سه‌شنبه ، ۲۰ اردیبهشت ماه ۹۹ جلسه ۲۲ام

$$d : \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M)$$

if $\omega \in \Lambda^1(M)$ $\qquad d\omega(X,Y) := X(\omega(Y)) - Y(\omega(X))$

برای تعریف

$$d\omega(fX,Y) \neq f\, d\omega(X,Y)$$

تعریف درست و صحیح‌تر $\rightarrow$ $d\omega(X,Y) := X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$

برای تعریف

$d\omega$ is bilinear, $\begin{cases} d\omega(X+X',Y) = d\omega(X,Y) + d\omega(X',Y) \\ d\omega(fX,Y) = \underline{f}\ \underline{d\omega(X,Y)} \end{cases}$

More properties: $\quad d\omega(X,Y) = -d\omega(Y,X)$

i) $\quad d(\omega+\theta) = d\omega + d\theta$

if $\omega = \omega_\mu dx^\mu$ $\qquad d\omega = \frac{1}{2}(d\omega)_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$$= \frac{1}{2}(d\omega)_{\mu\nu}\left[dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right]$$

$$(d\omega)(\partial_\mu, \partial_\nu) = (d\omega)_{\mu\nu}$$

از تعریف $\quad (d\omega)(\partial_\mu,\partial_\nu) = \partial_\mu[\omega(\partial_\nu)] - \partial_\nu[\omega(\partial_\mu)] - \omega[\underline{\underline{\partial_\mu,\partial_\nu}}]$

$$= \partial_\mu(\omega_\nu) - \partial_\nu(\omega_\mu) - \omega[0]$$

$$= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$$

$$\boxed{(d\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu.} \Rightarrow$$

از نظر مولفه‌ای تعریف $d\omega$ به این شکل است.

$$\omega = \omega_\mu dx^\mu$$

$$d\omega = \frac{\partial \omega_\mu}{\partial x^\alpha} dx^\alpha \wedge dx^\mu = \frac{1}{2}\left[\frac{\partial \omega_\mu}{\partial x^\alpha} - \frac{\partial \omega_\alpha}{\partial x^\mu}\right] dx^\alpha \wedge dx^\mu$$

تعریف $d: \Lambda^r(M) \to \Lambda^{r+1}(M)$

Def: $d\omega(X_1, X_2, X_3, \ldots X_{r+1}) = X_1[\omega(X_2, X_3, \ldots X_{r+1})]$

$$- X_2[\omega(X_1, X_3, X_4, \ldots X_{r+1})] + X_3[\omega(X_1, X_2, X_4, \ldots X_{r+1})]$$

$$\cdots (-1)^r X_{r+1}[\omega(X_1, X_2, \ldots X_r)] + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, X_2, \ldots \hat{X}_i \ldots \hat{X}_j \ldots)$$

$$\omega = \frac{1}{r!} \omega_{i_1 i_2 \ldots i_r} dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_r}$$

$$d\omega = \frac{1}{r!} \underbrace{\left(\frac{\partial}{\partial x^\alpha} \omega_{i_1 i_2 \ldots i_r}\right)} \underbrace{dx^\alpha \wedge dx^{i_1} \wedge \ldots dx^{i_r}}$$

Ex: in R^3:

$$A = A_\mu dx^\mu \longrightarrow dA = (\partial_\nu A_\mu) dx^\nu \wedge dx^\mu$$

$$= \frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) dx^\nu \wedge dx^\mu$$

بطور صریح:

$$A = A_1 dx + A_2 dy + A_3 dz.$$

$$dA = \left(\frac{\partial A_1}{\partial y} dy \wedge dx + \frac{\partial A_1}{\partial z} dz \wedge dx\right) + \left(\frac{\partial A_2}{\partial x} dx \wedge dy + \frac{\partial A_2}{\partial z} dz \wedge dy\right)$$

$$+ \left(\frac{\partial A_3}{\partial x} dx \wedge dz + \frac{\partial A_3}{\partial y} dy \wedge dz\right)$$

$$dA = \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) dy \wedge dz + \cdots \quad (1)$$

در $\mathbb{R}^4$: $A = A_\mu dx^\mu \qquad \mu = 0, 1, 2, 3.$

$$dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu = F$$ تانسور شدت میدان

$$F_{01} = \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0) = E_1$$

$$F_{12} = \frac{1}{2}(\partial_1 A_2 - \partial_2 A_1) = B_3.$$

$$F = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ & 0 & B_3 & -B_2 \\ & & 0 & B_1 \\ & & & 0 \end{bmatrix}$$

properties of d: 1) $d^2 = 0$ $\qquad \Lambda^r(M) \xrightarrow{d} \Lambda^{r+1}(M) \xrightarrow{d} \Lambda^{r+2}(M)$

