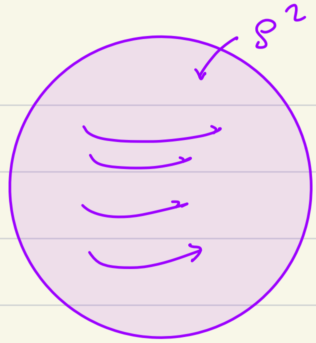


$\omega = \sin\theta \, d\theta \wedge d\varphi$

S^2 is not contractible.

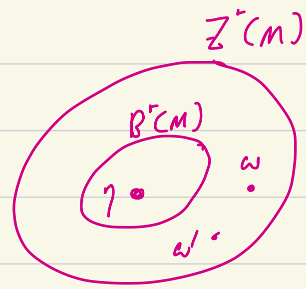
جلسه ی چهارم : ۱۸ آذر ۹۹

Manifold M $\qquad \Lambda^r(M) =$ همه‌ی فرم‌های r

$$Z^r(M) = \{ \omega \in \Lambda^r(M) \mid d\omega = 0 \} = \text{closed forms.}$$

$$B^r(M) \subset Z^r(M) = \{ \omega \in \Lambda^r(M) \mid \omega = d\eta \} = \text{exact forms}$$

$$\omega \sim \omega' \in Z^r(M) \quad \text{if} \quad \omega - \omega' = d\eta$$



r-form $\omega \in Z^r(M)$ $\qquad \omega = \frac{1}{r!} \underbrace{\omega_{i_1 \dots i_r}(\theta^1 \dots \theta^n)}\, d\theta^{i_1} \wedge \dots \wedge d\theta^{i_r}$

$$H^r(M) = \text{r-th Cohomology group of } M = Z^r(M) / B^r(M)$$

$$[\omega] \in H^r(M) = \{ \omega + d\eta \}$$

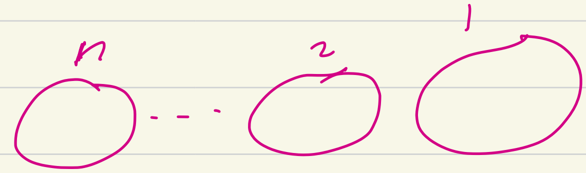
مثال :

$$\Lambda^0(M) = \{ \text{0-forms in } M \} = \{ \text{functions on } M \}.$$

$$Z^0(M) = \{ f \mid df = 0 \} = \{ \text{Const functions} \} \equiv \mathbb{R}$$

if M is connected.

بطور کلی if M has n-parts.



$$Z^0(M) = \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{\text{تعداد تکه‌ها} = n}$$

$$B^0(M) = \{ f \mid f = d(-1 \text{ form}) ?! \} = \phi$$

$$\Rightarrow \boxed{H^0(M) = \mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}.}$$

---

1 Sol: if ω = closed & ω' is also closed.
$\omega \in Z^p$, $\omega' \in Z^q$

Question: is $\omega \wedge \omega'$ closed?

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega' = 0.$$

2 Sol: if ω is closed & ω' is exact.

$\rightarrow$ $\omega \wedge \omega'$ is closed certainly.

Question: is $\omega \wedge \omega'$ exact?

Prof: Since ω is closed $\longrightarrow d\omega = 0$ Since ω' is exact $\longrightarrow \omega' = d\theta$

$$\omega \wedge \omega' = d(\omega \wedge \theta) \quad ?$$

چون درسته $\quad d(\omega\wedge\theta) = d\omega \wedge \theta + (-1)^p\,\omega \wedge d\theta$

$$= 0 + (-1)^p\,\omega \wedge \omega'$$

چون ω, ω' closed

بستگی به نماینده ها نداره $\quad H^r(M) := Z^r(M) \big/ B^r(M)$

$$\begin{cases} [\omega] + [\theta] := [\omega + \theta] \\ c\,[\omega] := [c\,\omega] \end{cases}$$

ابتدا خوش‌تعریفی:

باید چک کنیم

اگر $\begin{matrix} \omega \sim \omega' \\ \theta \sim \theta' \end{matrix} \longrightarrow \omega + \theta \sim \omega' + \theta'$

چون $\begin{matrix} \omega' \sim \omega \longrightarrow \omega' = \omega + d\alpha \\ \theta' \sim \theta \longrightarrow \theta' = \theta + d\beta \end{matrix} \longrightarrow \omega' + \theta' = \omega + \theta + d(\alpha + \beta)$

$\omega \sim \omega' \longrightarrow c\omega \sim c\omega'$ واضح درسته.

Proof: $\omega' = \omega + d\alpha \longrightarrow c\omega' = c\omega + d(c\alpha)$.

De Rahm. قضیه : if M is compact $\longrightarrow$ $H^r(M)$ is finite dimensional.

مثال: $M = \mathbb{R}$ $\quad \Lambda^1(M) = \{ f(x)dx \}$ = بی‌نهایت بعدی

$Z^1(M) = \Lambda^1(M)$ = بی‌نهایت بعدی

for any $\omega = f(x)dx$ $\quad d\omega = \frac{\partial f}{\partial x} dx \wedge dx = 0$

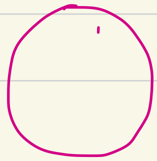
$B^1(M) = \{ \omega = f(x)dx \mid \omega = d\omega^\circ \}$.

