Entanglement Distribution with Separable States

Vahid Karimipour,
Sharif University of Technology
Tehran, Iran

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$$\rho_{AB} = \sum_i p_i \; \rho_A^i \otimes \rho_B^i$$

A and B are separable

T. S. Cubitt, F. Verstraete, W. Du¨r, and J. I. Cirac, Phys. Rev. Lett. 91, 037902 (2003).

The point is that the messenger remains separable at all stages.

$$\rho_{ABM} = \sum_i p_i \rho_A{}^i \otimes \rho_B^i \otimes \rho_M^i$$

But the initial state should be very peculiar.

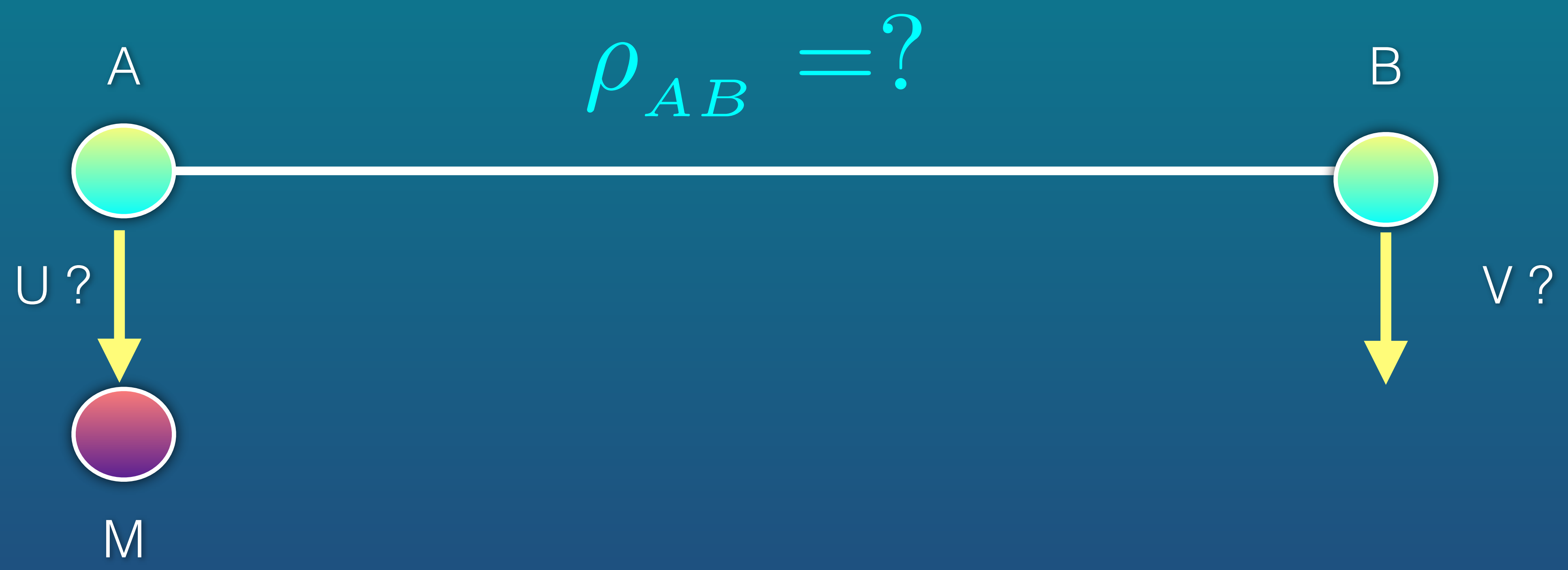
$$\rho_{ABM} = \frac{1}{6}\sum_{k=0}^{3} |\psi_k\rangle\langle\psi_k| \otimes |\psi_{-k}\rangle\langle\psi_{-k}| \otimes |0\rangle\langle 0| + \frac{1}{6}\sum_{i=0}^{1} |i,i,1\rangle\langle i,i,1|$$

$$|\psi_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{i\pi}{2}k}|1\rangle\right)$$

