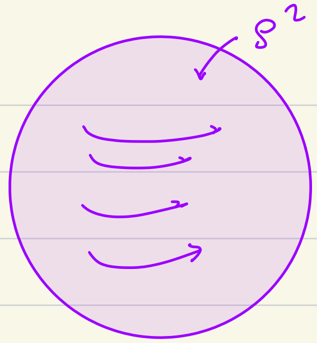


$\omega = \sin\theta \, d\theta \wedge d\varphi$

S^2 is not contractible.

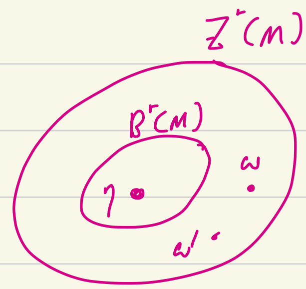
جلسه‌ی دوازدهم : ۱۸ آذر ۹۹

Manifold M $\qquad \Lambda^r(M) =$ همه‌ی فرم‌های r

$$Z^r(M) = \{ \omega \in \Lambda^r(M) \mid d\omega = 0 \} = \text{closed forms.}$$

$$B^r(M) \subset Z^r(M) = \{ \omega \in \Lambda^r(M) \mid \omega = d\eta \} = \text{exact forms.}$$

$$\omega \sim \omega' \in Z^r(M) \quad \text{if} \quad \omega - \omega' = d\eta$$



r-form $\omega \in Z^r(M)$ $\qquad \omega = \frac{1}{r!} \underbrace{\omega_{i_1 \dots i_r}(\theta^1 \dots \theta^n)}\, d\theta^{i_1} \wedge \dots d\theta^{i_r}$

$$H^r(M) = r\text{-th Cohomology group of } M = Z^r(M) / B^r(M)$$

$$[\omega] \in H^r(M) = \{ \omega + d\eta \}$$

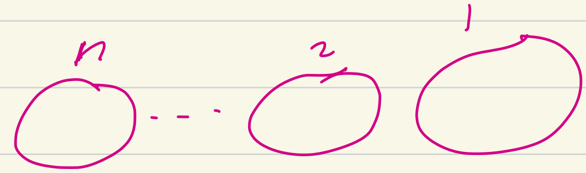
مثال :

$$\Lambda^0(M) = \{ 0\text{-forms in } M \} = \{ \text{functions on } M \}.$$

$$Z^0(M) = \{ f \mid df = 0 \} = \{ \text{Const functions} \} \equiv \mathbb{R}$$

if M is connected.

در حالت کلی if M has n-parts.



$$Z^0(M) = \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{\text{تعداد تکه‌ها} = n}$$

$$B^0(M) = \{ f \mid f = d(-1 \text{ form}) ?! \} = \phi$$

$$\Rightarrow \boxed{H^0(M) = \mathbb{R} \oplus \mathbb{R} \oplus \cdots \mathbb{R}.}$$

---

① لم: if $\underset{\in Z^p}{\omega}$ = closed & $\underset{\in Z^q}{\omega'}$ is also closed.

Question: is $\omega \wedge \omega'$ closed?

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega' = 0.$$

② لم: if ω is closed & ω' is exact.

$\rightarrow$ $\omega \wedge \omega'$ is closed certainly.

Question: is $\omega \wedge \omega'$ exact?

Proof: Since ω is closed $\longrightarrow d\omega = 0$ Since ω' is exact $\longrightarrow \omega' = d\theta$

$$\omega \wedge \omega' = d(\omega \wedge \theta) \quad ?$$

طبق فرمول $\quad d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta$

$$= 0 + (-1)^p \omega \wedge \omega'$$

چون ω, ω' closed

گروه کوهمولوژی دورام $\quad H^r(M) := Z^r(M) / B^r(M)$

$$\begin{cases} [\omega] + [\theta] := [\omega + \theta] \\ c[\omega] := [c\omega] \end{cases}$$

ابتدا خوش‌تعریفی:

باید چک بشه

از $\quad \begin{matrix} \omega \sim \omega' \\ \theta \sim \theta' \end{matrix} \longrightarrow \omega + \theta \sim \omega' + \theta'$

چون $\quad \begin{matrix} \omega' \sim \omega \longrightarrow \omega' = \omega + d\alpha \\ \theta' \sim \theta \longrightarrow \theta' = \theta + d\beta \end{matrix} \longrightarrow \omega' + \theta' = \omega + \theta + d(\alpha + \beta)$

$\omega \sim \omega' \longrightarrow c\omega \sim c\omega'$ $\quad$ و برای ضرب

Proof: $\quad \omega' = \omega + d\alpha \longrightarrow c\omega' = c\omega + d(c\alpha).$

De Rahm. قضیه : if M is compact $\longrightarrow$ $H^r(M)$ is finite dimensional.

