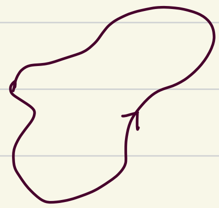
جلسه بیستم، ۹۹/۳/۱۱

Cohomology.

Homology:

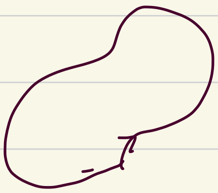


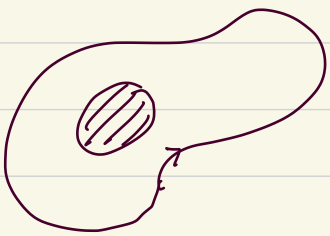
$Z_r(M) =$ زنجیرهای بسته $= \{ c_r \mid \partial c_r = 0 \}.$

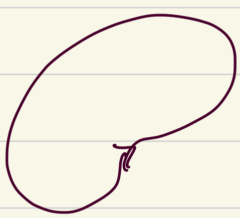
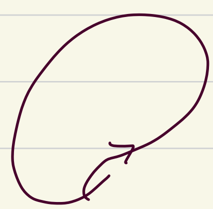
$\partial : C_r(M) \longrightarrow C_{r-1}(M)$

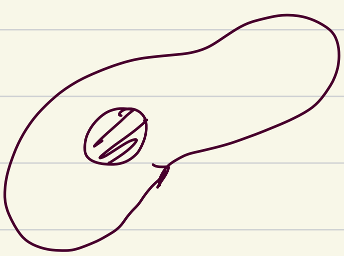
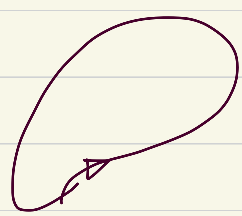
$\partial^2 = 0$

$B_r(M) = \{ c_r \mid c_r = \partial c_{r+1} \}.$

 $\in Z_r(M)$

 $\in Z_r(M)$

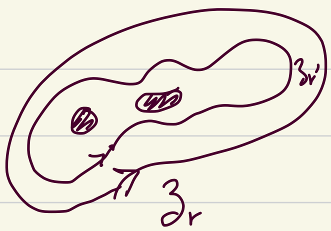
 $\in B_r(M)$ اگر:  $= \partial$ 

 $\notin B_r(M)$ اگر  $\neq \partial$?

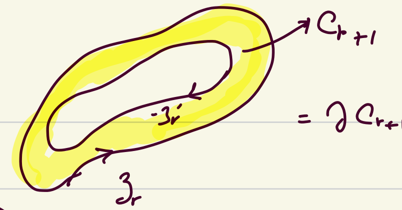
$B_r(M) \subset Z_r(M) \subset C_r(M)$

رابطهٔ هم‌ارزی: $Z_r(M)$ $\quad z_r \sim z_r'$ اگر $\quad z_r - z_r' \in B_r(M)$

$z_r - z_r' = \partial c_{r+1}$



$z_r \sim z_{r'}$ $\qquad z_r - z_{r'} =$

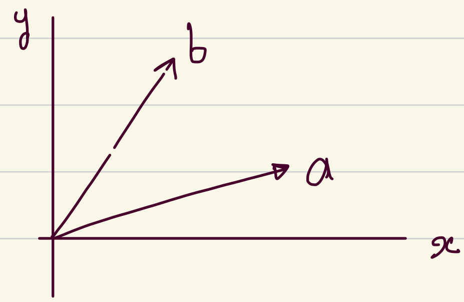


$= \partial c_{r+1}$

$$H_r(M) = Z_r(M) / B_r(M)$$

---

let us restrict ourself to R^n:



in R^2: $\qquad \omega = dx \wedge dy$

two form

Volume form.

$\omega = \omega_0(x, y)\, dx \wedge dy$

یک تابع حجم با جهت است.

با فرم دیفرانسیل اینگونه است.

$a = (a_1, a_2) \qquad b = (b_1, b_2) \qquad \omega(a, b) \in R$

$$\omega(a,b) = \frac{1}{2}\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu \left(a^\alpha \partial_\alpha, b^\beta \partial_\beta \right)$$

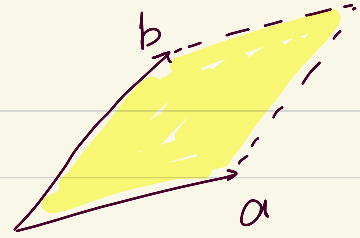
$$= \frac{1}{2}\omega_{\mu\nu}\, a^\alpha b^\beta \, (dx^\mu \wedge dx^\nu)(\partial_\alpha, \partial_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}\, a^\alpha b^\beta \left[\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right]$$

$$= \omega_{\mu\nu}\, a^\mu b^\nu$$

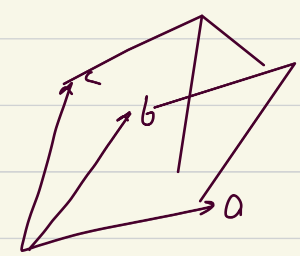
$$\omega = \omega_0\, dx \wedge dy\, (a, b) = \omega_0 \left[a_x b_y - a_y b_x \right]$$

$$\omega(a,b) = \omega_0 [\, a_x b_y - a_y b_x \,]$$



مقیاس مساحت از یک واحد نسبی حجم.