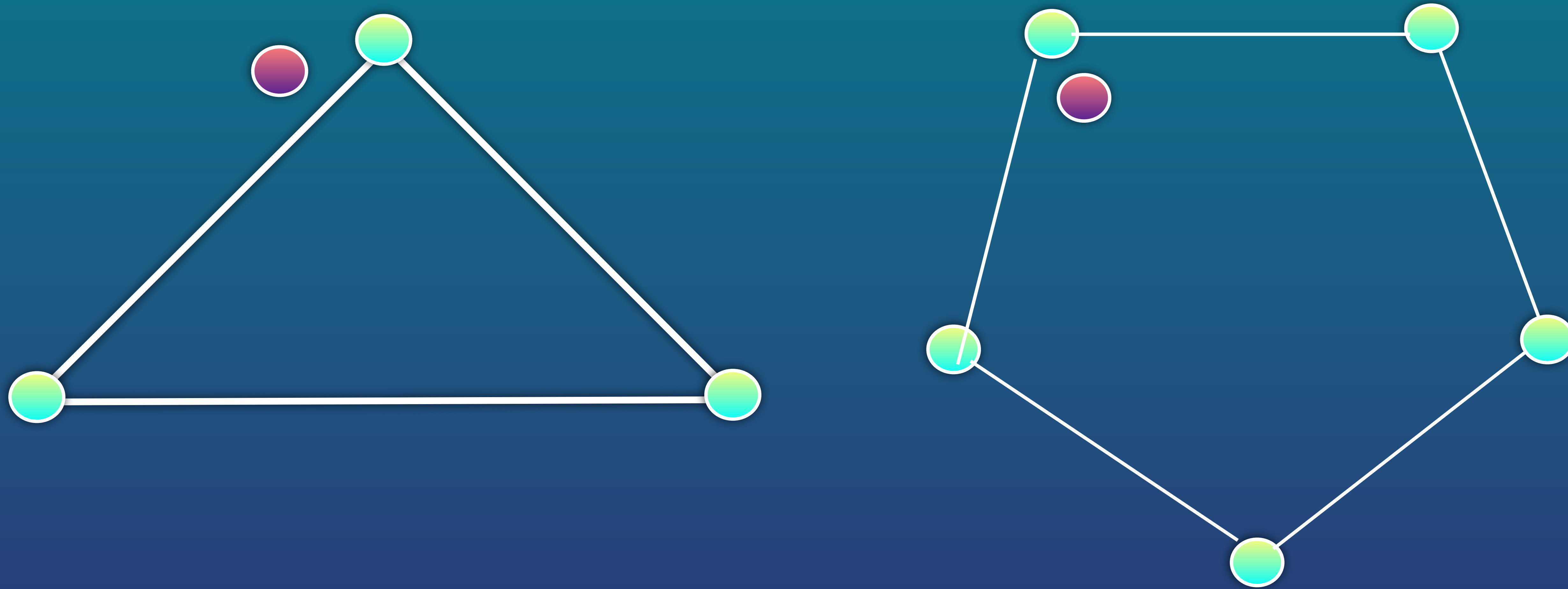
Our Goal:

To understand this result

Generalization to d-level states



Generalization to GHZ states



Let us start from the very beginning:

$$|\psi_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \alpha_k|1\rangle\right)$$

$$|\phi_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \beta_k|1\rangle\right)$$

$$|0\rangle$$

After Alice CNOT $$|\Psi_k\rangle_{AM} = \frac{1}{\sqrt{2}}\left(|00\rangle + \alpha_k|11\rangle\right)$$

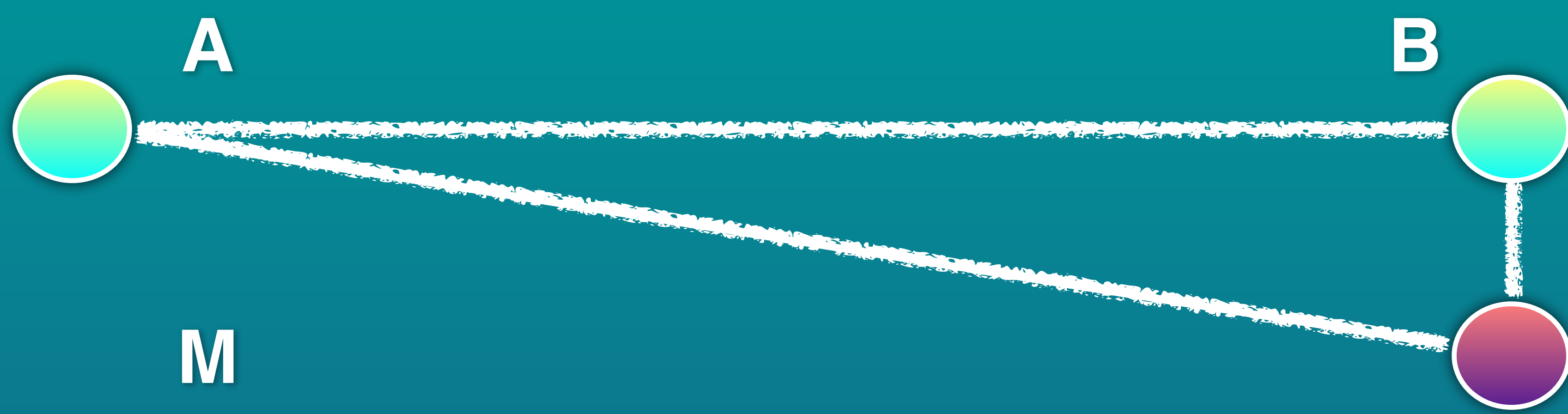
$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(|00\rangle + \alpha_k|11\rangle\right)\left(|0\rangle + \beta_k|1\rangle\right)$$

