سه‌شنبه ۲۳ اردیبهشت ۹۹ صفحه ۲۸

مثال: RP^2 = Real projective plane.

$RP^2 = R^3 / _{v \sim \lambda v}$ $\longrightarrow$ Coordinate charts: $R^3 = (x^1, x^2, x^3)$

بازنامی

$U_1: \ x_1 \neq 0 \qquad \xi^2_{(1)} = \frac{x^2}{x^1} \qquad \xi^3_{(1)} = \frac{x^3}{x^1} \longrightarrow \alpha = \frac{x^2}{x^1} \qquad \beta = \frac{x^3}{x^1}$

$U_2: \ x_2 \neq 0 \qquad \xi^1_{(2)} = \frac{x^1}{x^2} \qquad \xi^3_{(2)} = \frac{x^3}{x^2} \qquad \alpha' = \frac{x^1}{x^2} \qquad \beta' = \frac{x^3}{x^2}$

$U_3: \ x_3 \neq 0 \qquad \xi^1_{(3)} = \frac{x^1}{x^3} \qquad \xi^2_{(3)} = \frac{x^2}{x^3} \qquad \alpha'' = \frac{x^1}{x^3} \qquad \beta'' = \frac{x^2}{x^3}$

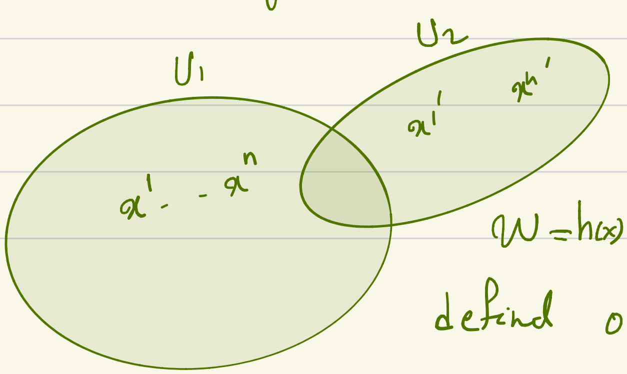
$$U_1 \cap U_2: \quad \alpha' = \frac{1}{\alpha} \qquad \beta' = \frac{\beta}{\alpha} \Rightarrow J = \left| \frac{\partial x'^i}{\partial x^i} \right| = \begin{vmatrix} \frac{\partial \alpha'}{\partial \alpha} & \frac{\partial \alpha'}{\partial \beta} \\ \frac{\partial \beta'}{\partial \alpha} & \frac{\partial \beta'}{\partial \beta} \end{vmatrix}$$

$$= \left\| \begin{matrix} -\frac{1}{\alpha^2} & 0 \\ \frac{-\beta}{\alpha^2} & \frac{1}{\alpha} \end{matrix} \right\| = \frac{-1}{\alpha^3} \not> 0 \longrightarrow RP^2 \text{ is not orientable.}$$

Definition: let M be an orientable manifold of dim $= n$.

we can define an n-form $\omega \mid \omega \neq 0$ over all M.

volume form.



$\omega = h(x) dx^1 \wedge dx^2 \wedge \ldots dx^n$ defined on U_1

$h(x) > 0$

on $U_1 \cap U_2$ $\quad w = h(x(x')) \dfrac{\partial x^1}{\partial x'^{i_1}} \dfrac{\partial x^2}{\partial x'^{i_2}} \cdots \dfrac{\partial x^n}{\partial x'^{i_n}} \underbrace{dx'^{i_1} \wedge \cdots \wedge dx'^{i_n}}$

$$dx'^{i_1} \wedge dx'^{i_2} \wedge \cdots \wedge dx'^{i_n} = \epsilon^{i_1 i_2 \cdots i_n}\, dx'^1 \wedge dx'^2 \wedge \cdots \wedge dx'^n \leftarrow$$

on $U_1 \cap U_2$ $\quad w = h(x(x')) \underbrace{\dfrac{\partial x^1}{\partial x'^{i_1}} \dfrac{\partial x^2}{\partial x'^{i_2}} \cdots \dfrac{\partial x^n}{\partial x'^{i_n}} \epsilon^{i_1 i_2 \cdots i_n}} dx'^1 \wedge \cdots \wedge dx'^n$

$$\omega = \underline{h'(x')}\ \underline{\left\| \frac{\partial x}{\partial x'} \right\|}\ dx'^1 \wedge dx'^2 \wedge \cdots \wedge dx'^n > 0$$

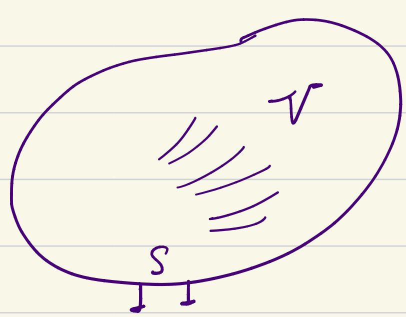
then we extend this to U_2 and then we continue on all M.

