Problem 1. Consider the inner product as a binary operation in a Hilbert space $H$; therefore it acts on the $H \times H$ and returns a member of the field. Prove that the inner product is a continuous map.

Problem 2. Let $T : X \rightarrow X$ be a bounded linear operator on a complex inner product space $X$. If $\langle x | T | x \rangle = 0$ for all $|x| \in X$, show that $T = 0$. Show that this does not hold in the case of real inner product space.

Problem 3. Show that a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ on a complex hilbert space $\mathcal{H}$ is Normal if and only if $||T^\dagger x|| = ||Tx||$ for all $x \in \mathcal{H}$. Using this, show that for a normal linear operator,

$$||T^2|| = ||T||^2$$

Hint: $T^\dagger$ is adjoint of $T$.

Problem 4. Consider the operator $T$ on the Hilbert space $\mathcal{H}$. For any given $\lambda \in \mathbb{C}$, we can associate the operator

$$T_\lambda = T - \lambda I$$

Where $I$ is the identity operator on $\mathcal{H}$. If $T_\lambda$ has an inverse, we denote it by $R_\lambda(T)$, that’s it,$

$$R_\lambda(T) = (T - \lambda I)^{-1}$$

and call it the resolvant operator of $T$. ( The name ”resolvant“ is appropriate, since $R_\lambda(T)$ help to solve the equation $T_\lambda x = y$. Thus, $x = R_\lambda(T)y$ provided $R_\lambda$ exists.)

Prove that:

1) The resolvant $R_\lambda$ satisfy the equation

$$R_\mu - R_\lambda = (\mu - \lambda)R_\mu R_\lambda$$

2) $R_\lambda(T)$ commutes with any operator that commutes with $T$.

3) $R_\mu R_\lambda = R_\lambda R_\mu$

Problem 5. Prove that an operator of the form

$$(Ax)(s) = \int_a^b k(s, t)x(t)dt$$

is hermitian in and only if

$$k(s, t) = k(t, s)^*$$