In homology we had: $\partial^2 = 0$

مثال: let $\omega = \frac{1}{r!}\omega_{i_1 i_2 \cdots i_r}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r}$

$$d\omega = \frac{1}{r!}\, \partial_\alpha \omega_{i_1 \cdots i_r}\, dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

$$d^2\omega = \frac{1}{r!}\, \underbrace{\partial_\beta \partial_\alpha \omega_{i_1 i_2 \cdots i_r}}\, \underbrace{dx^\beta \wedge dx^\alpha \wedge \cdots \wedge dx^{i_r}} = 0$$

$d^2 = 0$ $\underset{\Longleftarrow}{\text{نتیجه}}$ $\nabla \cdot \nabla \times A = 0 \qquad \nabla \times (\nabla \varphi) = 0$

$d : C^\infty(M) \longrightarrow \Lambda^1(M)$ $\qquad$ $df = \frac{\partial f}{\partial x^\mu} dx^\mu \equiv \nabla f.$

$\quad\;\, \| $

$\Lambda^0(M)$

$$\underline{d^2 f = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} dx^\nu \wedge dx^\mu = 0} \qquad \underline{\nabla \times \nabla f = 0.}$$

if $\quad A = A_x dx + A_y dy + A_z dz$ $\qquad$ $dA = (\partial_x A_y - \partial_y A_x) dx \wedge dy + \cdots$

$$\equiv \nabla \times A$$

$$\underbrace{A_{\mu\nu}}_{\text{متقارن}} \underbrace{B^{\mu\nu}}_{\text{پادمتقارن}} = A_{12} B^{12} + A_{21} B^{21} + \cdots$$

$$= A_{12} B^{12} + A_{12}(-B^{12}) = 0.$$

$\nabla \cdot (\nabla \times A) = 0$ $\qquad$ How this is a result of $d^2 = 0$. ?

در فضای ۳ بعدی $\qquad$ یک دوفرم $\equiv$ یک بردار

let B be a two-form in R^3, $\quad B = B_{xy}\, dx \wedge dy + B_{yz}\, dy \wedge dz +$

$$B_{xz}\, dx \wedge dz$$

$B_{xy} \equiv B_z \qquad B_{yz} \equiv B_x \qquad B_{xz} \equiv -B_y$

$$B = B_z\, \underline{dx \wedge dy} + B_x\, \underline{dy \wedge dz} + B_y\, dz \wedge dx.$$

$$dB = \frac{\partial B_z}{\partial z}\, dz \wedge dx \wedge dy + \frac{\partial B_x}{\partial x}\, dx \wedge dy \wedge dz + \frac{\partial B_y}{\partial y}\, dy \wedge dz \wedge dx$$

$$= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\equiv (\nabla \cdot B)$$

$$\text{if} \quad B = dA \to \boxed{B = (\nabla \times A)}$$

$$A = A_x\, dx + A_y\, dy + A_z\, dz \qquad dA = B = \underbrace{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)}_{B_z} dy \wedge dx + \cdots$$

$$d_{r+1}\, d_r = 0$$

تمرین : $d_3 d_2 = 0$ از روی تعریف صریح نشان دهید.

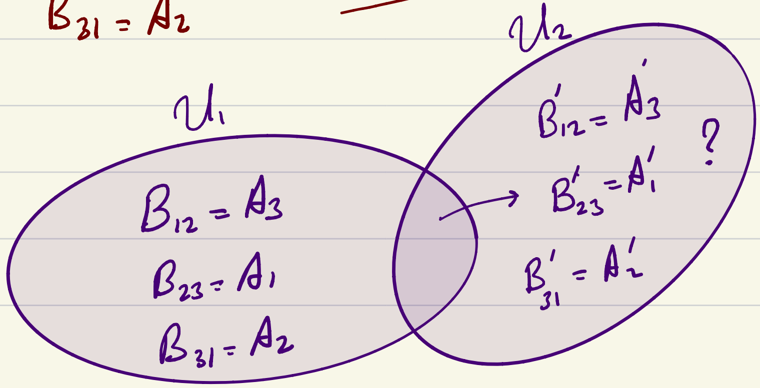
$$B = B_{12}\, dx^1 \wedge dx^2 + B_{23}\, dx^2 \wedge dx^3 + B_{31}\, dx^3 \wedge dx^1$$

[illegible] مختصات دکارتی باشد.