$f(x)dx = d \int_0^x f(x')dx' \longrightarrow Z^1(M) = B^1(M)$

$\rightarrow \quad H^1(\mathbb{R}) = Z^1(M) / B^1(M) = \{0\}$.

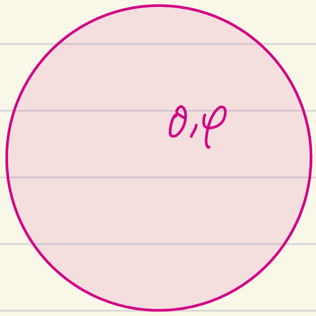
مثال ۲: $\quad M = S^1 \longrightarrow H^1(S^1) = \mathbb{R}$

De Rahm قضیه دی‌رام: $\quad H^r(M) = H_r(M)$ دقیقاً برقرار است

Ex. $\quad M = S^1$  $\rightarrow$ یک حلقه‌ای وجود دارد که مرز چیزی نیست. $\rightarrow H_1(S^1) = \mathbb{Z}$

$H^1(S^1)$

کره‌ی دوبعدی



$\omega = \omega(\theta,\varphi)\, d\theta \wedge d\varphi$

2- form.

(از قضیه‌ی دورام) $\longrightarrow H^2(S^2) = \mathbb{R} \longrightarrow$ هر 2-form بسته که exact نیستند با یک ضریب حقیقی از هم به دست می‌آیند

در نتیجه $\{0\} = H^1(S^2) \longrightarrow \forall$ ۱-فرم بسته exact است

$$\omega = \omega_1(\theta,\varphi)\, d\theta + \omega_2(\theta,\varphi)\, d\varphi \qquad \exists \; f : S^2 \to \mathbb{R} \mid$$

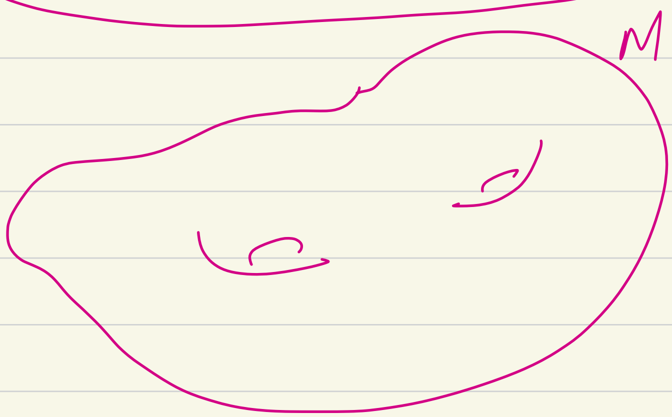
$$\omega = df$$

Since ω is closed $\rightarrow$ $d\omega = 0 \longrightarrow \dfrac{\partial \omega_1}{\partial \varphi} d\varphi \wedge d\theta + \dfrac{\partial \omega_2}{\partial \theta} d\theta \wedge d\varphi = 0$

$$\frac{\partial \omega_1}{\partial \phi} = \frac{\partial \omega_2}{\partial \theta} \longrightarrow \exists \; f(\theta,\varphi)$$

$$\omega = df$$

$$\omega = \omega_1(\theta^1,\theta^2)\,d\theta^1 + \omega_2(\theta^1,\theta^2)\,d\theta^2$$ 1- form تعریف



$d\omega = 0$ گوییم بسته

$$\Updownarrow$$

$$\frac{\partial \omega_1}{\partial \theta^2} = \frac{\partial \omega_2}{\partial \theta^1}$$

حال سوال اینست: آیا $\omega = d f(\theta_1,\theta_2)$ که $f(\theta^1,\theta^2)$ روی M تعریف کنیم؟

if $\Rightarrow H^1(M) = \{0\} \longrightarrow \omega = df.$

$H^1(M) \neq \{0\} \longrightarrow$ No Answer.

what if $H^1(M) = R \longrightarrow$

$\omega \sim \omega' \longrightarrow \omega' - \omega \in B^r(M) \longrightarrow [\omega] = \{\omega + B^r(M)\}.$

$[\omega] = \omega + d\alpha \qquad \alpha \in B^{r-1}(M)$

$\forall\ \omega$ which is closed $\qquad \omega = \lambda\omega_0 + d\alpha$

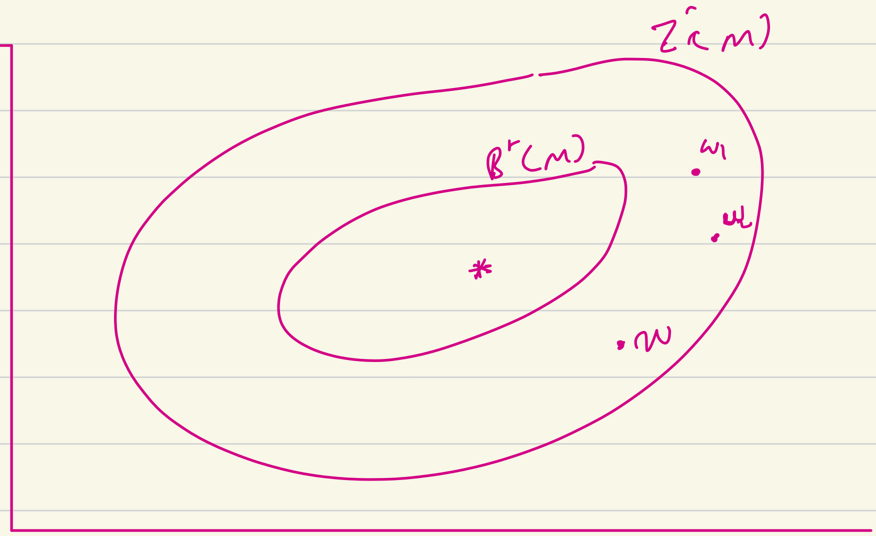
$\lambda \in R.$

$H^r(M) \cong \mathbb{R} \oplus \mathbb{R}$ $\qquad w \sim \lambda_1 w_1 + \lambda_2 w_2$

w_1, w_2 are fixed

$$w = \lambda_1 w_1 + \lambda_2 w_2 + d\theta .$$

در آنالیز ۲ و بیشتر قضیه (دورام).