The messenger is sent to Bob:

$$|\Psi_k^1\rangle_{AMB} = \tfrac{1}{2}\left(|000\rangle + \alpha_k|110\rangle + \beta_k|001\rangle + \alpha_k\beta_k|111\rangle\right)$$

We rearrange the indices for simplicity

$$|\Psi_k^1\rangle_{ABM} = \tfrac{1}{2}\left(|000\rangle + \alpha_k|101\rangle + \beta_k|010\rangle + \alpha_k\beta_k|111\rangle\right)$$



$$\alpha_k \beta_k = 1$$

From the previous page:

$$|\Psi_k^1\rangle_{ABM} = \frac{1}{2}\left(|000\rangle + \alpha_k|101\rangle + \beta_k|010\rangle + \alpha_k\beta_k|111\rangle\right)$$

After Bob CNOT operation, the state is:

$$|\Psi_k^2\rangle_{ABM} = \frac{1}{2}\left(|000\rangle + \alpha_k|101\rangle + \beta_k|011\rangle + |110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\phi_+\rangle|0\rangle_M + |\chi_k\rangle|1\rangle_M\right)$$

$$|\chi_k\rangle = \frac{1}{\sqrt{2}}\left(\alpha_k|10\rangle + \beta_k|01\rangle\right)$$

However the particle M has entangled itself

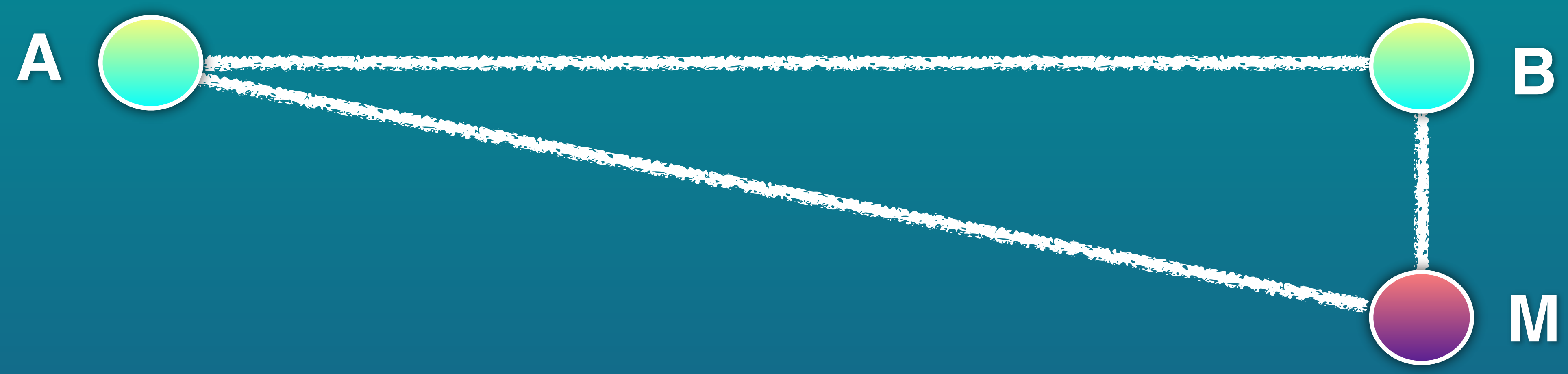
with A and B

in both stages of the process.

