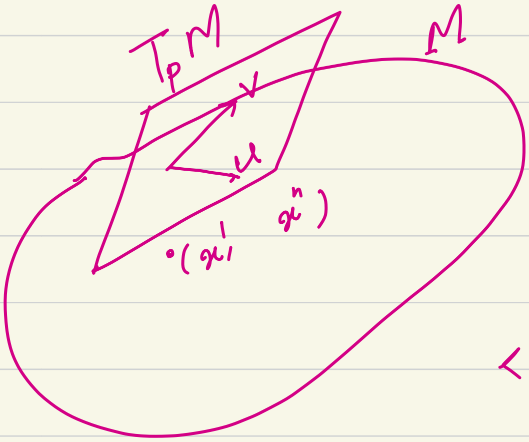
جلسه‌ی پنجم: ۲۵ خرداد ۹۹

Riemmanian Geometry.



$$(u,v) \in T_pM \longrightarrow R$$

$$\langle u,v \rangle = g(u,v)$$

$$\langle u,v \rangle = \text{bilinear form.} \quad g(u,v)$$

تانسور

g is called the metric tensor. $\quad g(u,u) \not> 0 \quad$ الزاما

if $g(u,x)=0 \quad \forall x \in T_pM \longrightarrow u=0$: غیر تبهگن است g ولی

Non-degenerate.

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu \qquad g \text{ is a tensor of rank } (0,2).$$

$$g(u,v) = g(u^\alpha \partial_\alpha, v^\beta \partial_\beta) = u^\alpha v^\beta \underbrace{g(\partial_\alpha, \partial_\beta)}$$

$$g(\partial_\alpha,\partial_\beta) = g_{\mu\nu}\, dx^\mu \otimes dx^\nu (\partial_\alpha, \partial_\beta) = g_{\alpha\beta}.$$

متریک $\longrightarrow$ Inner product $\longrightarrow$ طول / زاویه $\longrightarrow$ ...... , انی

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu'\nu'}\, dx^{\mu'} \otimes dx^{\nu'}$$

$$g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu'\nu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}\, dx^\mu \otimes dx^\nu$$

$$g_{\mu\nu} = g_{\mu'\nu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}$$

$g_{\mu\nu} \to g^{\mu\nu}$ is defined.

نیز زمانی تعریف می‌شود که g^{-1} وجود داشته باشد.

$g(u,u)=0$ $\quad g(u,x)=0 \ \forall x$

$u=0$

$$g = \begin{bmatrix} & & & \\ & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$g(u,u) = 0$

$$u = \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} \ \vdots \ \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}$$

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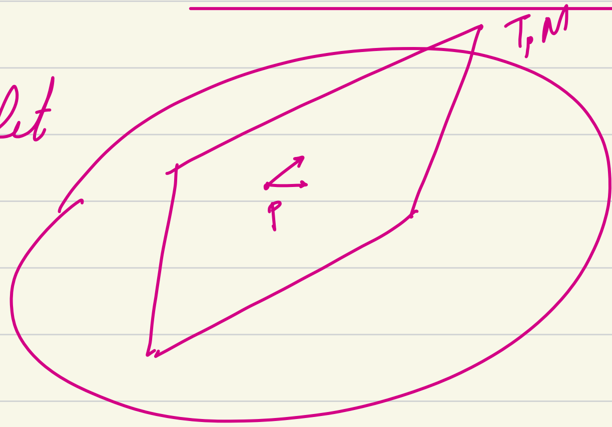
let



$g(u,u) = g_{\mu\nu} u^\mu u^\nu = \|u\|^2$

مثبت نیست الزاما

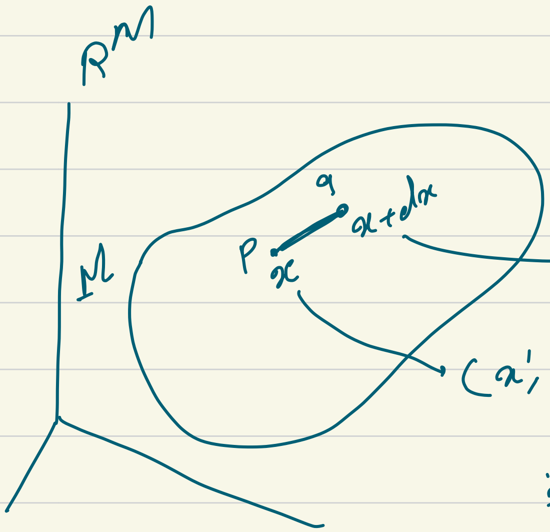
$$\cos\theta = \frac{g(u,v)}{\sqrt{g(u,u)}\sqrt{g(v,v)}}$$