ω is positive over all M. It is called a volume form.

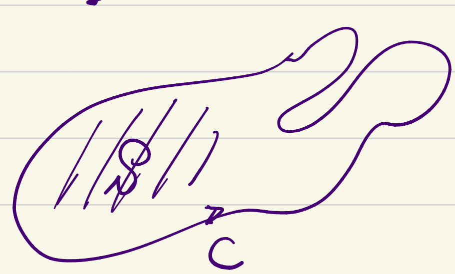
EXAMPLE: over S^2 $\quad w = \sin\theta\, d\theta \wedge d\varphi \qquad 0 \le \theta \le \pi$

$\omega > 0$ over all $S^2 - \{\text{North pole}\} - \{\text{South pole}\}$.

قضیه گاوس. $\displaystyle\int_{\partial V} A \cdot ds = \int_V (\nabla \cdot A)\, dv$

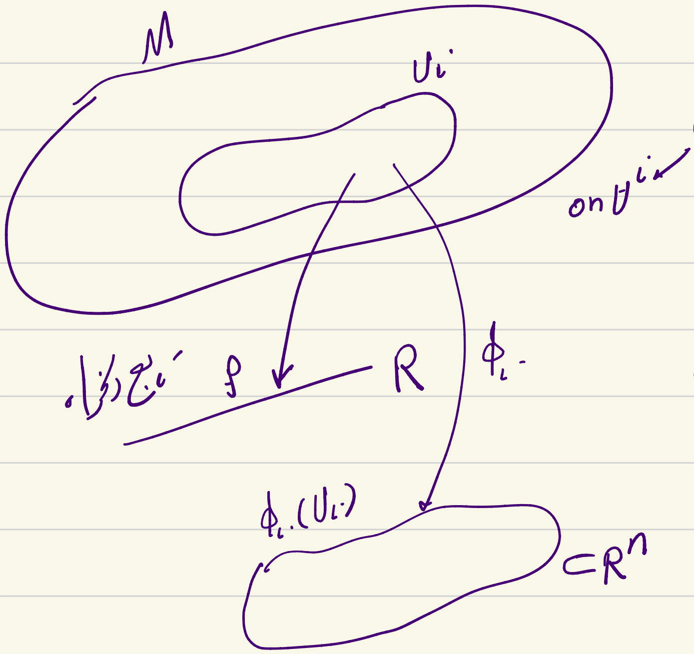


قضیه استوکس. $\displaystyle\int_{\partial S} A \cdot dl = \int_S (\nabla \times A) \cdot ds$



فضاهای کامپکت دیگر حالت خاص از یک فضاهایی هستند که روی هر مانیفلدی حاکم است.

تعریف انتگرال تابع روی یک چارت.



$\omega = h(x)\, dx^1 \wedge \dots dx^n$

$$\int_{U_i} f\omega := \int_{\phi_i(U_i)} f(x)h(x)\, dx^1 dx^2 \cdots dx^n$$

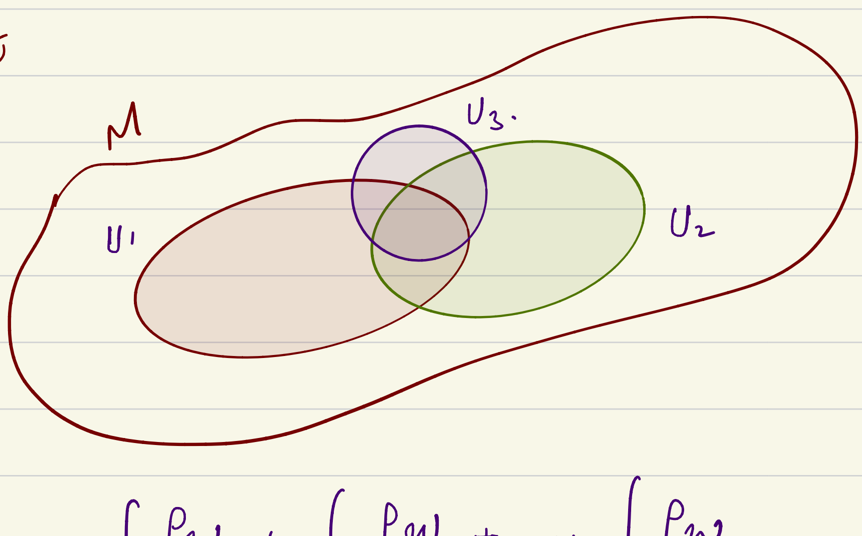
انتگرال n‌گانه معمولی است.