let $\omega \in \Lambda^r(M)$

$c \in C_r(M)$ — r-chains on M.

Define the following pairing: $\Lambda^r(M) \times C_r(M) \longrightarrow \mathbb{R}$

$$\langle \omega, c \rangle := \int_c \omega \in \mathbb{R} .$$

1) pairing is bi-linear.

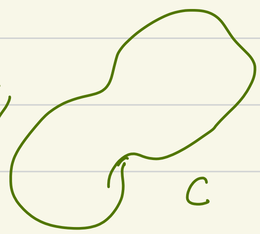
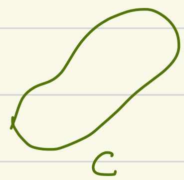
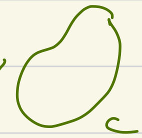
2) the pairing can be defined: $H^r(M) \times H_r(M) \longrightarrow \mathbb{R}$

$$\langle [\omega], [c] \rangle := \int_c \omega$$

باید نشان دهیم، تعریف بالا خالی از ابهام است = بستگی به انتخاب نماینده ندارد.

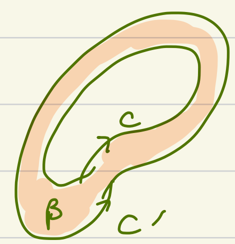
اثبات:

$$\omega \sim \omega' \quad \longrightarrow \text{باید نشان دهیم} \quad \int_C \omega = \int_C \omega'$$

$\langle \omega,$  $\rangle = \int_C \omega \qquad \langle \omega',$  $\rangle = \langle \omega,$  $\rangle$

$$\int_C \omega' = \int_C \omega + d\theta = \int_C \omega + \int_C d\theta = \int_C \omega + \int_{\underset{\overset{\shortparallel}{0}}{\partial C}} \theta = \int_C \omega$$

حالا فرض کنید $\quad C \sim C' \qquad \int_{C'} \omega \overset{?}{=} \int_C \omega$



Proof: $$\int_{C'} \omega = \int_{C+\partial\beta} \omega = \int_C \omega + \int_{\partial\beta} \omega$$

$$= \int_C \omega + \int_\beta \underset{=0}{d\omega} = \int_C \omega$$

Now let $\omega \sim \omega'$ & $C \sim C'$ $\qquad \int_{C'} \omega' = \int_{C+\partial\beta} \omega + d\theta =$

$$= \int_c \omega + \int_{\partial\beta} \omega + \int_c d\theta + \int_{\partial\beta} d\theta =$$

$$= \int_c \omega + \int_\beta \underbrace{d\omega}_{0} + \int_{\underbrace{\partial c}_{0}} \theta + \int_{\partial^2\beta = 0} \theta = \int_c \omega$$

پس یک pairing بین $H^r(M)$ و $H_r(M)$ تعریف کردیم که درست و صحیح است.

قضیه دورام:
الف: $H^r(M)$ و $H_r(M)$ متناهی‌البعد هستند.
ب: این pairing ناتبهگن است.

Form a basis for $H^r(M)$: $\{\omega_1, \omega_2, \ldots, \omega_k\}$.

" " " $H_r(M)$: $\{c_1, c_2, \ldots, c_k\}$.

$$\langle \omega_i, c_j \rangle = H_{ij}.$$

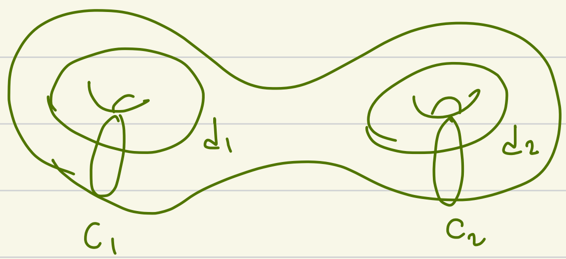
H_{ij} is invertible.

تغییر پایه میدهیم

$$\boxed{\langle \omega_i, c_j \rangle = \delta_{ij}.}$$

$$w'_i = S_{ik}\, w_k \qquad \langle w'_i, c_j \rangle = \delta_{ij}$$

$$S_{ik} \langle \omega_k , c_j \rangle = \delta_{ij} \longrightarrow S_{jk} H_{kj} = \delta_{ij} \longrightarrow \boxed{S = H^{-1}}$$



$\omega_1 \quad \omega_2 \quad \omega_3 \quad \omega_4$

$$\int_{c_j} w_i = \delta_{ij}$$

Some specific and detailed examples.

$$dw = dx \wedge dx + dy \wedge dy = 0$$

① Consider $\underline{M = R^2}$. $\quad \underline{w = x\,dx + y\,dy}$ $\quad$ w is defined on R^2 and is a closed form, $dw = 0$

w is also exact, $w = d(xy)$. xy is a function on R^2.