معیار تقریبی زاویه

مثال: ① در R^n با مختصات $(x^1, \dots, x^n)$

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i$$

Induced Metric ② متریک القائی



مختصات نقطه‌ی سر خط

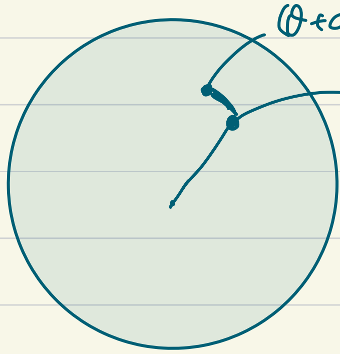
$(x^1+dx^1, x^2+dx^2, \ldots, x^n+dx^n) \rightarrow (y^1, y^2, \ldots, y^M)$

$(x^1, x^2, \ldots, x^n)$ $\qquad (dx^1, dx^2, \ldots, dx^n)$

مختصات نقطه‌ی سر خط $\qquad$ بردار جابجایی است.

$(y^1, y^2, \ldots, y^M)$ مختصات در فضا R^M

$$ds^2 = \text{طول بردار} = g_{ij}(y)\, dy^i dy^j \qquad i,j : 1 \text{ to } M$$



$(\theta+d\theta, \varphi+d\varphi)$ $\qquad (x+dx, y+dy, z+dz)$

(θ, φ) $\qquad (x, y, z)$

$$ds^2 = \underbrace{g_{ij}(y)\, \frac{\partial y^i}{\partial x^\mu} \frac{\partial y^j}{\partial x^\nu}}_{g_{\mu\nu}(x) \,=\, \text{Induced Metric.}} dx^\mu dx^\nu$$

از این قضیه استفاده میکنیم، هر M را میتوان در R^n (n: به اندازه‌ی کافی بزرگ) نشاند.

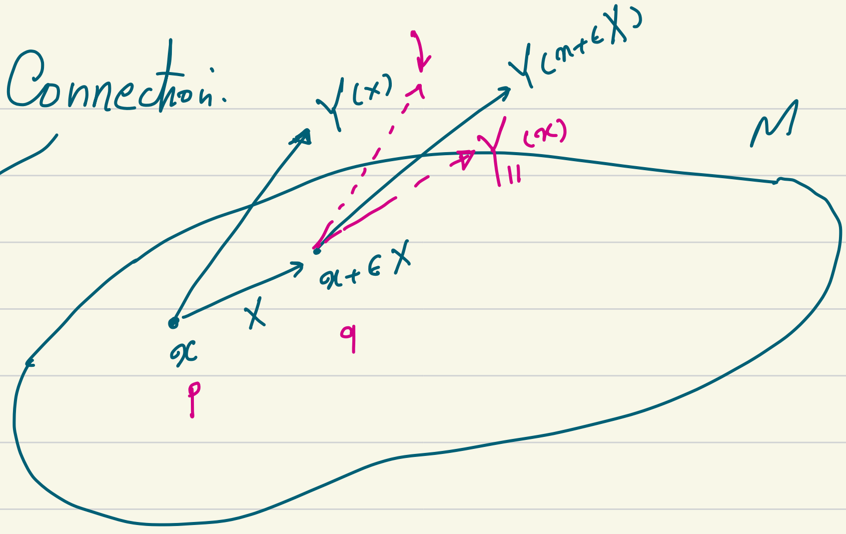
تمرین ①: متریک القائی کره‌ی S^n را بدست آورید. $S^n \subset R^{n+1}$.

متریک القائی بر سیلندر $S^1 \times R^n$

Parallel Transport & Connection.

انتقال موازی

همبند



$Y(x) \in T_x M$

$Y(x+\epsilon X) \in T_{x+\epsilon X} M$

$$\frac{Y(x+\epsilon X) - Y(x)}{\epsilon}$$ بی‌معنی است.