مثال: روی کره.

$$\int_{S^2} f\omega := \iint_{\substack{\theta \in [0,\pi] \\ \varphi \in [0,2\pi]}} f(\theta,\varphi)\, \underline{\sin\theta\, d\theta\, d\varphi}$$

No wedge product

تعریف انتگرال روی کل M



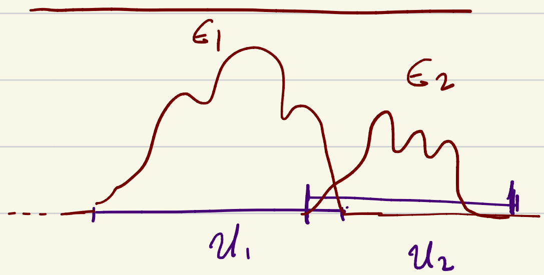
$$\int_M f\omega = \int_{U_1} f\omega + \int_{U_2} f\omega + \cdots \int_{U_N} f\omega$$

در این صورت f چند بار شمرده می‌شود.

[illegible] : M is paracompact if any point of M can be covered by a finite number of charts

Def: partition of unity: $\{\epsilon_i\}: M \to \mathbb{R}$ are continuous functions on M.

$1 = \sum \epsilon_i$

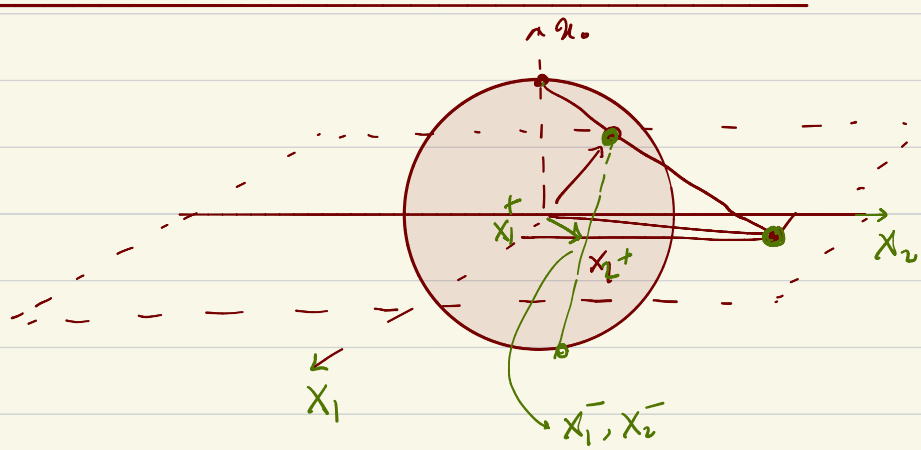


i) $0 \le \epsilon_i(p) \le 1$

ii) $\epsilon_i(p) = 0$ if $p \notin U_i$

iii) $\sum_i \epsilon_i(p) = 1 \quad \forall\, p \in M$

$$\int_M f\omega := \int_M f\omega \left(\sum_i \epsilon_i\right) = \sum_i \int_{U_i} f\omega\, \epsilon_i$$



Ex: $M = S^2$:

$U_N = S^2 - \{N\}$

$U_S = S^2 - \{S\}$.

$U_N: \quad X_1^+ = \dfrac{x_1}{1 - x_0} \qquad X_2^+ = \dfrac{x_2}{1 - x_0}$

U_S : $\quad X_1^- = \dfrac{x_1}{1+x_0} \qquad X_2^- = \dfrac{x_2}{1+x_0}$

$$\begin{cases} x_0 = \cos\theta \\ x_1 = \sin\theta\cos\varphi \\ x_2 = \sin\theta\sin\varphi \end{cases} \qquad \begin{aligned} X_1^+ &= \frac{\sin\theta\cos\varphi}{1-\cos\theta} = \frac{\sin\theta\cos\varphi}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}\cos\varphi \\ X_2^+ &= \cot\frac{\theta}{2}\sin\varphi \end{aligned}$$

in U_N $(\theta \neq 0)$ $\qquad X_1^+ = \cot\frac{\theta}{2}\cos\varphi \qquad X_2^+ = \cot\frac{\theta}{2}\sin\varphi$

$$Z^+ := X_1^+ + i X_2^+ = \cot\frac{\theta}{2}\, e^{i\varphi}$$

in U_S $(\theta \neq \pi)$ $\qquad X_1^- = \tan\frac{\theta}{2}\cos\varphi \qquad X_2^- = \tan\frac{\theta}{2}\sin\varphi$

