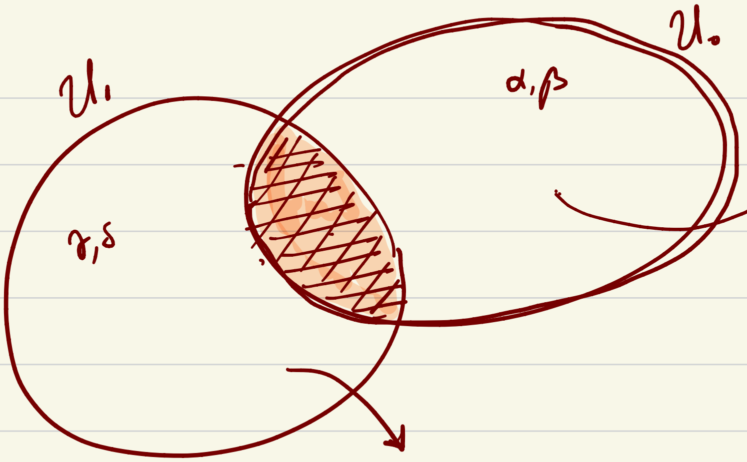


$$(f_* X) = \underbrace{\frac{-\alpha}{\beta}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}}.$$

$$(f_* X) = \underline{\frac{\delta^2}{\gamma}\frac{\partial}{\partial\gamma} + 2\gamma\frac{\partial}{\partial\delta}}$$

جلسهٔ بیست و سوم _ ۱۴ اردیبهشت ۹۹

Flow: $\quad \sigma(t): M \longrightarrow M \qquad \sigma(t)$ is a diffeomorphism:

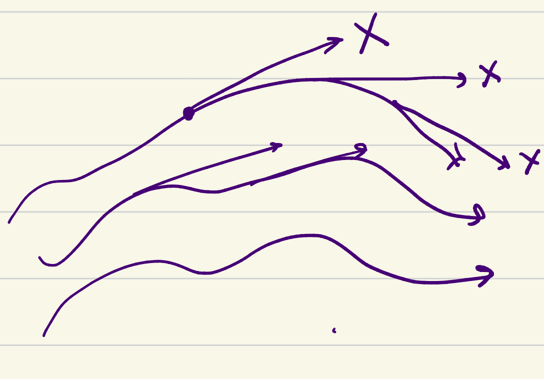
$\sigma: \mathbb{R}\times M \longrightarrow M \qquad \sigma$ is a diffeomorphism.

i) $\quad \sigma(t)\circ\sigma(s) = \sigma(t+s) \qquad \sigma(t, \sigma(s,x)) = \sigma(t+s,x).$

ii) $\quad \sigma(0) = id_M$

iii) $\quad \sigma(t)^{-1} = \sigma(-t). \qquad$ A one-parameter group of diffeomorphisms.

Flow $\longleftrightarrow$ Vector field

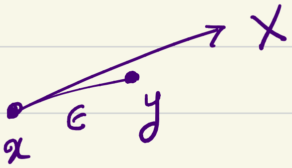


مختصات را با x^i نشان دهیم.

$$\sigma(t) : \sigma_t \qquad \begin{cases} x \xrightarrow{\sigma_t} y = \sigma_t(x) \\ x^\mu \longrightarrow y^\mu = \sigma_t^\mu(x) = \sigma_t^\mu(x^1, x^2, x^n) \end{cases}$$

if t is infinitesimal $= \epsilon$

$$\begin{cases} x \xrightarrow{\sigma_\epsilon} y = x + \epsilon X \quad \text{میدان برداری} \\ x^\mu \longrightarrow y^\mu = x^\mu + \epsilon X^\mu = \sigma_\epsilon^\mu(x) \end{cases}$$



$$\left.\frac{d\sigma_t^\mu}{dt}\right|_{t=0} = X^\mu \quad : \quad \left.\frac{d\sigma_\epsilon^\mu}{d\epsilon}\right|_{\epsilon \to 0} = X^\mu$$

$$y^\mu = x^\mu + \epsilon X^\mu(x) = (\underbrace{id + \epsilon X}) ^\mu(x)$$

for finite t:

$$y^\mu = \left(\left[id + \frac{t}{N} X \right]^N\right)^\mu(x) \equiv \underline{\underline{e^{tX} x^\mu}}$$

for $t = 2\epsilon$

$$\begin{aligned} y &= \sigma_\epsilon \circ (\sigma_\epsilon(x)) \\ &= \sigma_\epsilon[x + \epsilon X(x)] \\ &= [x + \epsilon X] + \epsilon X(\underbrace{x + \epsilon X}_{\text{نقطه}}) \end{aligned}$$