② $\underline{\text{Consider } M = R^2}$ $\qquad w = \dfrac{y\,dx - x\,dy}{x^2+y^2}$ $\quad$ w is NOT defined on R^2, but it is defined on $R^2 - \{0\}$ which has the topology of S^1

$$dw = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx - \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy$$

$$= -\left\{ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} dx \wedge dy .$$

$$= -\left\{ \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right\} dx \wedge dy = 0 \rightarrow w \text{ is closed on } R^2 - \{0\}.$$

$$dW = 0$$

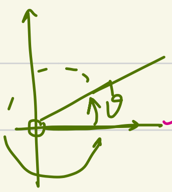
$\omega = df\,?$

We now ask if ω is exact? ω can be written as:

$$\omega = -\,d\,\underbrace{tg^{-1}\left(\frac{y}{x}\right)}\ , \quad \text{because:} \longrightarrow \; d\tan^{-1}\left(\frac{y}{x}\right) = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}\,dx + \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}\,dy$$

$$= \frac{-y\,dx + x\,dy}{x^2+y^2}$$

So it is true that $\omega = df$ where $f = -tg^{-1}\frac{y}{x}$ which is a zero-form, but, $\underbrace{f = \tan^{-1}\frac{y}{x}}$ is not a well-defined [single valued] function on $R^2 - \{0\}$.



To see this we use polar coordinate and then $f = \theta$ and obviously θ is not a single-valued [continuous function] on S^1.

Now what is the meaning of $\underbrace{H^1(S^1) \cong H^1(R^2 - \{0\}) = R}\ ? \Longrightarrow$ ①

صعب

Answer: this means that any $\underline{\text{closed 1-form}}$ $\omega \sim \alpha\, d\theta$ where α is a real number. Or any 1-form $\omega = df + \alpha\, d\theta$ where df is an exact form but as we know $d\theta$ is not. In fact given any closed one form ω to find α, we immediately calculate $\oint_C \omega$ where C is a circle around the origin $\rightarrow$ $\oint_C \omega$ gives a value $\alpha 2\pi$. then $\rightarrow$

$\omega = df + \alpha\, d\theta$. we find α. Now if for another form η we find to see if η is co-homologous to ω? i.e. if $\eta = \omega + df$

① صفحه بعد

$Z_1(S^1) =$ فرم‌های بسته $dw = 0$

$B_1(S^1)$

$w = df$

$$w = \underbrace{df}_{\text{exact}} + \alpha\, \underbrace{d\theta}$$

$$(w - df) = \alpha\, d\theta$$

کلاس کوهمولوژی را با یک عدد حقیقی مشخص و
تعیین می‌کند.

$$[w] = [df] + [\alpha\, d\theta]$$

$$[w] = 0 + \alpha [d\theta]$$

ضریب

مولد

$$\oint_{S^1} w = \int_{S^1} df + \alpha \int_{S^1} d\theta$$

$$= 0 + \alpha\, 2\pi$$

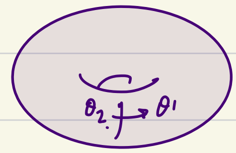
$$\boxed{\alpha = \frac{1}{2\pi} \oint_{S^1} w} \longleftarrow \oint_{S^1} w = \alpha\, 2\pi$$

To answer this question, we immediately calculate $\oint_C \eta$ if $\oint_C \eta = \oint_C \omega$

then η and ω are cohomologous, otherwise not.

---

(3) Example: Consider now $M = \Sigma_1$ = Torus with genus 1.

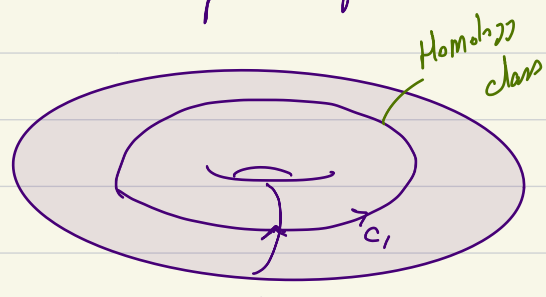


we know that $\underbrace{H_1(\Sigma_1) = \mathbb{Z} \oplus \mathbb{Z}}$. $\longrightarrow$ this means that $H^1(\Sigma_1) = \mathbb{R} \oplus \mathbb{R}$.

$\rightarrow$ Any closed 1-form on the torus $\omega = \underbrace{df} + \underbrace{\alpha d\tilde{\theta}_1 + \beta d\tilde{\theta}_2} \rightarrow$

$\alpha, \beta \in \mathbb{R}$

the equivalence class of any closed one-form is determined by the pair of real numbers (α, β). To find the class of each form we calculate $\oint_{C_1} \omega$ & $\oint_{C_2} \omega$

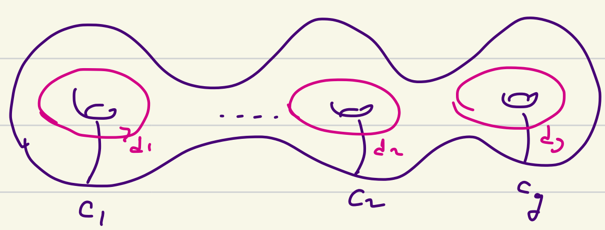


$$\oint_{C_1} \omega = \int_{\theta_1=0}^{2\pi} \omega = \int_{\theta_1=0}^{2\pi} df + \alpha d\theta_1 + \beta d\theta_2 = 0 + 2\pi\alpha_1 + 0 = 2\pi\alpha$$

$$\oint_{C_2} \omega = \dots = 2\pi\beta$$

Once α and β are calculated, the class of ω is determined.