$$Z^- = X_1^- + i X_2^- = \tan\frac{\theta}{2}\, e^{i\varphi}$$

in $U_N \cap U_S$: $\quad Z^+ = \dfrac{1}{Z^{-*}}$

on S^2: we had $\quad \omega = \sin\theta\, d\theta \wedge d\varphi$

on U_N: $\quad \omega = \sin\theta \left(\dfrac{\partial\theta}{\partial X_1^+} dX_1^+ + \dfrac{\partial\theta}{\partial X_2^+} dX_2^+ \right) \wedge \left(\dfrac{\partial\varphi}{\partial X_1^+} dX_1^+ + \dfrac{\partial\varphi}{\partial X_2^+} dX_2^+ \right)$

$$\omega = \sin\theta \begin{vmatrix} \dfrac{\partial\theta}{\partial X_1^+} & \dfrac{\partial\theta}{\partial X_2^+} \\ \dfrac{\partial\varphi}{\partial X_1^+} & \dfrac{\partial\varphi}{\partial X_2^+} \end{vmatrix} dX_1^+ \wedge dX_2^+$$

$$\omega = \sin\theta \, \frac{1}{\left|\frac{\partial X_1, X_2}{\partial \theta, \varphi}\right|} \, dx_1^+ \wedge dx_2^+$$

یادآوری

$$X_1^+ = \cot\frac{\theta}{2}\cos\varphi \qquad X_2^+ = \cot\frac{\theta}{2}\sin\varphi$$

$$J = \begin{vmatrix} \frac{\partial X_1^+}{\partial\theta} & \frac{\partial X_1^+}{\partial\varphi} \\ \frac{\partial X_2^+}{\partial\theta} & \frac{\partial X_2^+}{\partial\varphi} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}\frac{1}{\sin^2\frac{\theta}{2}}\cos\varphi & -\cot\frac{\theta}{2}\sin\varphi \\ \frac{-1}{2}\frac{1}{\sin^2\frac{\theta}{2}}\sin\varphi & \cot\frac{\theta}{2}\cos\varphi \end{vmatrix} = +\frac{1}{2}\left\{\frac{1}{\sin^2\frac{\theta}{2}}\right\}\cot\frac{\theta}{2}\left\{-1\right\}$$

$$J = -\frac{1}{2}\frac{\cos\theta/2}{\sin^3\theta/2} \Longrightarrow \omega = -2\sin\theta \, \frac{\sin^3\theta/2}{\cos\theta/2} \, dx_1^+ \wedge dx_2^+$$

$$\omega = -4\left(\sin\frac{\theta}{2}\right)^4 dX_1^+ \wedge dX_2^+ \qquad (X_1^+)^2 + (X_2^+)^2 = \cot^2\frac{\theta}{2}$$

$$1 + \cot^2\frac{\theta}{2} = \frac{1}{\sin^2\frac{\theta}{2}} \longrightarrow \omega_+ = -4\,\frac{1}{\left(1 + (X_1^+)^2 + (X_2^+)^2\right)^2}\, dX_1^+ \wedge dX_2^+$$

$\omega_+ = \omega$ on U_N.

برحسب مختصات مختلط بنویسیم: توجه: Z^+ را با Z می‌نویسیم.

$$Z = X_1 + iX_2 \qquad \bar{Z} = X_1 - iX_2 \qquad dZ \wedge d\bar{Z} = -2i\, dX_1 \wedge dX_2.$$

$$Z\bar{Z} = X_1^2 + X_2^2 \longrightarrow \boxed{\omega_+ = \frac{2i}{(1 + Z\bar{Z})^2}\, d\bar{Z} \wedge dZ}$$

ω_+ is always non-zero & well defined on U_N.

to find $\bar{w}$, we use change of coordinates:

For simplicity: $\quad Z_- \equiv \xi \qquad \xi = \frac{1}{\bar{Z}} \qquad z = \frac{1}{\bar{\xi}},\ \bar{z} = \frac{1}{\xi}.$

$$W_- = \frac{2i}{\left(1 + \frac{1}{\bar{\xi}\xi}\right)^2} \frac{-1}{\xi^2} d\xi \wedge \frac{-1}{\bar{\xi}^2} d\bar{\xi} = \frac{2i \ d\xi \wedge d\bar{\xi}}{(1+\xi\bar{\xi})^2}$$

on U_S : $\boxed{\omega_- = \frac{2i}{(1+\xi\bar{\xi})^2} d\xi \wedge d\bar{\xi}}$