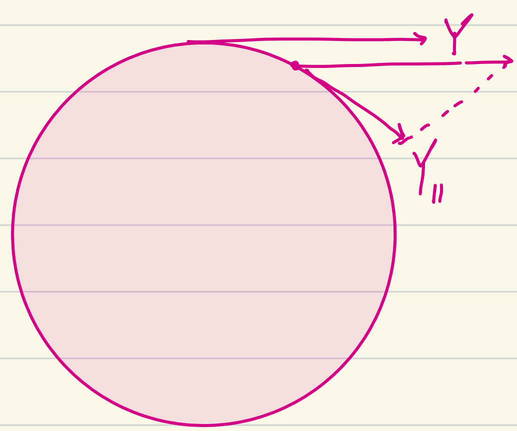
نقطه $p: \quad x^{\mu}$

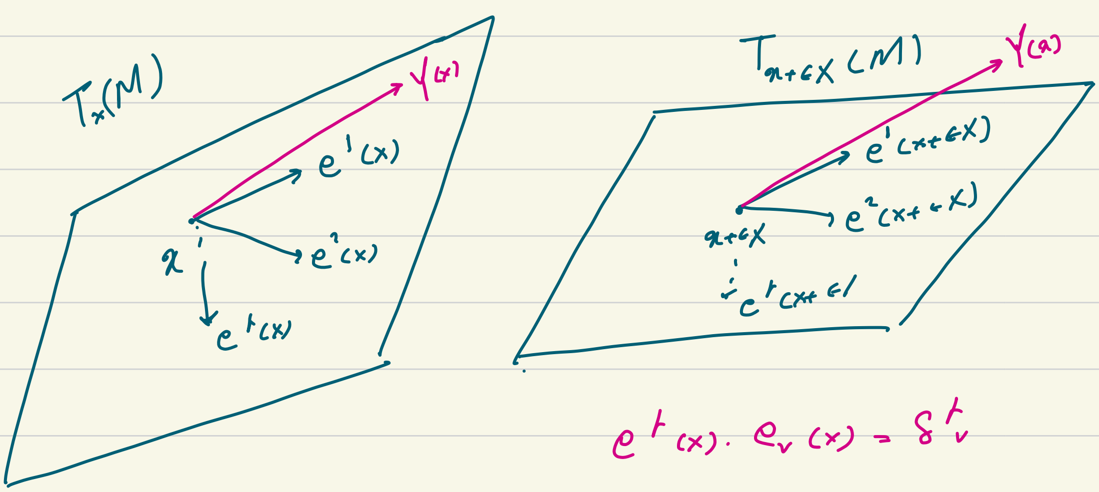
نقطه $q: \quad x^{\mu} + \epsilon X^{\mu}$

$$\nabla_X Y := \frac{Y(x+\epsilon X) - Y_{\parallel}(x)}{\epsilon}$$

$Y_{\parallel}(x) \in T_{x+\epsilon X} M$

$Y(x) = Y^i(x)\, e_i \longrightarrow R^n$ بردار پایه





$e^{\mu}(x) \cdot e_{\nu}(x) = \delta^{\mu}_{\nu} \qquad \forall x.$

$$Y^{\beta}_{\parallel}(x) = \left(Y^{\alpha}(x)\, e_{\alpha}(x)\right) \cdot e^{\beta}(x+\epsilon X)$$

$$= Y^{\alpha}(x)\, \underbrace{e_{\alpha}(x) \cdot \left\{ e^{\beta}(x)\right.} + \left.\epsilon X^{\nu} \frac{\partial e^{\beta}}{\partial x^{\nu}} \right\}$$

$$= Y^{\beta}(x) + \epsilon\, X^{\nu}\, Y^{\alpha}(x) \left(e_{\alpha}(x) \cdot \frac{\partial e^{\beta}}{\partial x^{\nu}}\right)$$

$$(\nabla_X Y)^{\mu} = \frac{Y^{\mu}(x+\epsilon X) - Y^{\mu}_{\parallel}(x)}{\epsilon \to 0} = \frac{Y^{\mu}(x+\epsilon X) - \left\{ Y^{\mu}(x) + \epsilon X^{\nu} Y^{\alpha} \left(e_{\alpha} \cdot \frac{\partial e^{\mu}}{\partial x^{\nu}}\right)\right\}}{\epsilon}$$

$$(\nabla_X Y)^{\mu} = X^{\alpha} \partial_{\alpha} Y^{\mu} - X^{\nu}\, Y^{\alpha} \underbrace{\left(e_{\alpha} \cdot \frac{\partial e^{\mu}}{\partial x^{\nu}}\right)}_{\text{Connection}}$$

$$\Gamma^{\mu}_{\alpha\nu}$$

$$(\nabla_X Y)^{\mu} = X^{\alpha} \partial_{\alpha} Y^{\mu} - X^{\nu} Y^{\alpha} \Gamma^{\mu}_{\alpha\nu}$$

$$(\nabla_X Y)^{\mu} = X^{\nu} \Big[\underbrace{\partial_{\nu} Y^{\mu} - \Gamma^{\mu}_{\alpha\nu} Y^{\alpha}}_{Y^{\mu}_{;\nu}} \Big].$$

$$Y^{\mu}_{;\nu} := Y^{\mu}_{,\nu} - \Gamma^{\mu}_{\alpha\nu} Y^{\alpha}$$

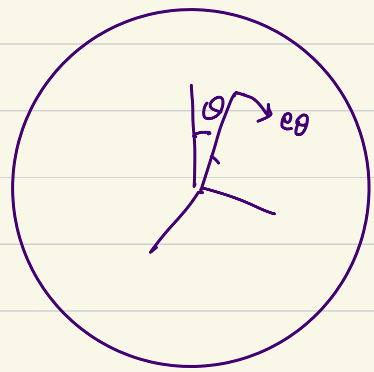
Connection Coefficient.