In stage 1: After Alice operation:

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(|00\rangle + \alpha_k|11\rangle\right)\left(|0\rangle + \beta_k|1\rangle\right)$$

In stage 2: After Bob's Operation



$$|\Psi_k^2\rangle_{AMB} = \frac{1}{\sqrt{2}}\left(|\phi_+\rangle|0\rangle_M + |\chi_k\rangle|1\rangle_M\right)$$

How to remove the entanglement?

For (1) we use Symmetrization

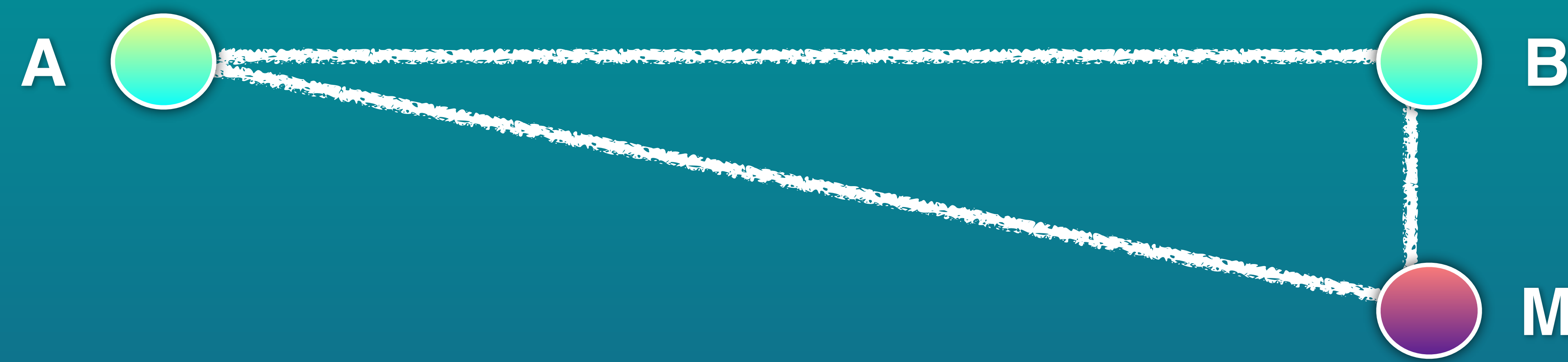
For (2) we use Mixing

Mixing

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad |\phi_-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$\frac{1}{2}\left(|\phi_+\rangle\langle\phi_+| + |\phi_-\rangle\langle\phi_-|\right) = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

(2)



$$|\Psi_k^2\rangle_{ABM} = \frac{1}{\sqrt{2}}\left(|\phi_+\rangle|0\rangle + |\chi_k\rangle|1\rangle\right)$$

$$\rho^2_{ABM} = \sum_k |\Psi_k^2\rangle\langle\Psi_k^2|$$

$$\rho^2_{ABM} = \sum_k \frac{1}{2}\left(|\phi_+\rangle\langle\phi_+| \otimes |0\rangle\langle 0| + |\chi_k\rangle\langle\chi_k| \otimes |1\rangle\langle 1|\right)$$
$$+\frac{1}{2}\left(|\phi_+\rangle\langle\chi_k|\otimes|0\rangle\langle 1|+|\chi_k\rangle\langle\phi_+|\otimes|1\rangle\langle 0|\right)$$

We demand that;

$$\sum_k |\chi_k\rangle = 0$$

$$|\chi_k\rangle = \frac{1}{\sqrt{2}}\left(\alpha_k|10\rangle + \beta_k|01\rangle\right)$$

$$\sum_k \alpha_k = 0$$

So after Bob's operation we have this:

$$\rho^2_{ABM} = \sum_k \frac{1}{2}\left(|\phi_+\rangle\langle\phi_+| \otimes |0\rangle\langle 0| + |\chi_k\rangle\langle\chi_k| \otimes |1\rangle\langle 1|\right)$$

(1): Removing entanglement in the first stage

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(|00\rangle + \alpha_k|11\rangle\right)\left(|0\rangle + \beta_k|1\rangle\right)$$

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(|000\rangle + \alpha_k|110\rangle + \beta_k|001\rangle + |111\rangle\right)$$

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(\sqrt{2}|GHZ\rangle + \alpha_k|110\rangle + \beta_k|001\rangle\right)$$

B is obviously separate from A and M.

So If we make the state symmetric with respect to B and M,

then M will be separate from A and B.

