Solution 1.

\[ ||A|| = \left\{ \max \frac{||Ax||_{\mathbb{C}^n}}{||x||_{\mathbb{C}^m}}, x \neq 0 \right\} \]

a)

\[ ||Ax||_1 = \sum_{j=1}^{m} \sum_{j=1}^{n} |a_{ij}| x_j \leq \max_j (|x_j| \sum_{j=1}^{m} |a_{ij}|) = \max_j (|x_j| \sum_{j=1}^{m} \sum_{j=1}^{n} |a_{ij}|) = ||x||_{\infty} \sum_{j=1}^{m} \sum_{j=1}^{n} |a_{ij}| \quad (1) \]

We can not go ahead more than this bound.

b) Let \( n \times m \) matrix \( A \) be represented in column format

\[ A = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_m \end{bmatrix} \]

\[ Ax = \sum_{j=1}^{m} \mathbf{A}_j x_j \]

\[ ||Ax||_1 = || \sum_{j=1}^{m} \mathbf{A}_j x_j ||_1 \leq \sum_{j=1}^{m} |x_j| ||\mathbf{A}_j||_1 \leq \max_j (|x_j| \sum_{j=1}^{m} ||\mathbf{A}_j||_1) = \max_j (|x_j| \sum_{j=1}^{m} \sum_{j=1}^{n} |a_{ij}|) = ||x||_{\infty} \sum_{j=1}^{m} \sum_{j=1}^{n} |a_{ij}| \quad (2) \]

We have found constant \( C = \max_j ||\mathbf{A}_j||_1 \)

such that

\[ ||Ax||_1 \leq C ||x||_1 , \quad \forall x \in \mathbb{C}^m \]

We must find at least one vector for which we have equality in Eq(2). But this is easy; consider a vector in \( \mathbb{C}^m \) that it \( j \)-th component equal to 1 and other component equal to zero. When we use this vector and compute \( ||Ax||_1 \) we have equality in Eq(2). So

\[ ||A||_{(1,1)} = \max_{1 \leq j \leq m} ||\mathbf{A}_j||_1 = \max_{1 \leq j \leq m} \sum_k |a_{kj}| \]

That is, the matrix 1-norm is the maximum of the column sums.

c)

\[ ||Ax||_{\infty} = \left\| \begin{array}{c} \sum_k a_{1k} x_k \\ \sum_k a_{2k} x_k \\ \vdots \\ \sum_k a_{nk} x_k \end{array} \right\|_{\infty} = \max_{1 \leq k \leq n} \left\| \begin{array}{c} \sum_k a_{1k} x_k \\ \sum_k a_{2k} x_k \\ \vdots \\ \sum_k a_{nk} x_k \end{array} \right\|_{\infty} \]
\[
\max_{1 \leq k \leq m} |x_k| \max_{1 \leq i \leq n} \sum_k |a_{ik}| = \left( \max_{1 \leq i \leq n} \sum_k |a_{ik}| \right) \|x\|_\infty
\]

(3)

Similar to part (b), we have found a constant

\[
C = \max_{1 \leq i \leq n} \sum_k |a_{ik}|
\]

for which

\[
\|Ax\|_\infty \leq C \|x\|_\infty \quad \forall x \in \mathbb{C}^m
\]

If we find non-zero vector for which equality in holds in Eq(3), \(\infty\)-norm of matrix becomes

\[
\|A\|_{(\infty, \infty)} = \max_{1 \leq i \leq n} \sum_k |a_{ik}|
\]

Equality prevails if \(x\) define to have the components:

\[
x_k = \begin{cases} 
\frac{a_{ik}}{|a_{ik}|} & a_{ik} \neq 0 \\
1 & a_{ik} = 0
\end{cases}
\]

Where \(i\) is the index for maximum row sum.

That is, the matrix \(\infty\)-norm is the maximum of the row sums.

**Solution 2.** If \(|x| \neq 0\) and \(P|x| \neq 0\), then the use of the Cauchy-Schwarz inequality implies that

\[
\|P|x|\| = \frac{\langle x|P^1 P|x| \rangle}{\|P|x|\|} \leq \frac{\|x\| \|P|x|\|}{\|P|x|\|} = \|x\|
\]

Therefore \(\|P\| \leq 1\). If \(P \neq 0\), then there is an \(|x| \in \mathcal{H}\) with \(P|x| \neq 0\) and

\[
\|P(P|x|)\| = \|P|x|\|
\]

So that \(\|P\| \geq 1\). Consequently \(\|P\| = 1\).

**Solution 3.** A linear functional \(\phi\) on a complex Hilbert space \(\mathcal{H}\) is bounded, or continuous, if there exists a constant \(M\) such that

\[
||\phi(x)|| = |\phi(x)| \leq M\|x\| \quad \forall x \in \mathcal{H}
\]

The norm of a bounded linear functional \(\phi\) is

\[
||\phi|| = \sup_{\|x\|=1} |\phi(x)|
\]

If \(y \in \mathcal{H}\), then

\[
\phi_y(x) = \langle y, x \rangle
\]

is bounded linear functional on \(\mathcal{H}\), with \(||\phi_y|| = ||y||\).

Now,

\[
\phi_n(f) = \frac{1}{\sqrt{2\pi}} \int_T e^{-inx} f(x) dx = \langle \frac{e^{inx}}{\sqrt{2\pi}}, f(x) \rangle
\]

Thus:

\[
||\phi_n|| = \left| \frac{e^{inx}}{\sqrt{2\pi}} \right| = \left( \frac{1}{2\pi} \int_0^{2\pi} e^{inx} e^{-inx} dx \right)^{\frac{1}{2}} = 1
\]