in R^3: $\qquad \omega = \omega_0\, dx \wedge dy \wedge dz$



$$\omega(a,b,c) = \omega_0 \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \text{حجم متوازی السطوح}$$

$$\omega = \omega_0\, dx \wedge dy \xrightarrow{\text{در تغییر مختصات}} \frac{\partial x}{\partial q^i}\frac{\partial y}{\partial q^j} \underbrace{dq^i \wedge dq^j}_{\epsilon^{ij}\, dq^1 \wedge dq^2} = \omega_0 \epsilon^{ij} \frac{\partial x}{\partial q^i}\frac{\partial y}{\partial q^j}\, dq^1 \wedge dq^2$$

$$\omega = \omega_0 \left\| \frac{\partial x}{\partial q} \right\| dq^1 \wedge dq^2$$

Ex.

$$x = r\cos\theta \qquad dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$y = r\sin\theta \qquad dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$\omega = dx \wedge dy = (dr\cos\theta - r\sin\theta\, d\theta) \wedge (dr\sin\theta + r\cos\theta\, d\theta)$$

$$= dr \wedge d\theta\, [\, r\cos^2\theta + r\sin^2\theta \,] = r\, dr \wedge d\theta.$$

Ex Ex.

In R^3: $\qquad \omega = dx \wedge dy \wedge dz \qquad x = r\sin\theta\cos\phi$

$y = r \sin\theta \sin\phi \qquad z = r\cos\theta \quad \longrightarrow \quad dz = dr\cos\theta - r\sin\theta\, d\theta$

$dx = \dots$

$dy = \dots$

$$\omega = dx \wedge dy \wedge dz = r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi .$$

[illegible]

یک یادآوری لازم : in R^3 : We have:

$0\text{-forms} \longleftrightarrow f \longleftrightarrow$ اسکالر

$1\text{-forms} \longleftrightarrow A = A_\mu dx^\mu \longleftrightarrow$ بردار

$2\text{-forms} \longleftrightarrow B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu \longleftrightarrow$ بردار $\left(B^\mu = \frac{1}{2!}\epsilon^{\mu\alpha\nu} B_{\alpha\nu}\right)$

$3\text{-forms} \longleftrightarrow \omega = \frac{1}{3!}\omega_{\mu\nu\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha = \omega_0 \, dx^1 \wedge dx^2 \wedge dx^3 . \longleftrightarrow$ تابع

$$\omega_0 = \omega_{123}$$

$$d : \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M)$$

$d_0 : \Lambda^0(M) \longrightarrow \Lambda^1(M) \qquad d_0 : f \longrightarrow df \qquad d_0 :$ تابع $\longrightarrow$ بردار

$d_1 : \Lambda^1(M) \longrightarrow \Lambda^2(M) \qquad d_1 : A \longrightarrow dA \qquad d_1 :$ بردار $\longrightarrow$ بردار

$d_2 : \Lambda^2(M) \longrightarrow \Lambda^3(M) \qquad d_2 : B \longrightarrow dB \qquad d_2$ بردار $\longrightarrow$ تابع

$$d_0 f = \frac{\partial f}{\partial x^i}\, dx^i$$

کرل

$$d_1 A = d_1\left(A_i\, dx^i\right) = \frac{\partial A_i}{\partial x^j}\, dx^j \wedge dx^i = \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) dx^j \wedge dx^i$$

$$= \nabla \times A.$$

$$d_2 B = d_2\left(B_{ij}\, dx^i \wedge dx^j\right) = \frac{\partial B_{ij}}{\partial x^k}\, dx^k \wedge dx^i \wedge dx^j$$

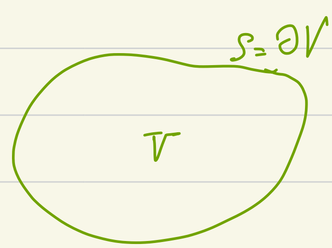
$$= \frac{\partial B_{ij}}{\partial x^k}\, \epsilon^{kij}\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= \left[\frac{\partial B_{12}}{\partial x^3} + \frac{\partial B_{23}}{\partial x^1} + \frac{\partial B_{31}}{\partial x^2}\right] dx^1 \wedge dx^2 \wedge dx^3$$

$$= \underbrace{\left(\frac{\partial \tilde{B}_1}{\partial x^1} + \frac{\partial \tilde{B}_2}{\partial x^2} + \frac{\partial \tilde{B}_3}{\partial x^3}\right)}_{\nabla \cdot \tilde{B}} dx^1 \wedge dx^2 \wedge dx^3$$