مثال: $M = \mathbb{R}$ $\quad \Lambda^1(M) = \{ f(x)dx \}$ = تمام یک فرمی ها

$Z^1(M) = \Lambda^1(M)$ = یک فرمی های بسته

for any $\omega = f(x)dx$ $\quad d\omega = \frac{\partial f}{\partial x} dx \wedge dx = 0$

$B^1(M) = \{ \omega = f(x)dx \mid \omega = d\omega^\circ \}$.

$f(x)dx = d \int_0^x f(x')dx' \longrightarrow Z^1(M) = B^1(M)$

$\longrightarrow \quad H^1(\mathbb{R}) = Z^1(M) / B^1(M) = \{0\}$.

مثال ۲: $M = S^1 \longrightarrow H^1(S^1) = \mathbb{R}$

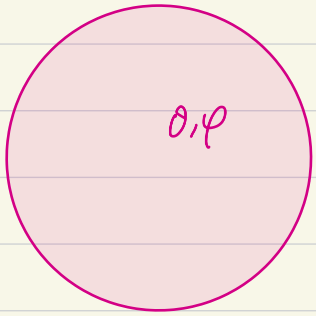
De Rahm قسمت دوم قضیه: $H^r(M) \cong H_r(M)$ دقیق‌تر دوگان هم‌اند.

Ex. $M = S^1$

$\longrightarrow$ ~~یک حلقه را دور یک نقطه نمی‌توان جمع کرد.~~ $\longrightarrow H_1(S^1) = \mathbb{Z}$

$H^1(S^1)$

مختصات دوبعدی



$\omega = \omega(\theta,\varphi)\, d\theta \wedge d\varphi$

2-form.

(از قضیه دورام) — $H^2(S^2) = \mathbb{R} \longrightarrow$ اگر 2-form ها نسبت به exact بودن با یک عدد حقیقی مشخص می‌شوند

در حالی که $\{0\} = H^1(S^2) \longrightarrow \forall$ 1-form بسته، exact اند.

$\omega = \omega_1(\theta,\varphi)\, d\theta + \omega_2(\theta,\varphi)\, d\varphi \qquad \exists\ f: S^2 \to \mathbb{R} \mid$

$\omega = df$

Since ω is closed $\rightarrow \quad d\omega = 0 \longrightarrow \dfrac{\partial \omega_1}{\partial \varphi}\, d\varphi \wedge d\theta + \dfrac{\partial \omega_2}{\partial \theta}\, d\theta \wedge d\varphi = 0$

$\dfrac{\partial \omega_1}{\partial \phi} = \dfrac{\partial \omega_2}{\partial \theta} \longrightarrow \exists\ f(\theta,\varphi)$

$\omega = df$

$$\omega = \omega_1(\theta^1, \theta^2)\, d\theta^1 + \omega_2(\theta^1, \theta^2)\, d\theta^2$$

1- form تعریف

M

$d\omega = 0$ گوییم بسته

$$\Updownarrow$$

$$\frac{\partial \omega_1}{\partial \theta^2} = \frac{\partial \omega_2}{\partial \theta^1}$$

حالا سوال این است : آیا $\omega = d f(\theta_1, \theta_2)$ که $f(\theta^1, \theta^2)$ روی M تعریف کرد؟

if $\Rightarrow H^1(M) = \{0\} \longrightarrow \omega = df.$

$H^1(M) \neq \{0\} \longrightarrow$ No Answer.

what if $H^1(M) = R \longrightarrow$

$\omega \sim \omega' \longrightarrow \omega' - \omega \in B^r(M) \longrightarrow [\omega] = \{\omega + B^r(M)\}.$