$$\Gamma^{\mu}_{\alpha\nu} = \left\langle e_{\alpha}, \frac{\partial e^{\mu}}{\partial x^{\nu}} \right\rangle \longleftarrow \Gamma^{\mu}_{\alpha\nu} := e_{\alpha} \cdot \frac{\partial e^{\mu}}{\partial x^{\nu}}$$

$\Gamma^{\mu}_{\nu\alpha}$: (θ, φ). در سطح کرهٔ دوبعدی:

$$\Gamma^{\theta}_{\theta\theta} := e^{\theta} \cdot \left[\frac{\partial e^{\theta}}{\partial \theta} \right]. \qquad e^{\theta} = e_{\theta}$$

$e_{\theta} = ?$ $\qquad e_{\theta} = \dfrac{\partial \vec{r}}{\partial \theta}$



$$\vec{r} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$e_{\theta} = (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta).$$

$$\frac{\partial e_{\theta}}{\partial \theta} = (-\sin\theta \cos\phi, -\sin\theta \sin\phi, -\cos\theta).$$

$$e^{\theta} \cdot \frac{\partial e_{\theta}}{\partial \theta} = (-\sin\theta \cos\theta + \sin\theta \cos\theta) = 0 \longrightarrow \Gamma^{\theta}_{\theta\theta} = 0$$

R^n $\qquad (e_1, e_2 \quad e_n) \in T_pM$

$\qquad (e^1, e^2 \quad e^n) \in T_pM^*$ $\qquad \langle e^i, e_j \rangle = \delta^i_j$

Properties of Covariant derivative.

① $\quad \Gamma^{\mu}_{\alpha\nu} = e_{\alpha} \cdot \dfrac{\partial e^{\mu}}{\partial x^{\nu}}$ $\qquad$ از $\; e_{\alpha} \cdot e^{\mu} = \delta^{\mu}_{\alpha}$

$$\boxed{\Gamma^{\mu}_{\alpha\nu} = -e^{\mu} \cdot \frac{\partial e_{\alpha}}{\partial x^{\nu}}} \longleftarrow \frac{\partial e_{\alpha}}{\partial x^{\nu}} \cdot e^{\mu} + e_{\alpha} \cdot \frac{\partial e^{\mu}}{\partial x^{\nu}} = 0$$

② $(\nabla_X Y)^\mu = (Y^\mu{}_{,\nu} - \Gamma^\mu{}_{\alpha\nu} Y^\alpha) X^\nu$

① $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

② $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

③ $[\nabla_X (fY)]^\mu = [(fY)^\mu{}_{,\nu} - \Gamma^\mu{}_{\alpha\nu} (fY)^\alpha] X^\nu = \big((f_{,\nu} Y^\mu + f Y^\mu{}_{,\nu}) - \Gamma^\mu{}_{\alpha\nu} f Y^\alpha\big) X^\nu$

$$= f_{,\nu} Y^\mu X^\nu + f\, Y^\mu{}_{;\nu}$$

$$\nabla_X (fY) = X(f) Y + f \nabla_X Y.$$

④ $\nabla_{fX} Y = f \nabla_X Y. \longrightarrow \nabla_X Y \neq \mathcal{L}_X Y.$ مشتق هموردا برعکس لی تانسوری است.

$\mathcal{L}_X Y = [X, Y]$ داشتیم

$$f \mathcal{L}_X Y \neq \mathcal{L}_{fX} Y$$

مشتق هموردای تانسورها:

$$\nabla_X f := X(f)$$

$\nabla_X Y$ قبلاً تعریف کردیم.

$$\nabla_X (Y \otimes Z) := \nabla_X Y \otimes Z + Y \otimes \nabla_X Z$$

$$\nabla_X (Y \otimes T) := \nabla_X Y \otimes T + Y \otimes \nabla_X T$$

بردار تانسور

$$(YZ)^{\mu\nu}{}_{;\alpha} = Y^{\mu}{}_{;\alpha} Z^{\nu} + Y^{\mu} Z^{\nu}{}_{;\alpha}$$

$$= \left(Y^{\mu}{}_{,\alpha} - \Gamma^{\mu}{}_{\alpha\beta} Y^{\beta}\right) Z^{\nu} + Y^{\mu}\left(Z^{\nu}{}_{,\alpha} - \Gamma^{\nu}{}_{\alpha\beta} Z^{\beta}\right)$$

$$(YZ)^{\mu\nu}{}_{;\alpha} = (Y^{\mu} Z^{\nu})_{,\alpha} - \Gamma^{\mu}{}_{\alpha\beta} Y^{\beta} Z^{\nu} - \Gamma^{\nu}{}_{\alpha\beta} Y^{\mu} Z^{\beta}$$