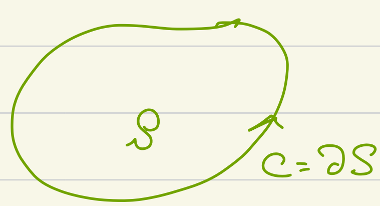
قضیهٔ گاوس،

$$\int_S \vec{A} \cdot d\vec{s} = \int_V (\vec{\nabla} \cdot \vec{A})\, dv$$



قضیهٔ استوکس:

$$\int_C \vec{A} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$$



Stoke's theorem. تعمیم به فضای دلخواه، فرم دلخواه:

در جلسه‌ی بعد توضیح می‌دهم.

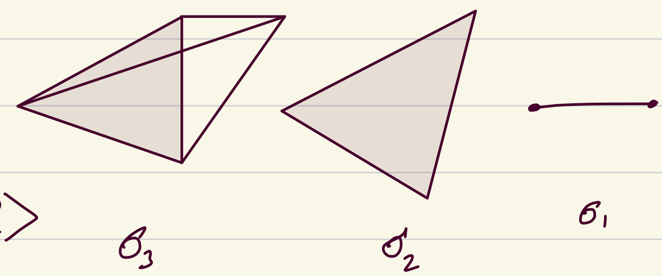
فضا در R^n

$$\sigma_r = r\text{-simplex} = \langle P_0, P_1, \dots P_r \rangle$$

$$= \left\{ \vec{x} \in R_n \;\middle|\; \vec{x} = \sum_{i=0}^{r} \lambda_i \vec{P}_i \,,\; \sum \lambda_i = 1 \right\}$$

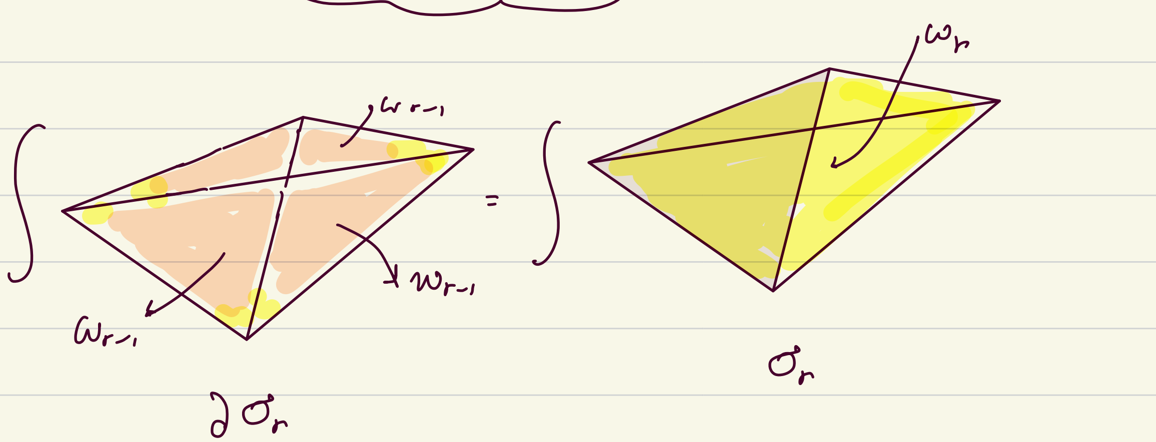
$$\partial : C_r \longrightarrow C_{r-1}$$

$$\partial \langle P_0 \dots P_r \rangle = \sum_{i=0}^{r} (-1)^i \langle P_0 \dots \hat{P}_i \dots P_r \rangle$$



Stoke's theorem:

$$\int_{\partial \sigma_r} \omega_{r-1} = \int_{\sigma_r} d\omega_{r-1}$$



قضیه گاوس:
$$\int_{S=\partial V} \vec{A} \cdot d\vec{s} = \int_{V} (\vec{\nabla} \cdot \vec{A})\, dv$$

قضیه استوکس
$$\oint_{C=\partial S} \vec{A} \cdot d\vec{l} = \int_{S} (\vec{\nabla} \times \vec{A}) \cdot ds$$

Proof: is left to you: Look it up in Nakahara.