$$B_{12} = A_3 \qquad B_{13} = A_1 \qquad B_{31} = A_2$$

B_{12}: 2-form مولفه‌ی

A_3: مولفه‌ی یک فرم / بردار



$$B = B_{ij}\, dx^i \wedge dx^j = B_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}\, dx^{i'} \wedge dx^{j'}$$

$$\epsilon^{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3) \text{ is even perm.} \\ -1 & \text{odd permut.} \end{cases} \qquad A^i := \epsilon^{ijk} B_{jk} \quad (*)$$

① $$B_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} B_{ij}$$

فوراً ببینیم آیا روابط ① ، ② با (*) سازگار هستند؟

② $$A^i = \frac{\partial x^i}{\partial x^{i'}} A^{i'}$$

② & (*) $\rightarrow$ $$\epsilon^{ijk} B_{jk} = \frac{\partial x^i}{\partial x^{i'}} \epsilon^{i'j'k'} B_{j'k'} \quad ④$$

④ , ① $\rightarrow$ $$\epsilon^{ijk} B_{jk} = \frac{\partial x^i}{\partial x^{i'}} \epsilon^{i'j'k'} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} B_{jk} \quad ⑤$$

آیا رابطه‌ی ⑤ یک اتحاد هست یا نه؟ فعلاً این رابطه با این شکل در R^3 برقرار است.

و مختصات دکارتی

$$(x', y', z') = R(x, y, z)$$

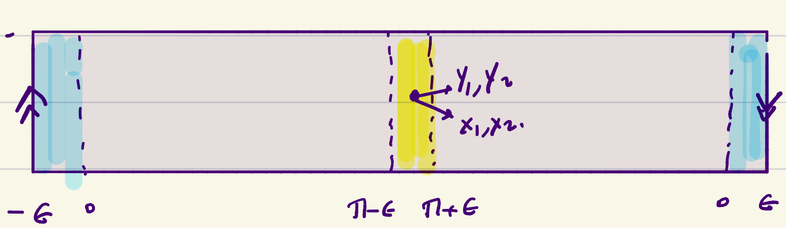
ماتریس دوران

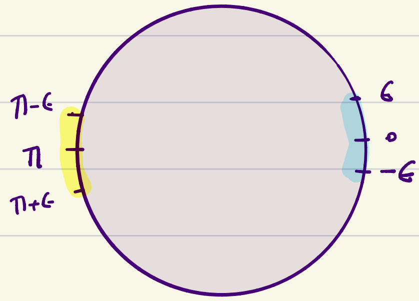
$$\underbrace{\underbrace{\epsilon^{ijk} = \epsilon^{i'j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}}}_{\epsilon^{ijk} \det \left| \frac{\partial x}{\partial x'} \right|} = \det R = 1}$$

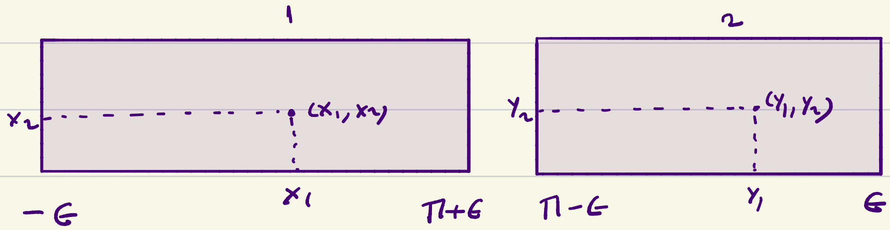
در حالت کلی پس از تعریف داریم:

$$\Lambda^r(M) \xleftrightarrow[\text{Hodge}]{*} \Lambda^{n-r}(M)$$

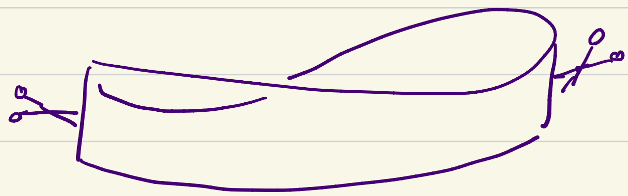
متر بر M ریمانی و جهت‌پذیر تعریف می‌شود.







در ناحیه زرد :
$$\begin{cases} y_1 = x_1 + c \\ y_2 = x_2 \end{cases} \longrightarrow \left(\frac{\partial y}{\partial x}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



ناحیه آبی:
$$\begin{cases} y_1 = x_1 + c_1' \\ y_2 = 1 - x_2 \end{cases} \longrightarrow \left(\frac{\partial y}{\partial x}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow$$ Mobius strip is not orientable