extension

$$T^{\mu\nu}{}_{;\alpha} = T^{\mu\nu}{}_{,\alpha} - \Gamma^{\mu}{}_{\alpha\beta} T^{\beta\nu} - \Gamma^{\nu}{}_{\alpha\beta} T^{\mu\beta}$$

$$T^{\mu\nu\gamma}{}_{;\alpha} = T^{\mu\nu\gamma}{}_{,\alpha} - \Gamma^{\mu}{}_{\alpha\beta} T^{\beta\nu\gamma} - \Gamma^{\nu}{}_{\alpha\beta} T^{\mu\beta\gamma} - \Gamma^{\gamma}{}_{\alpha\beta} T^{\mu\nu\beta}$$

let w = 1-form $\quad$ $w_{\mu} Y^{\mu}$ is a function.

Y = Vector

$$(\omega_{\mu} Y^{\mu})_{;\alpha} = (\omega_{\mu} Y^{\mu})_{,\alpha} \rightarrow$$

$$\omega_{\mu;\alpha} Y^{\mu} + \omega_{\mu} Y^{\mu}{}_{;\alpha} = \omega_{\mu,\alpha} Y^{\mu} + \omega_{\mu} Y^{\mu}{}_{,\alpha}$$

$$\omega_{\mu;\alpha} Y^{\mu} + \omega_{\mu}\left[\cancel{Y^{\mu}{}_{,\alpha}} - \Gamma^{\mu}{}_{\alpha\beta} Y^{\beta}\right] = \omega_{\mu,\alpha} Y^{\mu} + \cancel{\omega_{\mu} Y^{\mu}{}_{,\alpha}}$$

$$\omega_{\beta;\alpha} Y^{\beta} - \omega_{\mu} \Gamma^{\mu}{}_{\alpha\beta} Y^{\beta} = \omega_{\beta,\alpha} Y^{\beta}$$

$$\boxed{\omega_{\beta;\alpha} = \omega_{\beta,\alpha} + \Gamma^{\mu}{}_{\alpha\beta}\, \omega_{\mu}.}$$

$$(\omega_\alpha \theta_\beta)_{;\gamma} = \omega_{\alpha;\gamma}\,\theta_\beta + \omega_\alpha\,\theta_{\beta;\gamma}$$

$$= (\omega_{\alpha,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\,\omega_\nu)\,\theta_\beta + \omega_\alpha(\theta_{\beta,\gamma} + \Gamma^{\nu}_{\beta\gamma}\,\theta_\nu)$$

$$= (\omega_\alpha\theta_\beta)_{,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\,\omega_\nu\theta_\beta + \Gamma^{\nu}_{\beta\gamma}\,\omega_\alpha\theta_\nu$$

extension

$$S_{\alpha\beta;\gamma} = S_{\alpha\beta,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\,S_{\nu\beta} + \Gamma^{\nu}_{\beta\gamma}\,S_{\alpha\nu}$$

$$\Gamma^{\mu}_{\alpha\nu} = -e^{\mu}\cdot\frac{\partial e_\alpha}{\partial x^\nu}$$

این خیلی خاص است برای دکارتی
معرفی شده است.
و متقارن است. نسبت به اندیس پایینی.

$$e_\theta := \frac{\partial \vec r}{\partial \theta} \qquad e_\varphi := \frac{\partial \vec r}{\partial \varphi}$$

$$e_\alpha = \frac{\partial \vec r}{\partial x^\alpha} \longrightarrow \Gamma^{\mu}_{\alpha\nu} = -e^{\mu}\cdot\frac{\partial \vec r}{\partial x^\nu \partial x^\alpha} \longrightarrow \Gamma^{\mu}_{\alpha\nu} = \Gamma^{\mu}_{\nu\alpha}.$$

تبدیل‌پذیری $\Gamma^{\mu}_{\alpha\nu}$:

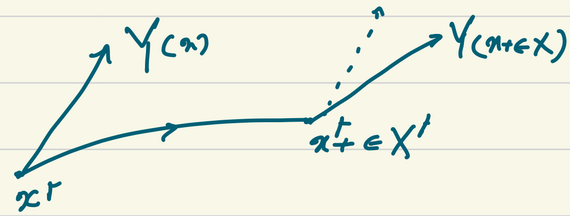
$$\Gamma^{\mu'}_{\alpha'\nu'} = -e^{\mu'}\cdot\frac{\partial e_{\alpha'}}{\partial x^{\nu'}} =$$