بردار

تمرین:

① $\mathcal{L}_X(Y) = ?$

② $\mathcal{L}_X(\omega) = ?$

③ $\mathcal{L}_X(f) = ?$

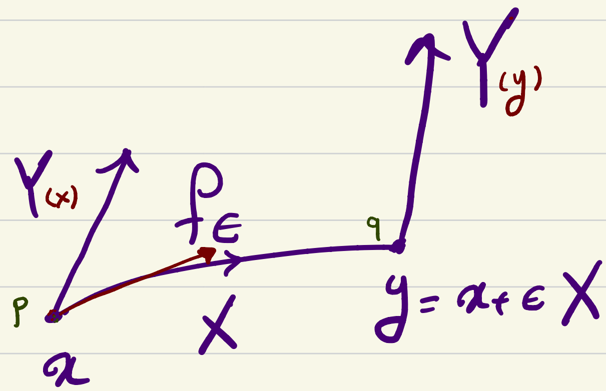
④ $\mathcal{L}_X(T) = ?$

① $\mathcal{L}_X(Y)$.

وابسته به X یک فلو وجود دارد f

$$\mathcal{L}_X(Y) = \frac{[(f_{-\epsilon})_* Y(y)](x) - Y(x)}{\epsilon}$$

مؤلفه‌های $\mathcal{L}_X(Y)$ را پیدا کنیم.



$$\mathcal{L}_X(Y) = \underbrace{[\mathcal{L}_X(Y)]^\mu}_{\text{مؤلفه‌های این را پیدا کنیم}} \frac{\partial}{\partial x^\mu}$$

$$y^\mu = x^\mu + \epsilon X^\mu$$

$$Y(x) = Y^\mu(x) \frac{\partial}{\partial x^\mu} \qquad Y(y) = Y^\mu(y) \frac{\partial}{\partial y^\mu}$$

$$Y(y) = Y^\mu(x + \epsilon X) \frac{\partial}{\partial y^\mu} \quad \in T_q M$$

$$y^\mu = x^\mu + \epsilon X^\mu$$

$$[(f_{-\epsilon})_* Y(y)](x) = \underbrace{Y^\mu(x + \epsilon X)} \underbrace{\frac{\partial x^\nu}{\partial y^\mu}} \frac{\partial}{\partial x^\nu}$$

$$= \underbrace{\left(Y^\mu(x) + \epsilon X^\alpha \frac{\partial Y^\mu}{\partial x^\alpha}\right)\left(\delta^\nu_\mu - \epsilon \frac{\partial X^\nu}{\partial x^\mu}\right)}_{[(f_{-\epsilon})_* Y]^\nu(x)} \frac{\partial}{\partial x^\nu}$$

$$= \left(Y^\nu + \epsilon X^\alpha \frac{\partial Y^\nu}{\partial x^\alpha} - \epsilon Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu}$$

$$\Rightarrow \boxed{(\mathcal{L}_X Y)^\nu = X^\alpha \frac{\partial Y^\nu}{\partial x^\alpha} - Y^\alpha \frac{\partial X^\nu}{\partial x^\alpha}} \; *$$

$$\boxed{\mathcal{L}_X Y = [X, Y]} \Rightarrow \quad \mathcal{L}_{fX}(Y) = [fX, Y] = f[X,Y] - Y(f) X$$

$\to \quad [ab, c] = a[b,c] + [a,c]b.$

$$\mathcal{L}_X(fY) = [X, fY] = f[X,Y] + X(f) Y.$$

$$[X, f] :\equiv X(f).$$

$$[X,f]g = X(fg) - fX(g) = X(f)g + \underbrace{f\cancel{X(g)} - \cancel{f X(g)}} = X(f)g$$

$$\boxed{[X, f] = X(f)}$$

Lemma:

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X. \qquad \text{proof:}$$

$$\mathcal{L}_{[X,Y]}(Z) = [[X,Y], Z] = -[[Y,Z],X] - [[Z,X],Y]$$

$$= [X, [Y, Z]] - [Y, [X, Z]]$$

$$= [X, \mathcal{L}_Y Z] - [Y, \mathcal{L}_X Z]$$

$$= \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z \Rightarrow L_{[X,Y]} = L_X L_Y - L_Y L_X .$$