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(\sqrt{2}|GHZ\rangle + \alpha_k|110\rangle + \beta_k|001\rangle\right)$$

We want this to be symmetric:

$$\rho_{AMB} = \sum_k |\Psi_k^1\rangle\langle\Psi_k^1|$$

The Term |GHZ><110| vanishes due to :

The Term |GHZ><001| vanishes due to :

$$\sum_k \alpha_k = 0$$

The Term |110><001| vanishes if we assume that:

$$\sum_k \alpha_k^2 = 0$$

$$|\Psi_k^1\rangle_{AMB} = \frac{1}{2}\left(\sqrt{2}|GHZ\rangle + \alpha_k|110\rangle + \beta_k|001\rangle\right)$$

But two terms remain which are not symmetric:: $\pi_{110} + \pi_{001}$

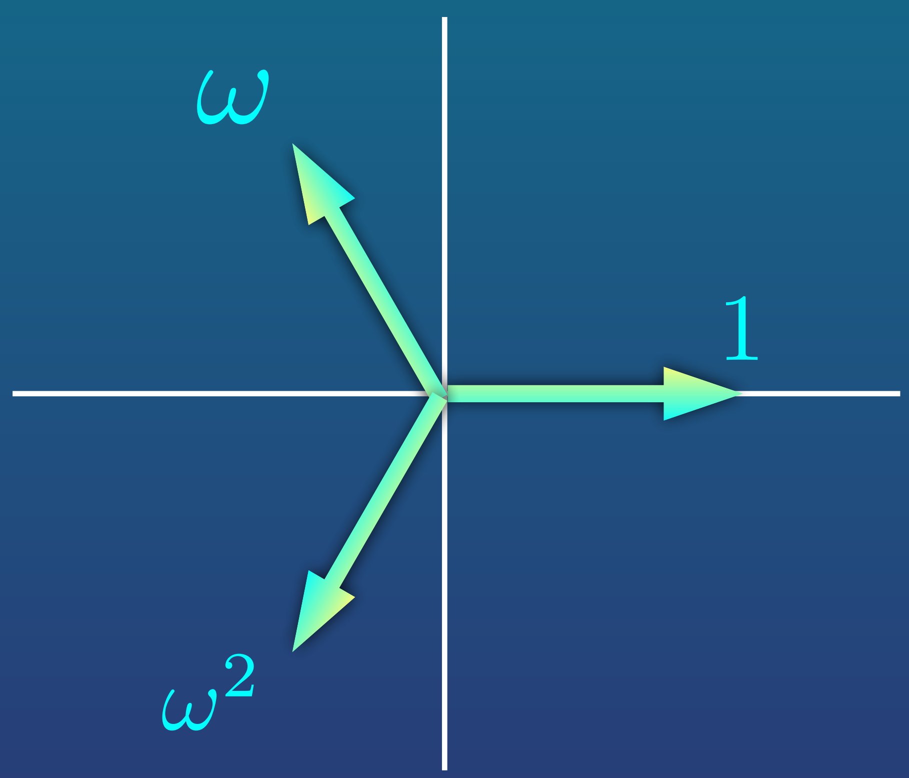
So we add the terms $\pi_{101} + \pi_{010}$