But not recommended.

$$\begin{cases} r = 0 \\ R^n = R^1 \end{cases} \qquad \int_{\partial\sigma^1} f = \int_{\sigma^1} df \quad (1)$$

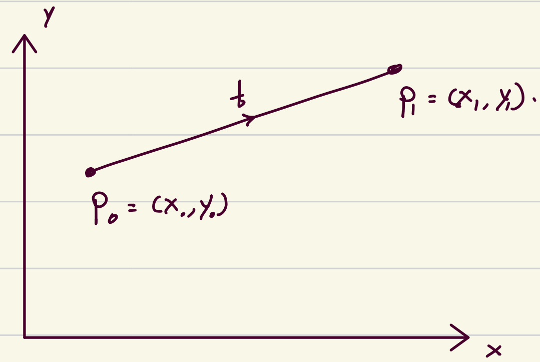
$\sigma^1 = \langle p_0, p_1 \rangle \qquad \partial\sigma^1 = \langle p_1 \rangle - \langle p_0 \rangle$

$$(1) \rightarrow \int_{\langle p_1 \rangle - \langle p_0 \rangle} f(x) = \int_{\langle p_0, p_1 \rangle} df \rightarrow f(p_1) - f(p_0) = \int_{p_0}^{p_1} df$$

قدم بعد،

in R^2 a zero form in R^2

$f(x,y)$.



$$\int_{\langle p_1 \rangle - \langle p_0 \rangle} f(x,y) = \int_{\langle p_0, p_1 \rangle} df$$

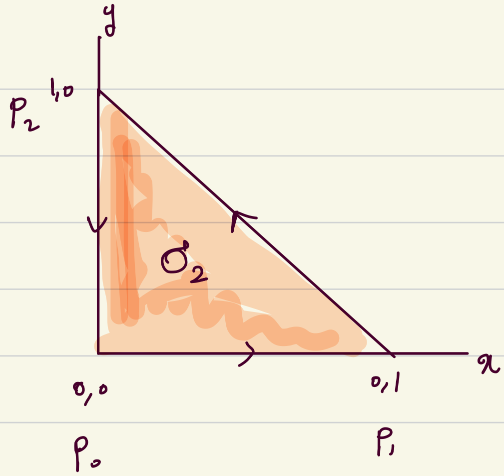
$f = f(x,y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\begin{cases} x(t) = (1-t)\, x_0 + t\, x_1 \\ y(t) = (1-t)\, y_0 + t\, y_1 \end{cases}$$

2 - Simplex in R^2

مرحله بعدی :



$$\sigma_2 = \langle P_0\, P_1\, P_2 \rangle$$

$$\partial\sigma_2 = \langle P_1\, P_2 \rangle - \langle P_0\, P_2 \rangle + \langle P_0\, P_1 \rangle$$

$$\int_{\partial\sigma_2} \omega = \int_{\sigma_2} d\omega$$

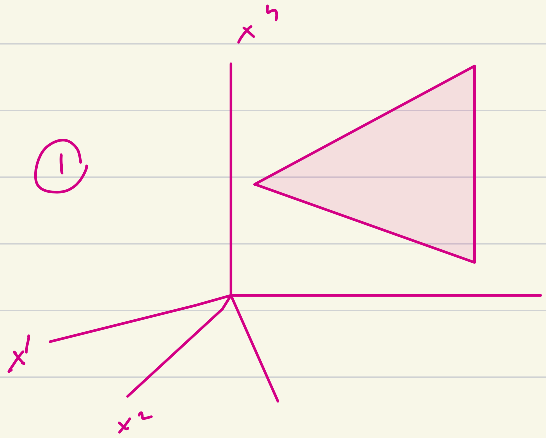
$$\omega = w_1(x,y)\,dx + w_2(x,y)\,dy$$

$$d\omega = \left(\frac{\partial w_1}{\partial y} - \frac{\partial w_2}{\partial x}\right) dx \wedge dy$$

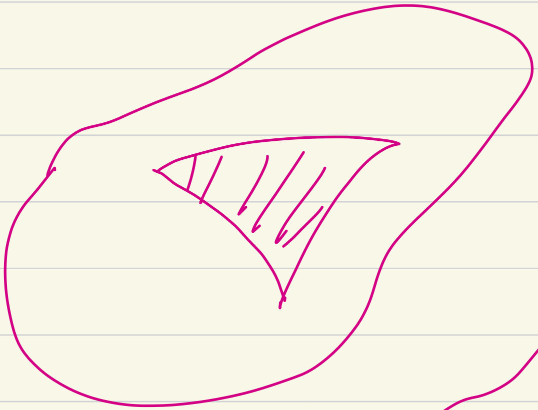
$\longrightarrow$ نکته مهم: قضیه را برای کمپلکس‌های سادکی در فضای R^n ثابت کنیم. حتی از این ساده‌تر: کمپلکس‌های سادکی σ^n در R^n.