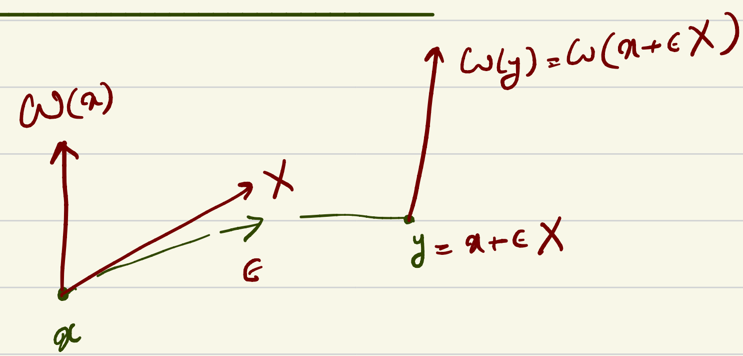
مثال:

$$J_x = y\partial_z - z\partial_y \qquad J_y = z\partial_x - x\partial_z \qquad J_z = x\partial_y - y\partial_x .$$

$$P_x = \partial_x \qquad P_y = \partial_y \qquad P_z = \partial_z .$$

$$\mathcal{L}_{P_x}(J_y) = [P_x, J_y] = [\partial_x, z\partial_x - x\partial_z] = -\partial_z = -P_z .$$

مشتق لی



$$\mathcal{L}_X \omega := \frac{1}{\epsilon}\left\{ [(f_\epsilon)^* \omega](x) - \omega(x) \right\}.$$

$$\omega(x) = \omega_\mu(x)\, dx^\mu \qquad \omega(y) = \omega_\nu(y)\, dy^\nu \longleftarrow$$

$$[(f_\epsilon^*)\omega] = \omega_\nu(x + \epsilon X)\frac{\partial y^\nu}{\partial x^\alpha} dx^\alpha$$

$$= \left[\omega_\nu(x) + \epsilon X^\beta \frac{\partial \omega_\nu}{\partial x^\beta}\right]\left[\delta^\nu_\alpha + \epsilon \frac{\partial X^\nu}{\partial x^\alpha}\right] dx^\alpha$$

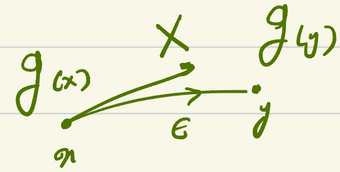
$$= \underbrace{\left[\omega_\alpha(x) + \epsilon X^\beta \frac{\partial \omega_\alpha}{\partial x^\beta} + \epsilon\, \omega_\nu \frac{\partial X^\nu}{\partial x^\alpha}\right]} dx^\alpha$$

$$\Rightarrow \boxed{(\mathcal{L}_X \omega)_\alpha = \left[X^\beta \frac{\partial \omega_\alpha}{\partial x^\beta} + \frac{\partial X^\beta}{\partial x^\alpha}\omega_\beta\right]}$$

قرارداد: $\frac{\partial \omega_\alpha}{\partial x^\beta} \Longrightarrow \omega_{\alpha,\beta} \qquad \frac{\partial X^\alpha}{\partial x^\beta} = X^\alpha_{,\beta}.$

$$\boxed{\begin{aligned}(\mathcal{L}_X \omega)_\alpha &= X^\beta \omega_{\alpha,\beta} + X^\beta_{,\alpha}\omega_\beta. \\ (\mathcal{L}_X Y)^\alpha &= X^\beta Y^\alpha_{,\beta} - X^\alpha_{,\beta} Y^\beta.\end{aligned}} \Longleftarrow$$

مشتق لی برای تابع: $(\mathcal{L}_X g) := \frac{g(x+\epsilon X) - g(x)}{\epsilon}$



$$= X^\alpha \partial_\alpha g = X(g).$$

$$\mathcal{L}_X(T) = \mathcal{L}_X(a \otimes b) = \frac{a' \otimes b' - a \otimes b}{\epsilon}$$

a' = push forward or pull back of a

$$\mathcal{L}_X(\tau) = \frac{a' \otimes b' - a \otimes b' + a \otimes b' - a \otimes b}{\epsilon}$$

اولاً

$$= \mathcal{L}_X(a) \otimes b + a \otimes \mathcal{L}_X(b).$$

$\tau \neq a \otimes b$

$\tau = \sum_i a_i \otimes b_i .$

با استفاده از خطی بودن مشتق بر τ کلی اثبات می‌شه.

