Solution 1. For a continuous map \( f(x, y) \), if \( x_n \to x \) and \( y_n \to y \) then, \( f(x_n, y_n) \to f(x, y) \).

\[
\left| \langle x_n, y_n \rangle - \langle x, y \rangle \right| = \left| \langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle \right| \leq \left| \langle x_n - x, y_n \rangle \right| + \left| \langle x, y_n - y \rangle \right| \\
\leq ||x_n - x|| ||y_n|| + ||y_n - y|| ||x|| \to 0
\]  

(1)

Consider the inner product as a binary operation in a Hilbert space \( \mathcal{H} \); therefore it acts on the \( \mathcal{H} \times \mathcal{H} \) and returns a member of the field. Prove that the inner product is a continuous map.

Solution 2. First, we can show that for every bounded Hermitian operator \( H \) if \( \langle x|H|x \rangle = 0 \), then \( H = 0 \) and vice versa.

For every normal operator we can write Spectral decomposition as:

\[
H = \sum_i \lambda_i |i\rangle \langle i|
\]

(3)

Where \( \lambda_i \in \mathbb{R} \) and \( |\lambda_i| \leq ||H|| < \infty \).

\[
\langle x|H|x \rangle = \langle x| \left( \sum_i \lambda_i |i\rangle \langle i| \right) |x \rangle = \sum_i \lambda_i |\langle x| i \rangle|^2
\]

(4)

\[
\langle x|H|x \rangle = 0 \implies \sum_i \lambda_i |\langle x| i \rangle|^2 = 0, \quad \forall |x\rangle \in X
\]

(5)

If we select \( |x\rangle = |i\rangle \), we can conclude that \( \lambda_i = 0 \) and this procedure show that all of eigenvalues is zero, consequently \( H = 0 \).

As we know, every complex number can be written as sum of real part and imaginary part. In the similar manner, every operator can be decompose to the hermitian part and antihermitian part:

\[
T_H = \frac{1}{2} (T + T^\dagger)
\]

(6)

\[
T_{AH} = \frac{1}{2} (T - T^\dagger)
\]

(7)

Now, as we know, the hermitian and antihermitian operator are normal. Thus \( \langle x|T_H|x \rangle = 0 \) and \( \langle x|T_{AH}|x \rangle = 0 \) give the result \( T_H = T_{AH} = 0 \).

For showing that this statement is no longer accurate for real inner product space, note that for the Rotation operator (90 degree) about the origin in \( \mathbb{R}^2 \), we always have \( \langle x|R|x \rangle = 0 \), but obviously \( R \neq 0 \).
Solution 3. a) Assume that $T$ is normal, thus $TT^\dagger = T^\dagger T$.

\begin{equation}
||Tx||^2 = \left(T|x\right)^\dagger \left(T|x\right) = \langle x|TT^\dagger |x\rangle = ||T^\dagger x||^2
\end{equation}

Conversely, if $||T^\dagger x|| = ||Tx||$ :

\begin{equation}
\langle x|T^\dagger T|x\rangle = \langle x|TT^\dagger |x\rangle \rightarrow \langle x|T^\dagger T - TT^\dagger |x\rangle = 0, \quad \forall |x\rangle \in \mathcal{H}. \implies T^\dagger T = TT^\dagger
\end{equation}

In the last conclusion, we used from the fact that $T^\dagger T - TT^\dagger$ is hermitian operator. (Last problem)

b) from the definition of $||T||$

\begin{equation}
||T^2|| = \max_{||x||=1} \langle T^2|x, T^2x \rangle = \max_{||x||=1} \langle x, T^\dagger T^2|x \rangle = ||T^\dagger T||
\end{equation}

$T$ is hermitian, so

\begin{equation}
||T^2|| = ||T||^2
\end{equation}

We know that $||AB|| \leq ||A|| \cdot ||B||$. Now

\begin{equation}
||T^2|| \leq ||T|| \cdot ||T|| = ||T||^2
\end{equation}

and

\begin{equation}
||T^2||^2 = \max_{||x||=1} \left(T^2|x\right)^\dagger \left(T^2|x\right) = \max_{||x||=1} \langle x|(T^\dagger)^2T^2|x \rangle = \max_{||x||=1} \langle x|T^\dagger TT^\dagger T|x \rangle
\end{equation}

\begin{equation}
||T^2||^2 = \max_{||x||=1} \langle x|T^\dagger TT^\dagger T|x \rangle \leq \max_{||x||=1} \langle x|T^\dagger T|x \rangle \langle x|T^\dagger T|x \rangle
\end{equation}

Solution 4. 1) 

\begin{equation}
(T_\lambda T_\mu)(R_\mu - R_\lambda) = T_\lambda T_\mu R_\mu - T_\lambda T_\mu R_\lambda = T_\lambda - T_\mu = (\mu - \lambda)I
\end{equation}

Multiplying two side to the $(T_\lambda T_\mu)^{-1} = R_\mu R_\lambda$ give

\begin{equation}
R_\mu - R_\lambda = R_\mu R_\lambda (\mu - \lambda)I = (\mu - \lambda)R_\mu R_\lambda
\end{equation}

2) Assume that $[A, T] = 0$. We must show that $[R_\lambda(T), A] = 0$.

Since commutator is a 2-linear map, $[A, T_\lambda]$, 

\begin{equation}
AT_\lambda - T_\lambda A = 0 \implies R_\lambda(AT_\lambda - T_\lambda A)R_\lambda = 0 \implies R_\lambda A - AR_\lambda = 0 \implies [R_\lambda(T), A] = 0
\end{equation}

3) Since $[T_\lambda, T_\mu] = 0$:

\begin{equation}
T_\lambda T_\mu = T_\mu T_\lambda \implies R_\mu R_\lambda(T_\lambda T_\mu) = R_\mu R_\lambda(T_\mu T_\lambda) = I
\end{equation}

Multiplying both side of last term by $R_\lambda R_\mu$ from left side

\begin{equation}
R_\mu R_\lambda = R_\lambda R_\mu
\end{equation}

Solution 5. In general $\langle x|H|y\rangle = \langle y|H^\dagger |x\rangle^*$ We can write the operation in general form (basis independent)

\begin{equation}
\mathbb{A}|x\rangle = \mathbb{A}\left( \int_D dt |t\rangle \langle t| \right) |x\rangle = \int_D dt \mathbb{A}|t\rangle \langle t|x\rangle
\end{equation}

The RHS is a vector that belong to our Hilbert space. It’s representation in the coordinate $\{|s\rangle\}$ is:
\[ Ax(s) = \langle s|A|x \rangle = \int_D dt \langle s|A|t \rangle \langle t|x \rangle = \int_D dt A(s,t)x(t) \]

Where

\[ A(s,t) = \langle s|A|t \rangle, \text{ and } x(t) = \langle t|x \rangle \]

We know that for hermitian operator, we have

\[ \langle s|A|t \rangle = \langle t|A|s \rangle^* \implies A(s,t) = A(t,s)^* \]