: صرفاً. با قضیه‌ی محلی ثابت کنیم

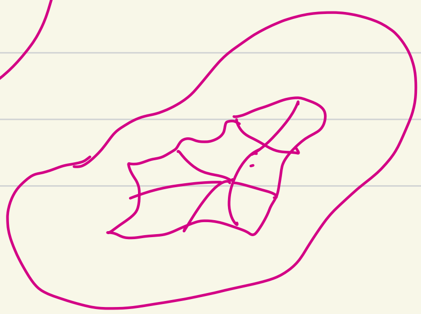
1



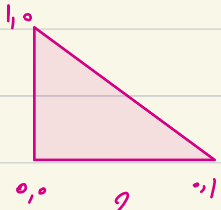
2



3 $\quad \sigma_r \longrightarrow \sum_i c_i\, \sigma_r^{(i)}$

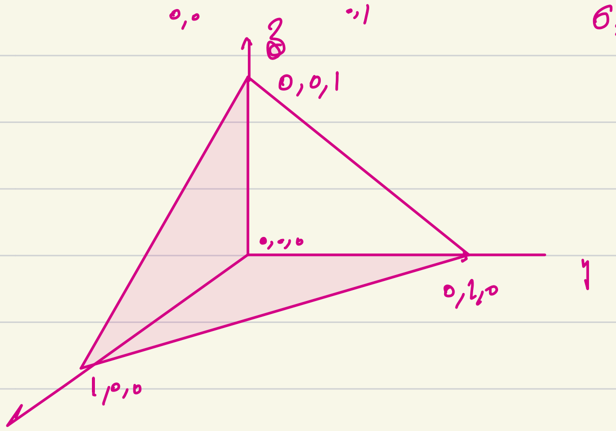


$$\int_0^1 f\,dx = f(1) - f(0)$$



$$\int_{\sigma_2} d\omega = \int_{\partial\sigma_2} \omega$$

$$\omega = \omega_1(x,y)\,dx + \omega_2(x,y)\,dy$$



$$\omega = \frac{1}{2}\omega_{ij}\,dx^i \wedge dx^j$$

$$= \omega_{12}\,dx \wedge dy + \omega_{23}\,dy \wedge dz + \omega_{31}\,dz \wedge dx$$

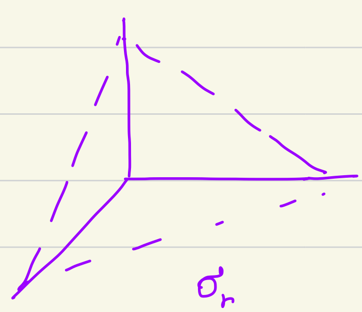
جلسه‌ی دوم - سیزده خرداد ۱۳۹۹

Cohomology Groups (Cont.)

Topology — Homology — Geometry

Cohomology

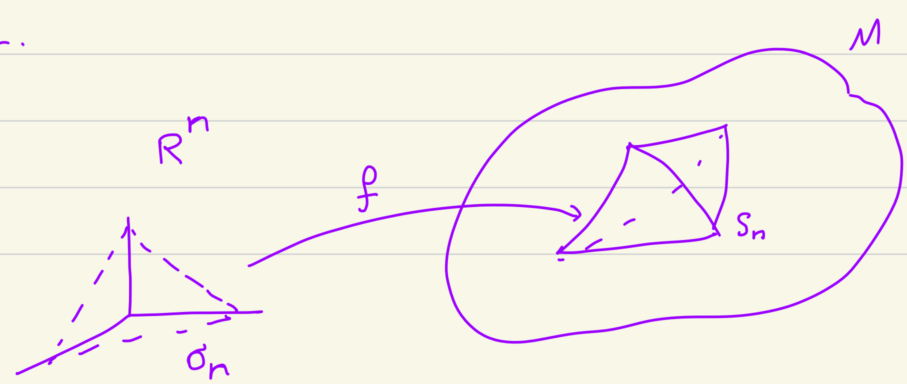
از جلسه‌ی گذشته



σ_r $\qquad$ $\partial\sigma_r$

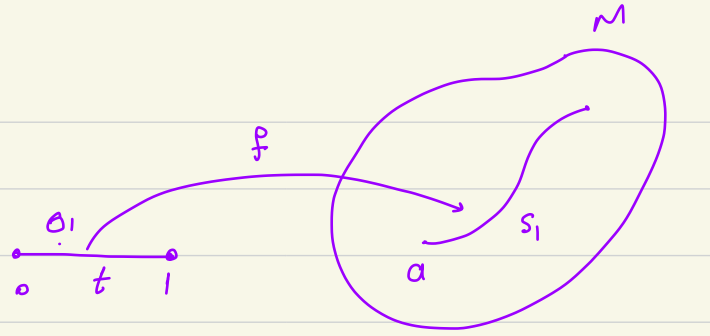
Stokes: $\displaystyle\int_{\sigma_r} d\omega_{r-1} = \int_{\partial\sigma_r} \omega_{r-1}$ $\qquad$ $d\omega_{r-1}$ $\qquad$ ω_{r-1}

$$\int_{S_n} \omega := ?$$



let w be a one-form on M.

$$w = \sum_i w_i(\theta)\, d\theta^i = w_i(\theta)\, d\theta^i$$



$$\int_{S_1} w = \int_{t=0}^{1} w_i(\theta) \frac{d\theta^i}{dt}\, dt = \int_{\sigma_1} f^* w$$