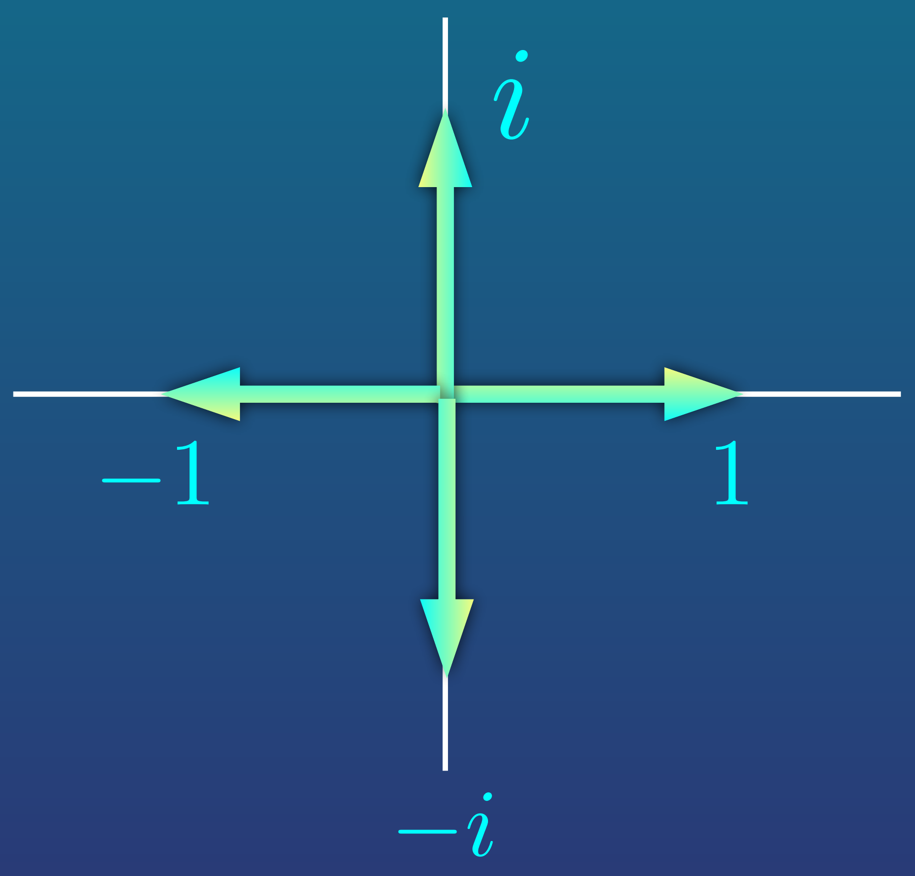
This addition does not affect the process?

The Solutions

$$\sum_k \alpha_k = 0$$

$$\sum_k \alpha_k^2 = 0$$





$$\rho_{ABM} = \frac{1}{6}\sum_{k=0}^{3} |\psi_k\rangle\langle\psi_k| \otimes |\psi_{-k}\rangle\langle\psi_{-k}| \otimes |0\rangle\langle 0| + \frac{1}{6}\sum_{i=0}^{1} |i,i,1\rangle\langle i,i,1|$$

$$\left|\psi_k\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{\frac{i\pi}{2}k}\left|1\right\rangle\right)$$

T. S. Cubitt, F. Verstraete, W. Du¨r, and J. I. Cirac,

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Distribution of d-level Bell States

$CNOT$ $CNOT^{-1}$

$$CNOT|i, j\rangle = |i, i + j\rangle$$

Distribution of d-level Bell States

$$|\psi\rangle_k = \frac{1}{\sqrt{d}}\left(|0\rangle + \alpha_k|1\rangle + \beta_k|2\rangle + \cdots \xi_k|d-1\rangle\right)$$

How many states we need?

What are the form of these states?

Distribution of d-level Bell States

$$|\psi_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{k s_j} |j\rangle$$

$$\omega = e^{\frac{2\pi i}{K}}$$

$$|\psi_k\rangle = \frac{1}{\sqrt{d}} \left(\omega^{s_0 k}|0\rangle + \omega^{s_1 k}|1\rangle + \omega^{s_2 k}|2\rangle + \cdots\right)$$

The number of states = K

Distribution of d-level Bell States

$$|\psi_k\rangle_A = \frac{1}{\sqrt{d}} \sum_j \omega^{k s_j} |j\rangle$$

$$|\psi_k\rangle_B = \frac{1}{\sqrt{d}} \sum_j \omega^{-k s_j} |j\rangle$$

$$\omega = e^{\frac{2\pi i}{K}}$$

$$|\Psi_k^1\rangle_{AM} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{k s_j} |j, j\rangle$$