In general for Σ_g = Torus with genus g



Each class is characterized by

Cohomology class of $H^1(\Sigma_g)$

$2g$ numbers, namely: $(\alpha_i ; \beta_i) = \frac{1}{2\pi}\left(\oint_{c_i} \omega , \oint_{d_i} \omega \right)$.

---

④: Example: let $M = S^2$ we know that $\underbrace{H_1(S^2) = 0}$ $\rightarrow$

we know this from the fact that $H_1(S^2)$ is the Abelian form of $\pi_1(S^2)$ (this is true for any manifold) and since $\pi_1(S^2) = \{e\} \rightarrow H_1(S^2) = 0$.

this means that any closed 1-form on S^2 must be exact!

In other words, you cannot find a form w such that $dw = 0$ but $w \neq df$.

$\omega = w_1(\theta_1, \theta_2)\, d\theta^1 + w_2(\theta_1, \theta_2)\, d\theta^2$

---

⑤: Example: Consider $\vec{B} = \frac{g}{4\pi r^2} \hat{r}$ A magnetic monopole.

We can calculate $\oint_{S^2} \vec{B} \cdot d\vec{s} = g$. ① if we now calculate $\vec{\nabla} \cdot \vec{B}$ we find that $\vec{\nabla} \cdot \vec{B} = 0$ if $r \neq 0 \longrightarrow \vec{\nabla} \cdot \vec{B} = 0$ over S^2.

we may then think that if $\vec{\nabla} \cdot \vec{B} = 0$ over S^2, then $\vec{B} = \vec{\nabla} \times \vec{A}$ over S^2! if this is the case, then we can write $\oint_{S^2} \vec{B} \cdot d\vec{s} = \int_{S^2} (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$

Stokes' theorem $= \int_{\partial S^2} \vec{A} \cdot d\vec{l} = 0$! which contradicts ①!

$\vec{E} = \dfrac{q}{4\pi r^2}\hat{r}$ $\qquad$ $\vec{E}$ is defined on $R^3 - \{0\} \equiv S^2$

$\vec{E}$ = 2-form on S^2 $\longleftrightarrow$ $W_{ij} = \epsilon_{ijk} E_k$.

$\underbrace{\vec{\nabla}\cdot\vec{E} = 0}$ $\quad$ S^2 روی $\longleftrightarrow$ $dW = 0$

W is closed. $\xrightarrow{\text{if}}$ $W = dA$

یعنی $\displaystyle\int_{S^2} w = \int_{S^2} dA = \int_{\partial S^2} A = 0.$ $\Longrightarrow$ $\displaystyle\int_{S^2} \vec{E}\cdot d\vec{s} = 0$

غلط $\displaystyle\int_{S^2} \vec{E}\cdot ds = \int_{S^2} (\nabla\times A) = \int_{\partial S^2} A = 0$

نادرست

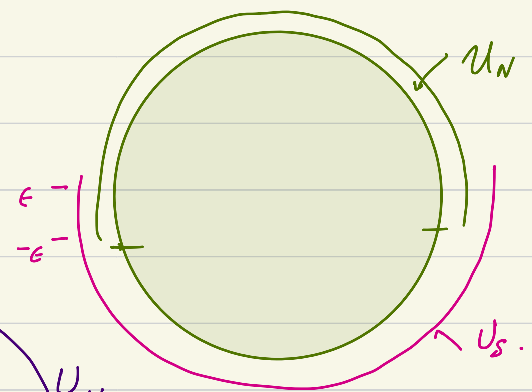
$\underbrace{\text{میدانیم: } q = \int_{S^2} \vec{E}\cdot d\vec{s} \text{ وحال آنکه}}$

جواب : $\quad H^2(S^2) = R \qquad H_2(S^2) = Z$

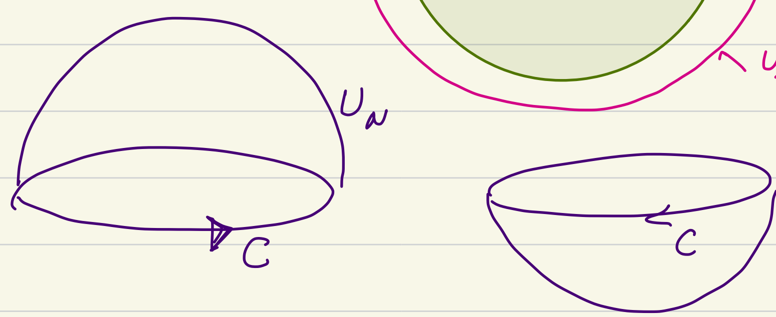
$\vec{E} \neq \vec{\nabla}\times\vec{A}$

How to resolve this contradiction? → the answer is that on S^2 if $\nabla\cdot B = 0$ you cannot say that $B = \nabla\times A$! Since S^2 is not simply connected. But you can cover S^2 with U_N & U_S: two open charts:

$$\rightarrow \quad \oint_{S^2} B\cdot ds \equiv \int_{U_N} B\cdot ds + \int_{U_S} B\cdot ds$$



$$\left.\begin{array}{ll} \rightarrow \text{ on } U_N: & B = \nabla\times A_N \\ \rightarrow \text{ on } U_S: & B = \nabla\times A_S \end{array}\right\}$$



$$\Longrightarrow \oint_{S^2} B\cdot ds = \int_{U_N} (\nabla\times A_N)\cdot ds + \int_{U_S} (\nabla\times A_S)\cdot ds \xrightarrow{\text{stokes}}$$

$$= \oint_C A_N\cdot dl + \oint_C A_S\cdot dl = \oint_C (A_N - A_S)\cdot dl$$

equator

$A_N - A_S$ is defined on $U_N \cap U_S$.