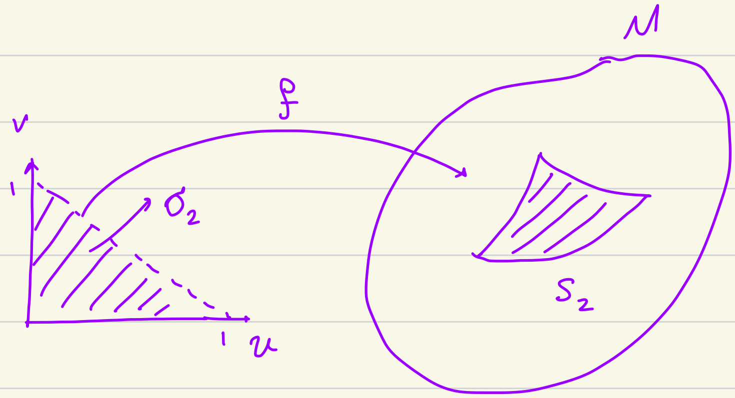
$S_1 : (\theta^1(t), \theta^2(t), \theta^n(t))$

$t=0 \longrightarrow a$

$t=1 \longrightarrow b$

$$w = \frac{1}{2} w_{ij}(\theta)\, d\theta^i \wedge d\theta^j$$

$\theta^i(u,v)$



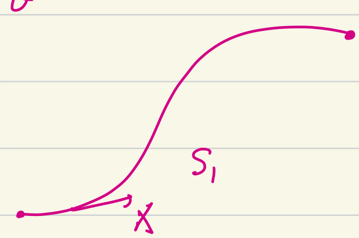
$$\int_{S_2} w = \int \frac{1}{2} w_{ij}(\theta)\, d\theta^i \wedge d\theta^j = \int_{\sigma_2} \frac{1}{2} w_{ij}(\theta(u,v)) \left(\frac{\partial\theta^i}{\partial u} du + \frac{\partial\theta^i}{\partial v} dv\right) \wedge \left(\qquad \right)$$

$$\int_{S_n} w_n = \int_{\sigma_n} f^* w_n.$$

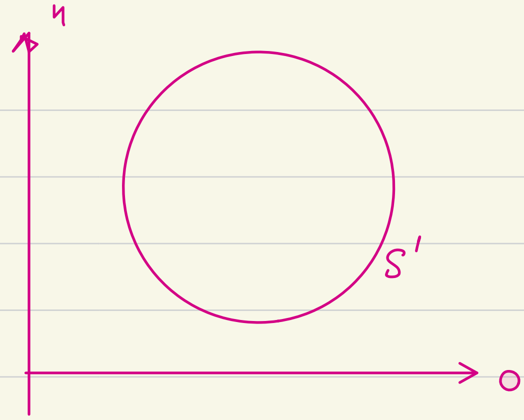
$i_X : \Lambda^2 \longrightarrow \Lambda^1$ if $w = w_{ij}(\theta)\, d\theta^i \wedge d\theta^j$

$X = X^i \frac{\partial}{\partial \theta^i}$ $i_X w = X^i(\theta)\, w_{ij}(\theta)\, d\theta^j$

$\int_{S^1} (i_X w)$ $i_X w = ?$



$\omega = w \, dx \wedge dy \qquad M = \mathbb{R}^2$



$$\int_{S^1} \omega = ? \qquad \int_{S^1} (i_X \omega)$$

$X = ?$ برداری روی S'

$$X = (-x\,\partial_y + y\,\partial_x) \qquad (i_X \omega)_\alpha = X^\beta \omega_{\alpha\beta}$$

$$\left.\begin{aligned} (i_X \omega)_1 &= X^1 \omega_{11} + X^2 \omega_{12} = 0 + xw = -xw \\ (i_X \omega)_2 &= X^1 \omega_{21} + X^2 \omega_{22} = y(-w) = -yw \end{aligned}\right\} \longrightarrow (i_X \omega) = -w\,(x\,dx + y\,dy)$$

$$\int_{S_1} i_X \omega = \int_{S^1} -w\,(x\,dx + y\,dy)$$

$$\int_{S_r} \omega_r = \int_{\sigma_r} f^* \omega \qquad f: \sigma_r \longrightarrow S_r$$