Lie derivative satisfies Leibnitz rule.

$$(\mathcal{L}_X \omega)_{\mu\nu} = X^{\alpha}{}_{,\beta}\, \omega_{\alpha\nu} + X^{\alpha}{}_{,\beta}\, \omega_{\mu\alpha} + X^{\alpha} \omega_{\mu\nu,\alpha}.$$

let $\omega_{\mu\nu} = a_\mu b_\nu \qquad (\mathcal{L}_X \omega)_{\mu\nu} = (\mathcal{L}_X a)_\mu b_\nu + a_\mu (\mathcal{L}_X b)_\nu$

$$= (X^{\alpha} a_{\mu,\alpha} + X^{\alpha}{}_{,\mu} a_\alpha) b_\nu + a_\mu (X^{\alpha}{}_{,\nu} b_\alpha + X^{\alpha} b_{\nu,\alpha})$$

$$= X^{\alpha}(\underbrace{a_{\mu,\alpha} b_\nu + a_\mu b_{\nu,\alpha}}) + X^{\alpha}{}_{,\mu}\, a_\alpha b_\nu + X^{\alpha}{}_{,\nu}\, a_\mu b_\alpha$$

$$= X^{\alpha} \omega_{\mu\nu,\alpha} + X^{\alpha}{}_{,\mu}\, \omega_{\alpha\nu} + X^{\alpha}{}_{,\nu}\, \omega_{\mu\alpha}.$$

$$(\mathcal{L}_X \omega)_{\alpha\beta\gamma} = X^{\mu}{}_{,\alpha}\, \omega_{\mu\beta\gamma} + X^{\mu}{}_{,\beta}\, \omega_{\alpha\mu\gamma} + X^{\mu}{}_{,\gamma}\, \omega_{\alpha\beta\mu} + X^{\mu} \omega_{\alpha\beta\gamma,\mu}$$

$$(\mathcal{L}_X ab)^{\mu\nu} = (\mathcal{L}_X a)^\mu b^\nu + a^\mu (\mathcal{L}_X b)^\nu$$

$$= (-X^\mu{}_{,\alpha} a^\alpha + X^\alpha a^\mu{}_{,\alpha}) b^\nu + a^\mu (-X^\nu{}_{,\alpha} b^\alpha + X^\alpha b^\nu{}_{,\alpha})$$

$$= -X^\mu{}_{,\alpha} a^\alpha b^\nu - X^\nu{}_{,\alpha} a^\mu b^\alpha + X^\alpha (a^\mu{}_{,\alpha} b^\nu + a^\mu b^\nu{}_{,\alpha})$$

$$\to (\mathcal{L}_X \tau)^{\mu\nu} = -X^\mu{}_{,\alpha} \tau^{\alpha\nu} - X^\nu{}_{,\alpha} \tau^{\mu\alpha} + X^\alpha \tau^{\mu\nu}{}_{,\alpha}.$$

1-form: ω is a linear operator on X. $\omega(X) \in \mathbb{R}$

$\omega \in T_p^* M$, $X \in T_p M$

if X is a vector field $\in \mathcal{X}(M)$ ω is a form field.

$\omega(X)$ is a function $\in C^\infty(M)$.

Let f be a function: df is a form field.

$df(X) := X(f) \to$

properties:

① $df(X+Y) \overset{?}{=} (df)(X) + (df)(Y)$

② $d(fg) = X(fg) = X(f)g + fX(g)$

$= g(df)(X) + f\,dg(X)$

$\rightarrow d(fg) = (df)g + f\,dg.$

③ $dx^i(\partial_j) = \partial_j(x^i) = \delta^i_j$

جلسه بیست‌وچهارم، پنجشنبه چهارم اردیبهشت ۹۹

Differential form.

T_pM = tangent space of M at p

T_p^*M = Cotangent space of M at p.

$v, w, X, Y, \ldots \in T_pM$

$\uparrow$
vectors

$X = X^\mu \partial_\mu$

$\omega, \alpha, \beta, \theta, \ldots$

$\omega = \omega_\mu dx^\mu$

$\omega(X) = \omega_\mu dx^\mu (X^\alpha \partial_\alpha) = \omega_\mu X^\mu.$

$T_pM^* \ni \alpha, \beta, \omega, \ldots$ = 1-form.