Now we note that $\nabla\times(A_N - A_S) = 0$ on C, since both of them should yield the same B. So $A_N - A_S$ is a closed form →

if we argue that $A_N - A_S = \underbrace{\nabla\varphi}_{\text{on } C}$ then this again leads to

$$\oint_{S^2} B.ds = \oint_C (A_N - A_S).dl = \oint_C \nabla\varphi.dl = 0 \quad ! \; ? \; !!!$$

which is not acceptable. the resolution is that although

$\nabla \times (A_N - A_S) = 0$ ($A_N - A_S$ is an closed form on C) $\rightarrow$

we cannot say that $A_N - A_S$ is exact, i:e that $A_N - A_S = \nabla\varphi$

In fact from what we learnt from Example 1, we can say

that ? $\underbrace{A_N - A_S}_{\text{exact}} = \nabla\varphi + \alpha\, \underline{\underline{d\theta}}$ $\rightarrow$ $\oint_{S^2} B.ds = \alpha \int_C d\theta = 2\pi\alpha$. $\rightarrow q$

In this way we resolve the apparent contradiction.

Consider now the same magnetic field $\vec{B} = \frac{g}{r^2}\hat{r}$ in $\underbrace{R^3 - \{0\}}$.

we have $\vec{\nabla}\cdot\vec{B} = 0$. Now consider the open neighborhoods to be

as follows: $\underbrace{U_N = R^3 - \{\text{negative } z\text{-axis}\}}$. $U_S = R^3 - \{\text{positive } z \text{ axis}\}$.

U_N & U_S are open, simply connected and their union cover R^3.

→ in U_N : $\exists A_N$: $\vec{B} = \nabla \times \vec{A}_N$

in U_S : $\exists A_S$: $\vec{B} = \nabla \times \vec{A}_S$

in $U_N \cap U_S = R^3 - \{z\text{-axis}\}$. $\quad \nabla \times (\vec{A}_N - \vec{A}_S) = 0$

$\vec{A}_N - \vec{A}_S = df + \alpha\, d\theta$ $\quad$ the rest of the argument is as before

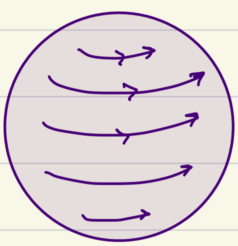
exact form.

→ $\oint_{S^2} \vec{B} \cdot d\vec{s} = 2\pi\alpha$.

question: What are the explicit forms of A_N & A_S?

Let $\underbrace{A_N = A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}}$

we want $\underbrace{\vec{\nabla} \times \vec{A}_N = \vec{B}_N}$ on U_N. We use curl in spherical coordinates:



$$\vec{\nabla} \times \vec{A}_N = \frac{1}{r^2 \sin\theta} \begin{bmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\, \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \underset{0}{\underset{\parallel}{A_r}} & \underset{0}{\underset{\parallel}{rA_\theta}} & r\sin\theta\, A_\varphi \end{bmatrix} = \frac{g}{r^2} \hat{r}$$

take → $\vec{A} = (0, 0, A_\varphi(r, \theta))$

$$\to \quad \vec{\nabla} \times \vec{A}_N = \frac{1}{r^2 \sin\theta} \left\{ \left(r\cos\theta\, A_\varphi + r\sin\theta \frac{\partial A_\varphi}{\partial \theta} \right) \hat{r} \right.$$

$$\left. - r\hat{\theta} \left\{ \underbrace{\sin\theta\, A_\varphi + r\sin\theta \frac{\partial A_\varphi}{\partial r}}_{0} \right\} \right\}$$

we want $B_{\hat{\theta}} = 0$ so we demand $A_\varphi + r\frac{\partial A_\varphi}{\partial r} = 0 \rightarrow$

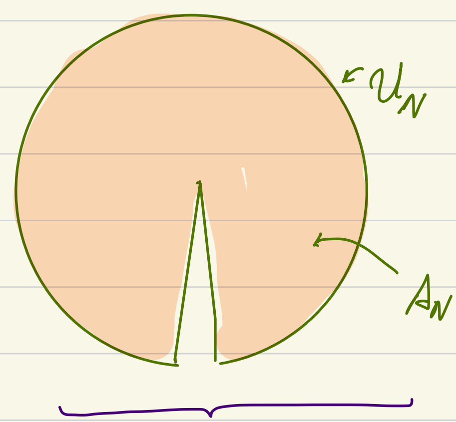
$\rightarrow A_\varphi = r^{-1} f(\theta)$, we want $\left\{ r\cos\theta\, A_\varphi + r\sin\theta \frac{\partial A_\varphi}{\partial \theta} \right\} \frac{1}{r^2 \sin\theta} = \frac{g}{r^2}$

$\rightarrow \rightarrow A_\varphi(r,\theta) = \frac{1}{r} f(\theta) \rightarrow \quad \cos\theta\, f(\theta) + \sin\theta\, f'(\theta) = g\sin\theta .$

$\rightarrow \frac{d}{d\theta}\left(\sin\theta\, f(\theta) \right) = g\sin\theta \rightarrow \quad \sin\theta\, f(\theta) = -g\cos\theta + C$

$\rightarrow f(\theta) = \frac{C - g\cos\theta}{\sin\theta} \rightarrow \quad \boxed{\vec{A}_N(r,\theta,\varphi) = \frac{+1}{r} \frac{C - g\cos\theta}{\sin\theta}\, \hat{\varphi}}$

How to determine C ?