$$= -\frac{\partial x^{\mu'}}{\partial x^{\mu}}\,e^{\mu}\cdot\frac{\partial}{\partial x^{\nu'}}\Big[\frac{\partial x^\alpha}{\partial x^{\alpha'}}\,e_\alpha\Big] = -\frac{\partial x^{\mu'}}{\partial x^{\mu}}\,e^{\mu}\cdot\Big\{\frac{\partial^2 x^\alpha}{\partial x^{\nu'}\partial x^{\alpha'}}\,e_\alpha + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\,\frac{\partial e_\alpha}{\partial x^{\nu'}}\Big\}$$

$$\Gamma^{\mu'}_{\alpha'\nu'} = -\frac{\partial x^{\mu'}}{\partial x^{\mu}}\,\frac{\partial^2 x^{\mu}}{\partial x^{\nu'}\partial x^{\alpha'}} + \frac{\partial x^{\mu'}}{\partial x^{\mu}}\,\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\,\frac{\partial x^{\nu}}{\partial x^{\nu'}}\,\underbrace{\Big(-e^{\mu}\cdot\frac{\partial e_\alpha}{\partial x^\nu}\Big)}_{\Gamma^{\mu}_{\alpha\nu}}$$

جلسه‌ی ششم ؛ ۲۷ خرداد ۱۳۹۹

فهرست → $\cancel{\nabla_X}$, $\cancel{\Gamma}$, $\cancel{\text{geo}}$, $\cancel{\text{comp.}}$, Riem. Ricci. R, Torsion.



$$\nabla_X Y := \frac{Y(x+\epsilon X) - Y_{\parallel}(x)}{\epsilon \to 0}$$

1) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$X \quad Y \to \nabla_X Y$

2) $\nabla_{X+X'}(Y) = \nabla_X Y + \nabla_{X'} Y$

3) $\nabla_X (fY) = X(f)\, Y + f\, \nabla_X Y$

4) $\nabla_{fX}(Y) = f\, \nabla_X Y$ $\longrightarrow$ then we extend by Leibnitz rule.

$$\nabla_X Y \overset{(1,2)}{=} \nabla_{X^\mu e_\mu} (Y^\nu e_\nu) \overset{4}{=} X^\mu \nabla_{e_\mu} (Y^\nu e_\nu) \overset{3}{=}$$

$$X^\mu \{ \underbrace{e_\mu (Y^\nu)} \, e_\nu + Y^\nu \underbrace{\nabla_{e_\mu} (e_\nu)} \}.$$

$$\underbrace{X^\mu (Y^\nu_{,\mu} \, e_\nu + Y^\nu \Gamma^\alpha_{\mu\nu} e_\alpha)} \qquad \to \Gamma^\alpha_{\mu\nu} e_\alpha$$

$$X^\mu \underbrace{\nabla_{e_\mu} Y}_{Y_{;\mu}.} = X^\mu \{ Y^\alpha_{,\mu} \, e_\alpha + Y^\nu \Gamma^\alpha_{\mu\nu} e_\alpha \}$$

$$= X^\mu \{ \underbrace{Y^\alpha_{,\mu} + \Gamma^\alpha_{\mu\nu} Y^\nu}_{Y^\alpha_{;\mu}.} \} e_\alpha$$

$$\Gamma^\alpha_{\mu\nu} e_\alpha := \nabla_{e_\mu} (e_\nu)$$

$Y^{\alpha}{}_{;\mu}$ is a tensor of type $\binom{1}{1}$.

$Y^{\alpha}{}_{,\mu}$ is not a tensor.

$$\nabla_X (Y \otimes Z) \longrightarrow \nabla_{e_\mu} (Y \otimes Z) = (\nabla_{e_\mu} Y) \otimes Z + Y \otimes \nabla_{e_\mu} Z$$