$(T_p^*M \otimes T_p^*M) = \text{Span}\{dx^\mu \otimes dx^\nu\}$

if M is n-dimensional $\quad T_p^*M \otimes T_p^*M$ is n^2-dimensional.

$(T_p^*M)^r = T_p^*M \otimes \cdots T_p^*M = \text{Span}\{ dx^{\mu_1} \otimes \cdots dx^{\mu_r} \}$ is n^r-dim.

2-forms: $\Lambda^2(M) =$ Antisymmetric Subspace of $T_p^*M \otimes T_p^*M$.

$$\Lambda^2(M) = \text{Span}\{ \underbrace{dx^i \wedge dx^j}_{\text{wedge.}} = dx^i \otimes dx^j - dx^j \otimes dx^i \}$$

$\Lambda^2(M)$ is $\binom{n}{2} = \frac{n(n-1)}{2}$.

Example: $M = R^3$ $\quad \Lambda^2(R) = \text{Span}\{ dx \wedge dy,\ dy \wedge dz,\ dx \wedge dz \}$.

$M = S^2$ $\quad \Lambda^2(S^2) = \text{Span}\{ d\theta \wedge d\varphi \}$ $\quad$ یک‌سری.

$dx^\mu \wedge dx^\mu = 0$

A general two form: $\quad \omega = \sum_{\mu<\nu} \underset{\sim}{\omega_{\mu\nu}}\, dx^\mu \wedge dx^\nu = \frac{1}{2} \sum_{\mu,\nu} \omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$

R^3 در: $\quad \omega = \omega_{12}\, dx \wedge dy + \omega_{13}\, dx \wedge dz + \omega_{23}\, dy \wedge dz$!

$\omega = \frac{1}{2} \omega_{12}\, dx \wedge dy + \frac{1}{2} \omega_{21}\, dy \wedge dx + \cdots$ $\qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$.

$\frac{1}{2} \omega_{12}\, dx \wedge dy + \frac{1}{2} (-\omega_{12})(-dx \wedge dy) + \cdots = \omega_{12}\, dx \wedge dy + \cdots$

for 1-form $\quad \omega(X) \in R \quad \omega$ is a linear functional.

$\omega(cX+Y) = c\,\omega(X) + \omega(Y)$.

for 2-forms: $\omega(X,Y) = -\omega(Y,X)$.

$\omega = \omega_{\mu\nu}\, dx \wedge dy \;(X,Y)$

$dx \wedge dy\,(X,Y) = \begin{pmatrix} dx \otimes dy \\ - dy \otimes dx \end{pmatrix}(X,Y)$

$= - dy \wedge dx\,(X,Y)$

A two form is an anti-symmetric bilinear functional on $T_pM \otimes T_pM$.

$\omega \in \Lambda^2(M)$ $\qquad \omega(X,Y) = -\omega(Y,X)$.

An r-form is an anti-symmetric multi-linear functional on $T_p(M)^{\otimes r}$.

$\omega \in \Lambda^r(M)$ $\qquad \omega(X_{p(1)}, X_{p(2)}, \ldots X_{p(r)}) = (-1)^{|p|}\,\omega(X_1, X_2, \ldots X_r)$.

EXAMPLE: in R^3: local coordinates (r, θ, φ).

$\omega = c_1\, dr \wedge d\theta + c_2\, dr \wedge d\varphi + c_3\, d\theta \wedge d\varphi$

numbers or functions $f(r,\theta,\varphi)$

$\dim \Lambda^r(M) = \binom{n}{r}$

0-forms: functions $f, g, h, \ldots$

1-forms: $\omega = \omega_\mu dx^\mu$

2-forms $\omega = \frac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$

...

in S^2

0-forms: $f(\theta, \varphi)$.