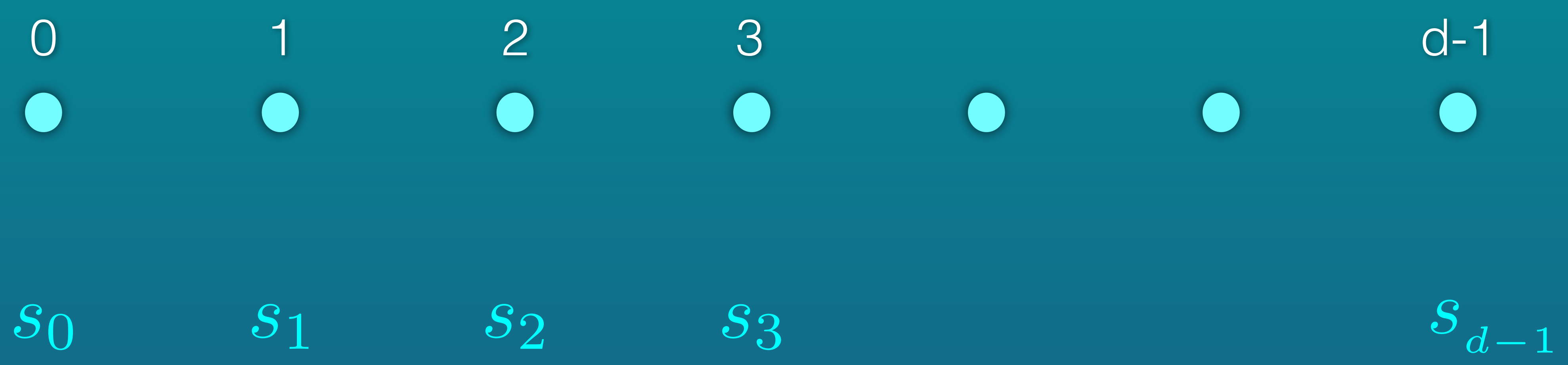
$$|\Psi_k^1\rangle_{AMB} = \frac{1}{\sqrt{d}} \sum_{j,j'} \omega^{(s_j - s_{j'})k} |j, j, j'\rangle$$

$$|\Psi_k^2\rangle_{AMB} = \frac{1}{\sqrt{d}} \sum_{j,j'} \omega^{(s_j - s_{j'})k} |j, j - j', j'\rangle$$

a: $s_j < \frac{K}{2}$

b: $s_i = s_j \longrightarrow i = j$

c: $s_i + s_j = s_k + s_l \longrightarrow \begin{cases} (i,j) = (k,l) \\ (i,j) = (l,k) \end{cases}$



$$s_i + s_j \neq s_k + s_l$$

For d=3 (Qutrits)

	0	1	2	
s_i	0	1	2	✗
	0	1	3	✓

$$s_i + s_j \neq s_k + s_l$$

s_i 0 1 3 $s_j < \frac{K}{2}$

$$|\psi^k\rangle_A = \frac{1}{\sqrt{3}}(|0\rangle + \omega^k|1\rangle + \omega^{3k}|2\rangle)$$

K=7 ω is the 7th root of unity.

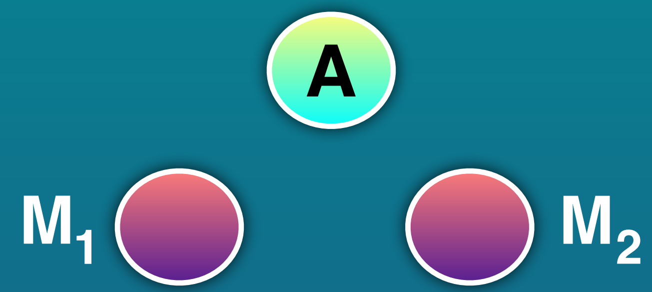
	0	1	2	3	
s_i	0	1	3	4	✗
	0	1	3	5	✗
	0	1	3	6	✗
	0	1	3	7	✓

$$s_i + s_j \neq s_k + s_l$$

Distribution of GHZ states

$\alpha_k \beta_k \gamma_k = 1$

$|0\rangle + \alpha_k |1\rangle \longrightarrow |000\rangle + \alpha_k |111\rangle$



$|0\rangle$ $|0\rangle$

$$|\Psi_k^1\rangle_{AM_1M_2BC} = (|000\rangle + \alpha_k |111\rangle)(|0\rangle + \beta_k |1\rangle)(|0\rangle + \gamma_k |1\rangle)$$

$|0\rangle + \gamma_k |1\rangle$

$|0\rangle + \beta_k |1\rangle$

$$|\Psi_k^2\rangle = \frac{1}{2}\left(|00\rangle|GHZ\rangle + |01\rangle|\chi_k^{01}\rangle + |10\rangle|\chi_k^{10}\rangle + |11\rangle|\chi_k^{11}\rangle\right)$$

$$|\chi_k^{01}\rangle = \frac{1}{\sqrt{2}}\left(\gamma_k|000\rangle + {\gamma_k}^{-1}|111\rangle\right)$$

$$|\chi_k^{10}\rangle = \frac{1}{\sqrt{2}}\left(\beta_k|000\rangle + {\beta_k}^{-1}|111\rangle\right)$$

$$|\chi_k^{11}\rangle = \frac{1}{\sqrt{2}}\left(\alpha_k|000\rangle + {\alpha_k}^{-1}|111\rangle\right)$$

The results of measurement of the messengers determine the type of GHZ state obtained!