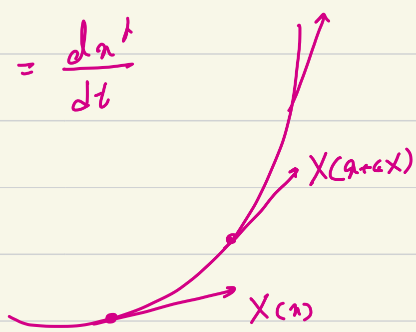
Geodesic

R^3

$$\boxed{\nabla_X X = \lambda(x) X}$$

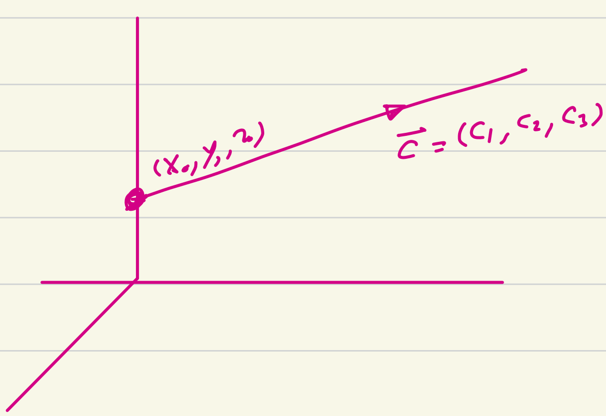
تعریف ژئودزی

$$X^\mu = \frac{dx^\mu}{dt}$$



مثال: معادلات خط راست

$$x(t) = c_1 t + x_0$$
$$y(t) = c_2 t + y_0$$
$$z(t) = c_3 t + z_0$$



$$\nabla_X X = 0 \qquad t = s^2$$

$$x(s) = c_1 s^2 + x_0$$
$$y(s) = c_2 s^2 + y_0$$
$$z(s) = c_3 s^2 + z_0$$

$$X^i = \frac{dx^i}{ds} \rightarrow \vec{X} = 2(c_1 s, c_2 s, c_3 s)$$

$$\nabla_X X = \lambda(s) X \qquad X^\mu \nabla_{e_\mu}(X^\nu e_\nu) = \lambda(s) X$$

$$X^\mu \left\{ X^\alpha{}_{,\mu}\, e_\alpha + X^\nu \Gamma^\alpha_{\mu\nu}\, e_\alpha \right\} = \lambda(s) X^\alpha e_\alpha$$

$$X^\mu X^\alpha{}_{,\mu} + X^\mu X^\nu \Gamma^\alpha_{\mu\nu} = \lambda(s) X^\alpha$$

$$X^\mu := \frac{dx^\mu}{ds}$$

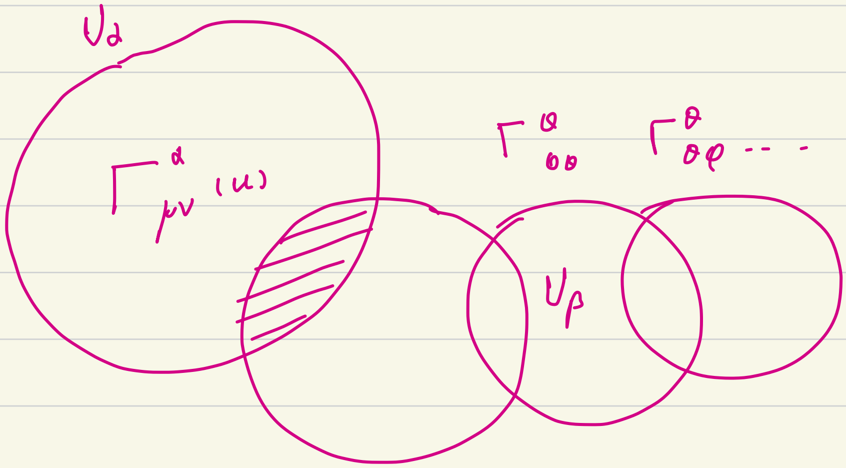
$$\underbrace{\frac{dx^\mu}{ds}\frac{\partial X^\alpha}{\partial x^\mu}} + \dot{x}^\mu \dot{x}^\nu \Gamma^\alpha_{\mu\nu} = \lambda(s)\dot{x}^\alpha$$

$$= \frac{dX^\alpha}{ds}$$

$$= \frac{d^2 x^\alpha}{ds^2} = \ddot{x}^\alpha$$

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \lambda(s)\dot{x}^\alpha$$

معادلهٔ ژئودزیک



تمرین: نشان دهید که می‌توان پارامتر را طوری انتخاب کرد

$\lambda(s) = 0$ شود.

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

$$\dot{x}^\mu = \frac{dx^\mu}{dt} = \frac{ds}{dt}\frac{dx^\mu}{ds} =: \frac{ds}{dt} x'^\mu$$

بیشتر شبیه حرکت آزاد است : حرکت بر حسب t حل کنیم.

$\nabla_X Y$ تعریف کردیم. $\nabla_X w := ?$ ω is a one-form

با قاعده لایب نیتز

$$\nabla_X \omega \Rightarrow \quad \nabla_{e_\mu} e^\nu = ?$$

$$\langle e^\nu, e_\mu \rangle = \delta^\nu_\mu \longrightarrow \langle \nabla_{e_\alpha} e^\nu, e_\mu \rangle + \underbrace{\langle e^\nu, \nabla_{e_\alpha} e_\mu \rangle} = \nabla_{e_\alpha} \delta^\nu_\mu = 0$$

$$\langle C^\nu_{\alpha\beta} e^\beta, e_\mu \rangle + \langle e^\nu, \Gamma^\beta_{\alpha\mu} e_\beta \rangle = 0$$

$$C^\nu_{\alpha\mu} + \Gamma^\nu_{\alpha\mu} = 0$$

$$\nabla_{e_\alpha} e^\nu = C^\nu_{\alpha\beta} e^\beta \longleftarrow C^\nu_{\alpha\mu} = -\Gamma^\nu_{\alpha\mu}$$

$$\nabla_{e_\alpha} e^\nu = -\Gamma^\nu_{\alpha\beta} e^\beta \qquad \nabla_{e_\alpha} e_\mu = \Gamma^\beta_{\alpha\mu} e_\beta$$