1-form $\omega = c_1 d\theta + c_2 d\varphi$

2-form $\omega = c\, d\theta \wedge d\varphi$

3-form 0

Definition: let $\omega \in \Lambda^{p}(M)$, $\theta \in \Lambda^{q}(M)$.

$$\omega \wedge \theta \in \Lambda^{p+q}(M) \qquad \omega = \frac{1}{p!}\omega_{\mu_1 \cdots \mu_p}\, dx^{\mu_1}\wedge \cdots dx^{\mu_p}$$

Exterior Product or Wedge Product $\qquad \theta = \frac{1}{q!}\theta_{\nu_1 \cdots \nu_q}\, dx^{\nu_1}\wedge \cdots dx^{\nu_q}$

$$\omega \wedge \theta := \frac{1}{p!\,q!}\,\omega_{\mu_1 \cdots \mu_p}\,\theta_{\nu_1 \cdots \nu_q}\,\underbrace{(dx^{\mu_1}\wedge \cdots dx^{\mu_p})\wedge(dx^{\nu_1}\cdots dx^{\nu_q})}$$

مثال:
$$\omega = c_\mu dx^\mu = c_1 dx + c_2 dy + c_3 dz.$$
$$\theta = a_1 dx + a_2 dy + a_3 dz.$$

$$\omega\wedge\theta = c_1 a_2\, dx\wedge dy + c_2 a_1\, dy\wedge dx + c_1 a_3\, dx\wedge dz + c_3 a_1\, dz\wedge dx$$
$$+ c_2 a_3\, dy\wedge dz + c_3 a_2\, dz\wedge dy$$
$$= (c_1 a_2 - c_2 a_1)\, dx\wedge dy + (c_1 a_3 - c_3 a_1)\, dx\wedge dz +$$
$$(c_2 a_3 - c_3 a_2)\, dy\wedge dz.$$

مثال:
$$\begin{cases} \omega = \omega_{12}\, \underline{dx\wedge dy} + \omega_{13}\, dx\wedge dz + \omega_{23}\, dy\wedge dz. \\ \theta = \theta_1 dx + \theta_2 \underline{dy} + \theta_3 dz. \end{cases}$$

$$\omega\wedge\theta = \omega_{12}\theta_3\, dx\wedge dy\wedge dz + \omega_{13}\theta_2\, \underline{dx\wedge dz\wedge dy} + \omega_{23}\theta_1\, dy\wedge dz\wedge dx$$
$$= (\omega_{12}\theta_3 - \omega_{13}\theta_2 + \omega_{23}\theta_1)\, dx\wedge dy\wedge dz \in \Lambda^3(R^3).$$

Properties of wedge product:

1) $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ if $\alpha \in \Lambda^p$ $\beta \in \Lambda^q$

$\alpha = \alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots dx^{i_p}$ $\beta = \beta_{j_1 \ldots j_q} \underline{dx}^{j_1} \wedge \ldots dx^{j_q}$

2) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

3) $\alpha \wedge \alpha = 0$ if $\alpha \in \Lambda^{2r+1}(M)$.

---

if α is an even form: $\alpha = c_1 dx \wedge dy + c_2 dz \wedge dt \in \Lambda^2(\mathbb{R}^4)$.

$\alpha \wedge \alpha = c_1 c_2 dx \wedge dy \wedge dz \wedge dt + c_2 c_1 dz \wedge dt \wedge \underbrace{dx \wedge} dy$

$= 2 c_1 c_2 \, dx \wedge dy \wedge dz \wedge dt.$

Abstract definition of Exterior product.

let $\alpha \in \Lambda^p(M)$ $\beta \in \Lambda^q(M)$.

$$(\alpha \wedge \beta)(X_1, X_2, \ldots X_p, X_{p+1} \ldots X_{p+q}) :=$$

$$\sum_{\pi} \frac{1}{p!\,q!} (-1)^{|P|} (\alpha \otimes \beta)(X_{\pi(1)}, X_{\pi(2)} \ldots X_{\pi(p+q)})$$

Sio: let $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$

$$\beta = \beta_{12}\, dx \wedge dy + \beta_{23}\, dy \wedge dz + \beta_{13}\, dx \wedge dz.$$

$$(\alpha \wedge \beta)(X, Y, Z) = \frac{1}{2!}\{ (\alpha \otimes \beta)(X, Y, Z) + (\alpha \otimes \beta)(Y, Z, X) +$$

$$(\alpha \otimes \beta)(Z, X, Y) - (\alpha \otimes \beta)(Y, X, Z) - (\alpha \otimes \beta)(Z, X, Y)$$

$$- (\alpha \otimes \beta)(X, Z, Y) \}.$$

Contraction of a p-form and a vector.