قضیه: $d f^* \omega = f^* d\omega$ proof by a simple example:

let $\omega = \omega_i(\theta)\, d\theta^i$ $\quad *\quad$ $\theta^i : \theta^1 \cdots \theta^n$

$$d\omega = \frac{\partial \omega_i(\theta)}{\partial \theta^j}\, d\theta^j \wedge d\theta^i \qquad \begin{cases} f: N \longrightarrow M \\ (t^1 \cdots t^m) \longrightarrow (\theta^1 \cdots \theta^n) \end{cases}$$

$$f^* d\omega = \underbrace{\frac{\partial \omega_i(\theta(t))}{\partial \theta^j} \frac{\partial \theta^j}{\partial t^\alpha}}_{\frac{\partial \omega_i}{\partial t^\alpha}} \frac{\partial \theta^i}{\partial t^\beta}\, dt^\alpha \wedge dt^\beta \qquad ①$$

$$* \rightarrow \quad f^*\omega = \omega_i(\theta(t)) \frac{\partial \theta^i}{\partial t^\alpha}\, dt^\alpha$$

$$d f^*\omega = \frac{\partial}{\partial t^\beta}\left[\omega_i(\theta(t)) \frac{\partial \theta^i}{\partial t^\alpha} \right] dt^\beta \wedge dt^\alpha \qquad ②$$

① = ② با جابجایی ترتیب

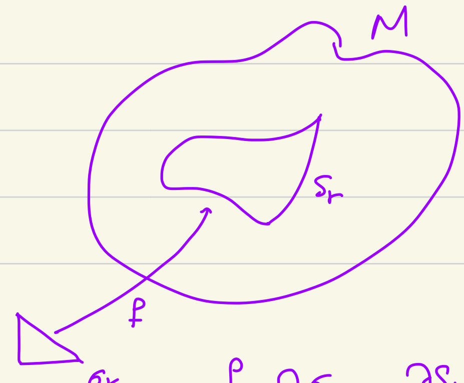
$$= \frac{\partial \omega_i(\theta(t))}{\partial t^\beta} \frac{\partial \theta^i}{\partial t^\alpha}\, dt^\beta \wedge dt^\alpha$$

$$\longrightarrow \boxed{d f^*\omega = f^* d\omega}$$

تمرین: با همین نوع استدلال نشان دهید که این اتحاد برای هر n فرمی برقرار است.

$$\int_{\partial S_{r+1}} \omega_r = \int_{\partial \sigma_r} f^*\omega \xrightarrow[\text{theorem}]{\text{Stokes}} \int_{\sigma_r} d(f^*\omega)$$

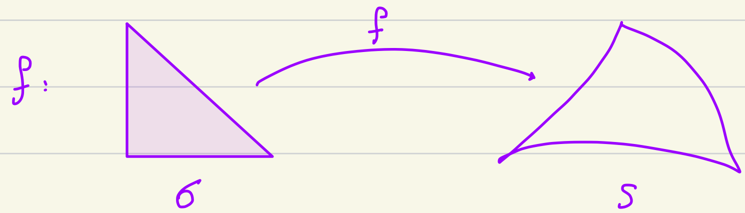
$\partial \sigma_r$ ↑ standard simplex



$f: \partial \sigma_r \longrightarrow \partial S_r$

اگر را تغییر بدیم

$$= \int_{\sigma_r} f^*(d\omega) = \int_{S_{r+1}} d\omega \longrightarrow \left\{ \int_{\partial S_{r+1}} \omega_r = \int_{S_{r+1}} d\omega_r \right.$$



قضیه استوکس

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega \longrightarrow \begin{cases} f^* d\omega = d f^* \omega \\ \int_{\partial S} \omega = \int_{\partial\sigma} f^*\omega \end{cases}$$ For any f

---

Def: A r-form ω is closed if $d\omega = 0$.

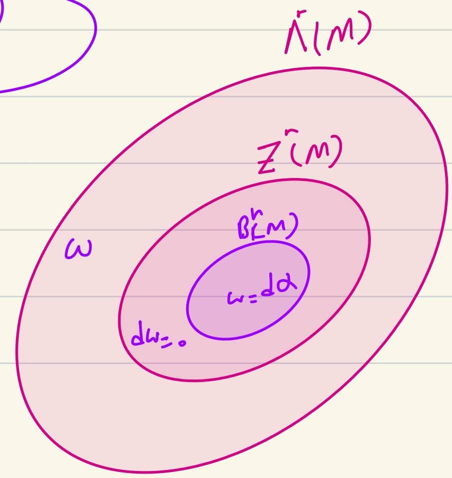
$$\Lambda^r(M) \supset Z^r(M) = \{\omega \mid d\omega = 0\}.$$

Def: An r-form is exact if $\exists\, \omega' \mid \omega = d\omega'$

$$\Lambda^r(M) \supset B^r(M) = \{\omega \mid \omega = d\omega'\}.$$

$$\boxed{B^r(M) \subseteq Z^r(M) \subseteq \Lambda^r(M)}$$

why if ω is exact $\longrightarrow$ ω is closed. ?



Theorem: if ω is Exact $\longrightarrow$ ω is closed.

زیرا $d^2 = 0$.

proof: if ω is exact $\rightarrow$ $\omega = d\alpha \longrightarrow d\omega = d^2\alpha = 0$.