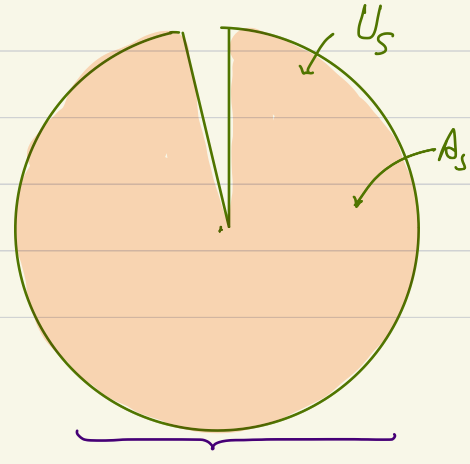


if $\theta \rightarrow 0$ $f(\theta)$ should be continuous. so

$C = g \rightarrow \quad \vec{A}_N(r,\theta,\varphi) = \frac{g}{r}\left(\frac{1-\cos\theta}{\sin\theta} \right) \hat{\varphi}$

Note that: $\vec{A}_N$ is not defined on the negative z-axis.

Consider now U_S: By the same argument we obtain:



$\vec{A}_S(r,\theta,\varphi) = \frac{1}{r} \frac{C - g\cos\theta}{\sin\theta}\, \hat{\varphi}$

to determine C, we note that A_S should be

$$\vec{B} = \frac{g}{r^2}\hat{r} = \vec{\nabla}\times\vec{A}_N = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta\,A_\varphi \end{vmatrix}$$

$$\vec{A}_N = A_r\,\hat{r} + A_\theta\,\hat{\theta} + A_\varphi\,\hat{\varphi}$$

دوران ناوردایی : نسبت به φ تغییر نمی‌کند

Ansatz: $$\vec{A}_N = A_\varphi(r,\theta,\varphi)\,\hat{\varphi} = A_\varphi(r,\theta)\,\hat{\varphi}$$

$$\frac{g}{r^2}\hat{r} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & r\sin\theta\,A_\varphi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left\{ \hat{r}\left[r\cos\theta\,A_\varphi + r\sin\theta\,\frac{\partial A_\varphi}{\partial\theta}\right] - r\hat{\theta}\,\underbrace{\left[\sin\theta\,A_\varphi + r\sin\theta\,\frac{\partial A_\varphi}{\partial r}\right]}_{=0}\right.$$

صفر

$$\boxed{A_\varphi = r^{-1} f(\theta)} \longleftarrow A_\varphi + r\frac{\partial A_\varphi}{\partial r} = 0$$

$$r\frac{\partial A_\varphi}{\partial r} = -r^{-1} f(\theta) \longrightarrow$$

r ها :
$$\frac{1}{r^2\sin\theta}\left\{ r\cos\theta\,\frac{1}{r}f(\theta) + r\sin\theta\,\frac{1}{r}f'(\theta)\right\} = \frac{g}{r^2}$$

$$\longrightarrow \quad \cot\theta\, f + \frac{df}{d\theta} = g$$

$$\cos\theta\, f(\theta) + \sin\theta \frac{df}{d\theta} = g \sin\theta \longrightarrow \frac{d}{d\theta}\left[\sin\theta\, f(\theta)\right] = g\sin\theta$$

$$\rightarrow \quad \sin\theta\, f(\theta) = -g\cos\theta + C \longrightarrow f(\theta) = \frac{C - g\cos\theta}{\sin\theta}$$

$$\rightarrow \quad \boxed{\vec{A}_N(r,\theta) = \frac{c - g\cos\theta}{r\sin\theta}\,\hat{\varphi}}$$

شرط پیوستگی در

$U_N \Rightarrow \theta = 0$

$C = g$

$$\vec{A}_N(r,\theta) = \frac{g(1-\cos\theta)}{r\sin\theta}\,\hat{\varphi}$$

to find $\vec{A}_S$, we demand continuity at the line $\theta = \pi$

$$A_S(r,\theta) = \frac{-g(1+\cos\theta)}{r\sin\theta}\,\hat{\varphi}.$$

Continuous on the negative z-axis, i.e. on $\theta = \pi$. $\longrightarrow$ $C = -g$

$$\rightarrow \boxed{\vec{A}_S(r,\theta,\varphi) = \frac{-g}{r}\,\frac{(1+\cos\theta)}{\sin\theta}\,\hat{\varphi}} \longleftarrow \quad A_N(r,\theta,\varphi) = \frac{g}{r}\,\frac{(1-\cos\theta)}{\sin\theta}\,\hat{\varphi}$$

Now on $U_N \cap U_S$: $\vec{A}_N(r,\theta,\varphi) - \vec{A}_S(r,\theta,\varphi) = \dfrac{2g}{r\sin\theta}\,\hat{\varphi} = \nabla(2g\hat{\varphi})$