Def: let $X \in T_pM$ & $\omega \in \Lambda^p(M)$.

$$(i_X \omega) \in \Lambda^{p-1}(M)$$

$$(i_X \omega)(X_1, X_2, X_{p-1}) = \omega(X, X_1, X_2, X_{p-1})$$

Properties.

$$i_{X+Y}(\omega) = i_X(\omega) + i_Y(\omega).$$

$$\omega = \frac{1}{p!} \omega_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p}.$$

$$\boxed{\omega(\partial_{i_1}, \partial_{i_2}, \dots \partial_{i_p}) = \omega_{i_1 i_2 \dots i_p}}$$

$$\omega = \frac{1}{2}\,\omega_{12}\,\underbrace{dx^1\wedge dx^2} + \frac{1}{2}\,\omega_{21}\,dx^2\wedge dx^1 + \cdots .$$

$$\omega(\underbrace{\partial_1,\partial_2}) = \frac{1}{2}\omega_{12}\,[\,1\,] + \frac{1}{2}\omega_{21}[\;-1\;] = \omega_{12}$$

$$dx^1\wedge dx^2 = \underbrace{dx^1 \otimes dx^2} - dx^2\wedge dx^1$$

$$(i_X\omega)_{i_1 i_2 \ldots i_{p-1}} = \omega(X, \partial_{i_1}, \partial_{i_2}, \ldots \partial_{i_{p-1}}) =$$

$$\omega(X^\alpha\partial_\alpha, \partial_{i_1}, \partial_{i_2} \ldots \partial_{i_{p-1}}) =$$

$$\rightarrow (i_X\omega)_{i_1 i_2 \ldots i_{p-1}} = X^\alpha\,\omega_{\alpha i_1 i_2 \ldots i_{p-1}}$$

Exterior derivative. Let $\omega \in \Lambda^1(M)$ $\quad d\omega \in \Lambda^2(M)$

Def: $\quad d\omega(X,Y) = X[\omega(Y)] - Y[\omega(X)]$

i) $d\omega$ has the right place; it acts on two vector fields

ii) $d\omega$ is bi-linear.

iii) $d\omega$ is anti-symmetric.

iv) $(d\omega)_{i_1,i_2}$ contain $\partial_j\omega_{i_1}$ --- etc.

v) $\underline{d}\omega(fX,Y) = f\, d\omega(X,Y)$. این رابطه مطلوب ماست

test of v):

$$d\omega(fX,Y) = (fX)\omega(Y) - Y(\omega(fX))$$

$$= f\,X(\omega(Y)) - Y[f\,\omega(X)]$$

$$= f\,X(\omega(Y)) - \underbrace{Y(f)\omega(X)}_{\text{اضافی است.}} - f\,Y(\omega(X)).$$

چطور تصحیح کنم؟

$$d\omega(X,Y) := X\omega(Y) - Y\omega(X) - \omega[X,Y]$$

test:

$$d\omega(fX,Y) = fX(\omega(Y)) - Y\,\omega(fX) - \omega[fX,Y]$$

✓ $$= f\,X(\omega(Y)) - Y[f\omega(X)] - \omega(f[X,Y] - Y(f)X)$$

✓ $$= f\,X(\omega(Y)) - \underbrace{Y(f)\omega(X)} - f\,Y(\omega(X)) - \omega(f[X,Y]) + \underbrace{Y(f)\omega(X)}$$

$$= f\,d\omega(X,Y).$$

v) $d(f\omega) \stackrel{?}{=} \underset{\hat{?}}{(df)}\omega + f(d\omega)$ let us check:

$$d(f\omega)(X,Y) = X((f\omega)(Y)) - Y((f\omega)(X))$$

$$= X[f.\omega(Y)] - Y[f.\omega(X)] = X(f)\,\omega(Y) + f.X(\omega(Y)) - Y(f)\,\omega(X) - f.Y(\omega(X))$$

$$= X(f)\,\omega(Y) - Y(f)\,\omega(X) + f\,d\omega(X,Y). \quad (1)$$

But $X(f) = df(X)$

(1) $[d(f\omega)](X,Y) = (df)(X)\,\omega(Y) - \omega(X)\,(df)(Y) + f\,d\omega(X,Y).$

$$= (df \otimes \omega)(X,Y) - (\omega \otimes df)(X,Y) + f\,d\omega(X,Y)$$

$$= \{(df \wedge \omega) + f\,d\omega\}(X,Y).$$