Question: if ω is closed $\xrightarrow{?}$ ω is exact?

if $d\omega = 0 \xrightarrow{?} \omega = d\alpha$?

در فیزیک: if $\nabla \times \vec{A} = 0 \xrightarrow{\text{آیا}} \vec{A} = \nabla \varphi$?

" if $\nabla \cdot B = 0 \xrightarrow{\text{آیا}} \vec{B} = \nabla \times \vec{A}$?

عدم تک قطبی

vector potential

قضیه پوانکاره: if U is a $\underbrace{\text{simply connected Region}}_{\text{Contractible}}$ of M.

then every closed form defined on U is exact.

U

if $d\omega = 0 \xrightarrow{\text{then}} \omega = d\alpha$

ω باید سراسر U تعریف بشه.

Example ① in R^2: $\omega = x\,dy + y\,dx$.

$$d\omega = dx\wedge dy + dy\wedge dx = 0 \qquad \omega = d(xy)$$

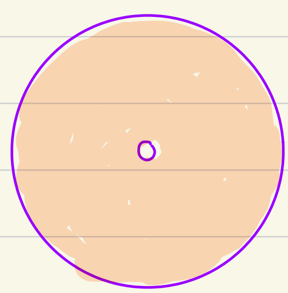
Example ② in R^2: $\omega = \dfrac{y\,dx - x\,dy}{(x^2+y^2)}$

$$d\omega = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy\wedge dx - \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx\wedge dy$$

$$= \underbrace{\left\{ \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\right\}}_{a} dx\wedge dy$$

$$a = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \begin{cases} 0 & \text{if } (x,y)\neq(0,0) \\ & \end{cases}$$

ω is closed on $R^2 - \{0\}$ = Annulus



Is ω exact?

$$\omega = \frac{y\,dx - x\,dy}{x^2+y^2} \qquad \omega = d\left(\tan^{-1}\frac{y}{x}\right) \; ?$$

$$d\left(\tan^{-1}\frac{y}{x}\right) = \frac{\partial}{\partial x}\left[\tan^{-1}\frac{y}{x}\right] dx + \frac{\partial}{\partial y}\left[\tan^{-1}\frac{y}{x}\right] dy$$

$$d\left(tg^{-1}\frac{y}{x}\right) = \frac{\frac{-y}{x^2}}{1+\frac{y^2}{x^2}}\,dx + \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}\,dy$$

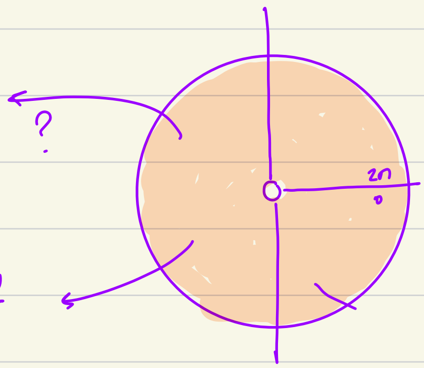
$$= \frac{-y\,dx + x\,dy}{x^2+y^2} \longrightarrow \boxed{\omega = d\left(-tg^{-1}\frac{y}{x}\right)}$$

$$\omega \neq d(-\theta)$$

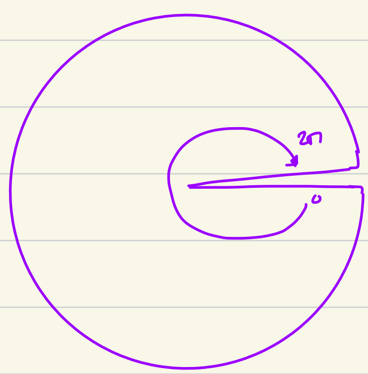
(r, θ)

$\begin{cases} \theta \in [0, 2\pi) \\ r > 0 \end{cases}$?

$\omega = \dfrac{y\,dx - x\,dy}{x^2+y^2}$



θ is not a single valued function even in the colored regions (Annulus).



$\underbrace{[0, 2\pi)}$

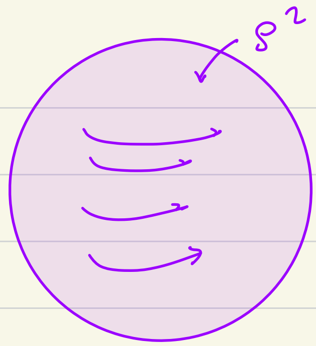
$\omega = d\theta$

Theorem 6.3. (**Poincaré's lemma**) If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.

مثال: on S^2 with coordinates θ & φ.

$$\omega = \sin\theta\, d\theta \wedge d\varphi \longrightarrow d\omega = 0$$

حل: آیا $\omega = d\alpha$? $\quad \omega = d(-\cos\theta\, d\varphi)$



$\omega = \sin\theta \, d\theta \wedge d\varphi$

S^2 is not contractible.