$U_N \cap U_S = R^3 - (z\text{-axis})$ and this space has the topology of a circle. we know $\nabla \times (A_N - A_S) = 0 \rightarrow A_N - A_S$ is closed

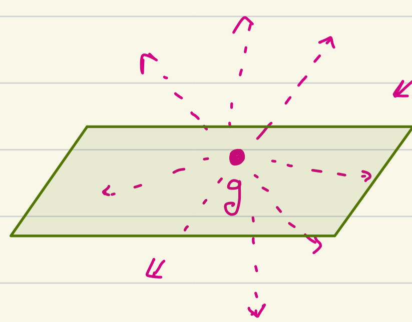
$$\rightarrow \quad A_N - A_S = \underbrace{df}_{\text{exact}} + \alpha\, d\theta \quad \rightarrow \quad \oint_C (A_N - A_S)\cdot dl = 2\pi\alpha.$$

for some real number α. In fact we can find α by explicit integration:

$$\oint (\vec{A}_N - \vec{A}_S)\cdot dl = \oint \frac{2g}{r\sin\theta}\,\hat{\varphi}\cdot[\,r\sin\theta\, d\varphi]\,\hat{\varphi} = 4g\pi$$

Now Bring in quantum mechanics.

Consider an electron moving in the field of this monopole.



Magnetic Monopole.

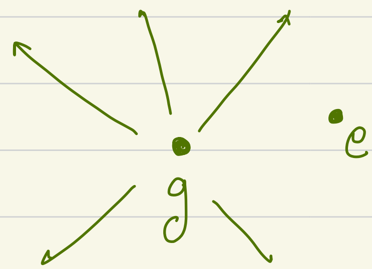
$$\vec{\nabla} V = \left[\hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\theta} \frac{\partial}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \hat{\varphi} \right] V$$

therefore $\quad \vec{A}_N - \vec{A}_S = \dfrac{2g}{r \sin\theta} \hat{\varphi} = \nabla (2g\varphi)$

But φ is not a function in $V_N \cap V_S$.

Bring Q.M. into account

let an electron moves in the field of magnetic Monopole



$$\frac{P^2}{2m} \psi = i\hbar \frac{\partial}{\partial t} \psi \qquad \text{for free particle}$$

$$\frac{\left(\vec{P} - \frac{e}{c} \vec{A}\right)^2}{2m} \psi = i\hbar \frac{\partial \psi}{\partial t} \qquad \text{for a particle moving in an EM field.}$$

But we know that for one Magnetic field B, there are different A's. $\quad A, A' \longrightarrow B$

We can show that the difference appears only in the phase of ψ.

$$A' = A + \nabla\chi \longrightarrow \psi' = e^{\frac{ie\chi}{\hbar c}}\psi$$

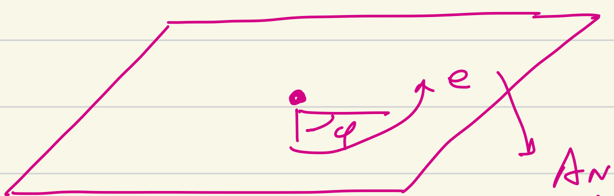
تمرین: اگر ψ درست باشد؛ A هم درسته، ψ' درست باشد؛ A' هم درسته.

استفاده از این تبدیل در حضور Magnetic Monopole.

Mag. Monopole $\to A_N$, A_S

Both define the magnetic field on $U_N \cap U_S$

$= R^3 - z$-axis.



در صفحه‌ی xy-plane.

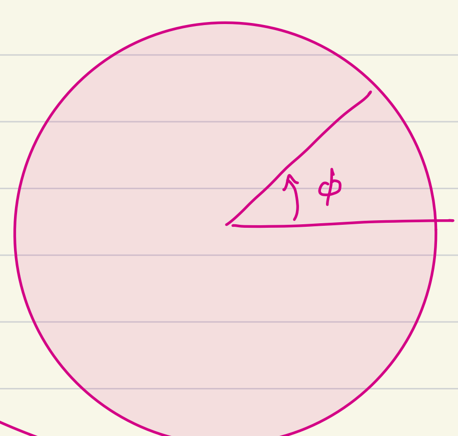
A_N, A_S

$$\psi'(r,\phi) = e^{\frac{ie\chi}{\hbar c}}\psi(r,\phi)$$

what is χ: $\chi = 2g\phi$ (We had)

$$\leadsto \psi'(r,\phi) = e^{\frac{i2eg}{\hbar c}\phi}\psi(r,\phi)$$

فاز



یک دور بزنیم

ψ باید تک‌مقدار باشد

$$\frac{2eg}{\hbar c} = n$$

یعنی

$$e = n\left(\frac{\hbar c}{2g}\right)$$

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{we have,} \quad \psi' = \psi e^{i\alpha\chi} = \psi e^{\frac{ie}{\hbar c}\chi}.$$

Now: on equator: $A_N - A_S = \nabla\left(\frac{2g}{\ }\varphi\right)$.

$$\rightarrow \quad \psi_N = \psi_S\, e^{\left(\frac{ie}{\hbar c}\right)2g\phi} \rightarrow \begin{cases} \psi_N(0) = \psi_S(0) \\ \psi_N(2\pi) = \psi_S(2\pi)\, e^{\frac{ie}{\hbar c}2g\,2\pi}. \end{cases}$$

Dirac Quantization of electric & magnetic $\leftarrow$ $\frac{2ge}{\hbar c} = n$