$[\omega] = \omega + d\alpha \qquad \alpha \in B^{r-1}(M)$

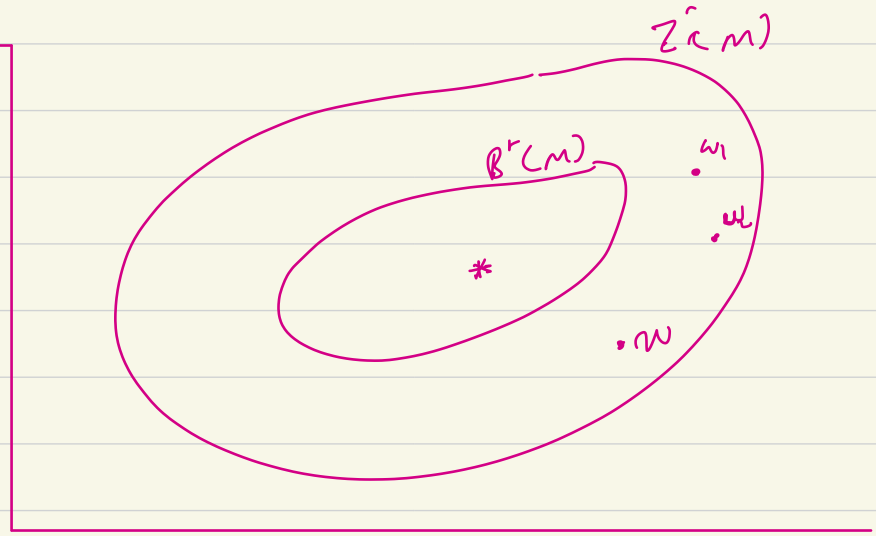
$\forall\ \omega$ which is closed $\qquad \omega = \underbrace{\lambda}_{R.} \omega_0 + d\alpha$

$H^r(M) \cong \mathbb{R} \oplus \mathbb{R}$ $\qquad w \sim \lambda_1 w_1 + \lambda_2 w_2$

w_1, w_2 are fixed

$$w = \lambda_1 w_1 + \lambda_2 w_2 + d\theta.$$

درآلیه و بیانگر فضای دوران.



let $\omega \in \Lambda^r(M)$

$c \in C_r(M)$ $\leftarrow$ r- chains on M.

Define the following pairing: $\Lambda^r(M) \times C_r(M) \longrightarrow \mathbb{R}$

$$\langle \omega, c \rangle := \int_c \omega \in \mathbb{R}.$$

1) pairing is bi-linear.

2) the pairing can be defined: $H^r(M) \times H_r(M) \longrightarrow \mathbb{R}$

$$\langle [\omega], [c] \rangle := \int_c \omega$$

باید نشان دهیم که تعریف بالا خالی از اشکال است = بستگی به انتخاب نماینده ندارد.

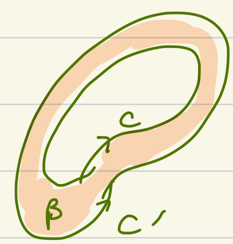
اثبات:

$$\omega \sim \omega' \longrightarrow \text{باید نشان دهیم} \quad \int_C \omega = \int_C \omega'$$

$$\langle \omega, C \rangle = \int_C \omega \qquad \langle \omega', C \rangle = \langle \omega, C \rangle$$

$$\int_C \omega' = \int_C \omega + d\theta = \int_C \omega + \int_C d\theta = \int_C \omega + \int_{\underset{\parallel}{\partial C}\atop 0} \theta = \int_C \omega$$

حالا فرض کنید $\quad C \sim C' \qquad \int_{C'} \omega \overset{?}{=} \int_C \omega$



Proof: $$\int_{C'} \omega = \int_{C+\partial\beta} \omega = \int_C \omega + \int_{\partial\beta} \omega$$

$$= \int_C \omega + \int_\beta \underbrace{d\omega}_{=0} = \int_C \omega$$

Now let $\omega \sim \omega'$ & $C \sim C'$ $$\int_{C'} \omega' = \int_{C+\partial\beta} \omega + d\theta =$$

$$= \int_C w + \int_{\partial\beta} w + \int_C d\theta + \int_{\partial\beta} d\theta =$$

$$= \int_C w + \int_\beta \underbrace{dw}_{0} + \int_{\underbrace{\partial c}_{0}} \theta + \int_{\partial^2\beta = 0} \theta = \int_C w$$

پس یک pairing بین $H^r(M)$ و $H_r(M)$ تعریف کرده‌ایم که درست و صحیح است.

قضیه‌ی دورام: الف: $H_r(M)$, $H^r(M)$ محدودالبعد هستند.

ب: این pairing ناتبهگن است.

Form a basis for $H^r(M)$: $\{ w_1, w_2 \cdots w_k \}$.

" " " $H_r(M)$: $\{ c_1, c_2, \cdots c_k \}$.

$$\langle w_i, c_j \rangle = H_{ij}$$

H_{ij} is invertible.

تعریف پایه‌ی دوگان.

$$\boxed{\langle w_i, c_j \rangle = \delta_{ij}}$$