اندیس مشتق‌گیری به عنوان اندیس اول پایین میره ۲ اندیس دیگه از تانسور با
شرط جمع (اینشتین) تعیین میشه. ← { برای بردار علامت + / برای فرم‌ها علامت − .

$$\nabla_{e_\alpha} e^\beta = -\Gamma^\beta_{\alpha\mu} e^\mu$$

$$\nabla_{e_\alpha} e_\beta = \Gamma^\mu_{\alpha\beta} e_\mu$$

Connection which are compatible with Metric.

$$\nabla_X g = 0 \quad \forall X \longrightarrow g_{\mu\nu;\alpha} = 0 \quad \forall\ \alpha, \mu, \nu.$$

$$\nabla_\alpha g_{\mu\nu} = 0 \longrightarrow \partial_\alpha g_{\mu\nu} - \Gamma_{\alpha\mu}{}^{\beta} g_{\beta\nu} - \Gamma_{\alpha\nu}{}^{\beta} g_{\mu\beta} = 0$$

$$\left\{\begin{aligned} &g_{\mu\nu,\alpha} - \cancel{\underline{\Gamma_{\alpha\mu\nu}}} - \cancel{\underline{\underline{\Gamma_{\alpha\nu\mu}}}} = 0 \quad (1)\\ &g_{\mu\alpha,\nu} - \Gamma_{\nu\mu\alpha} - \cancel{\underline{\underline{\Gamma_{\nu\alpha\mu}}}} = 0 \quad (2)\\ &g_{\alpha\nu,\mu} - \cancel{\underline{\Gamma_{\mu\alpha\nu}}} - \Gamma_{\mu\nu\alpha} = 0 \quad (3)\end{aligned}\right.$$

چون $\Gamma_{\alpha\mu}{}^{\beta} = \Gamma_{\mu\alpha}{}^{\beta}$

بعضی از عبارات شبیه هم می‌شوند: که حذف شده‌اند.

با خط زیر مشخص شده‌اند.

$$(2) + (3) - (1)$$

$$g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} = +2\Gamma_{\mu\nu\alpha}$$

$$\Gamma_{\mu\nu\alpha} = \frac{1}{2}\left(-g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu}\right)$$

$$\nabla_{e_\mu} e_\nu = \Gamma_{\mu\nu}{}^{\alpha} e_\alpha$$ $\Longleftrightarrow$ قرارداد مرجع به جای اندیس‌ها Γ

Definition: $\quad T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$

$$T: \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

$$T(X+X', Y) = T(X,Y) + T(X',Y)$$
$$T(X, Y+Y') = T(X,Y) + T(X,Y')$$

دوخطی است.

$$T(fX,Y) = f\, T(X,Y).$$

$$T(fX,Y) = \nabla_{fX} Y - \nabla_Y fX - [fX,Y]^{*}$$
$$= f\nabla_X Y - (\cancel{Y(f)X} + f\nabla_Y X) - (f[X,Y] - \cancel{XY(f)})$$
$$= f\, T(X,Y).$$

$$[X, fY] = [X,f]Y + f[X,Y] \qquad \underbrace{[X,f] = ?}$$ is not meaningless.

In fact: $[X, f] = X(f)$

why: $\quad [X,f]g = (Xf - fX)(g) = X(fg) - fX(g)$

$$= X(f)g + fX(g) - fX(g) = X(f)g$$

$$[X,f] = X(f)$$ کی؟

$$T(X,Y) = X^\mu Y^\nu\, T(e_\mu, e_\nu)$$

$$T(e_\mu, e_\nu) = \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu - [e_\mu, e_\nu]$$

$$= \Gamma_{\mu\nu}{}^{\alpha} e_\alpha - \Gamma_{\nu\mu}{}^{\alpha} e_\alpha = (\Gamma_{\mu\nu}{}^{\alpha} - \Gamma_{\nu\mu}{}^{\alpha}) e_\alpha .$$

Torsion free $\longrightarrow$ $\Gamma_{\mu\nu}{}^{\alpha} = \Gamma_{\nu\mu}{}^{\alpha}$.

$X(f) := X^\mu \partial_\mu f$ $\qquad$ $e_\mu(f) = (\partial_\mu f) \longrightarrow [e_\mu, e_\nu] f = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) f$

$= 0$.

But $[X, Y] \neq 0$. $\leftarrow$ $[e_\mu, e_\nu] = 0$

Always.