Removing the entanglement after stage 2 by mixing:

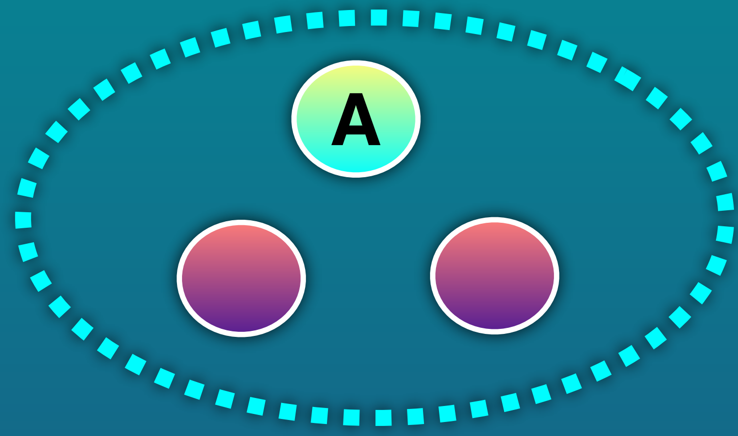
$$\sum_k \alpha_k = 0$$

$$\sum_k \beta_k = 0$$

$$\sum_k \gamma_k = 0$$

$$\sum_k \frac{\alpha_k}{\beta_k} = \sum_k \frac{\alpha_k}{\gamma_k} = \sum_k \frac{\beta_k}{\gamma_k} = 0$$

Symmetrization



All local operations are done by A on the messengers

Messengers are separate from B and C

We make this state symmetric with respect to
$(M_1 \longleftrightarrow B)$ and $(M_2 \longleftrightarrow C)$

Removing the entanglement after stage 1 by symmetrization

$$\sum_k \alpha_k^2 = \sum_k \beta_k^2 = \sum_k \gamma_k^2 = 0$$

Removing the entanglement after stage 1 by symmetrization

$$\sum_k \alpha_k = \sum_k \beta_k = \sum_k \gamma_k = 0$$

$$\sum_k \frac{\alpha_k}{\beta_k} = \sum_k \frac{\alpha_k}{\gamma_k} = \sum_k \frac{\beta_k}{\gamma_k} = 0$$

$$\sum_k \alpha_k^2 = \sum_k \beta_k^2 = \sum_k \gamma_k^2 = 0$$

Solution

$$\sum_k \alpha_k = 0 \qquad \sum_k \alpha_k^2 = 0 \qquad \sum_k \frac{\alpha_k}{\beta_k} = 0$$

$$\sum_k \beta_k = 0 \qquad \sum_k \beta_k^2 = 0 \quad \sum_k \alpha_k^4 = 0 \qquad \sum_k \frac{\alpha_k}{\gamma_k} = 0$$

$$\sum_k \alpha_k^3 = 0 \quad \sum_k \gamma_k = 0 \qquad \sum_k \alpha_k^6 = 0 \quad \sum_k \gamma_k^2 = 0 \qquad \sum_k \frac{\beta_k}{\gamma_k} = 0 \quad \sum_k \alpha_k^5 = 0$$

$$\beta_k = \alpha_k^2 \qquad \gamma_k = \alpha_k^{-3}$$

$$\alpha_k \beta_k \gamma_k = 1$$

Solution

$$\sum_k \alpha_k = 0$$

$$\sum_k \alpha_k^2 = 0$$

$$\sum_k \alpha_k^4 = 0$$

$$\sum_k \alpha_k^3 = 0$$

$$\sum_k \alpha_k^6 = 0$$

$$\sum_k \alpha_k^5 = 0$$

$$\alpha_k = \omega^k$$

$$1+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$$

$$1+\omega^2+\omega^4+\omega^6+\omega^1+\omega^3+\omega^5=0$$

$$1+\omega^3+\omega^6+\omega^2+\omega^5+\omega^1+\omega^4=0$$

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$$\omega = e^{\frac{2\pi i}{7}}$$

We can distribute GHZ states with probability $\frac{1}{7}$

Summary:

There is a systematic way to produce
Bell states and GHZ GHZ(n)
states in any dimension, by using only separable
messengers.

Thanks for your attention