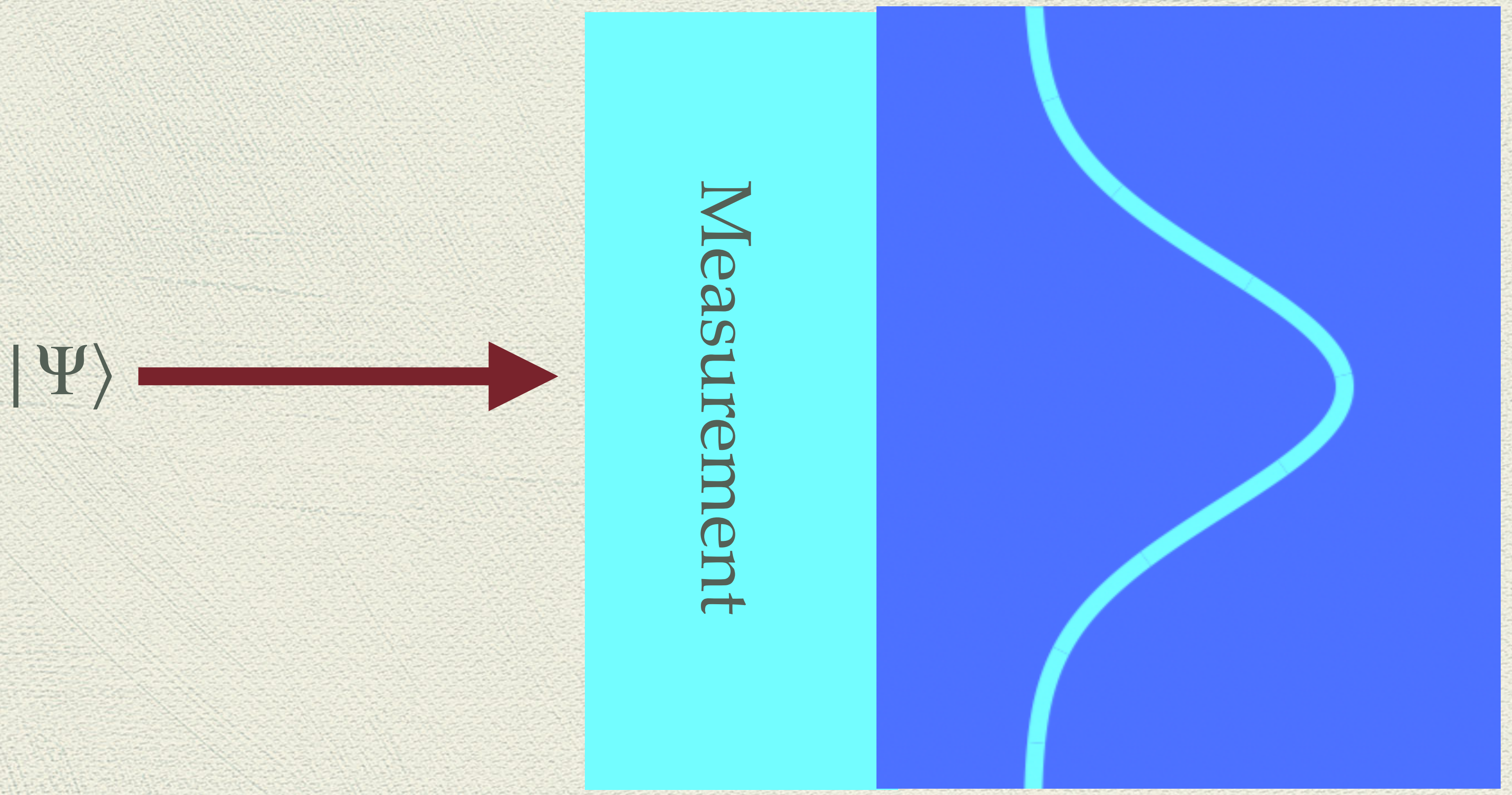
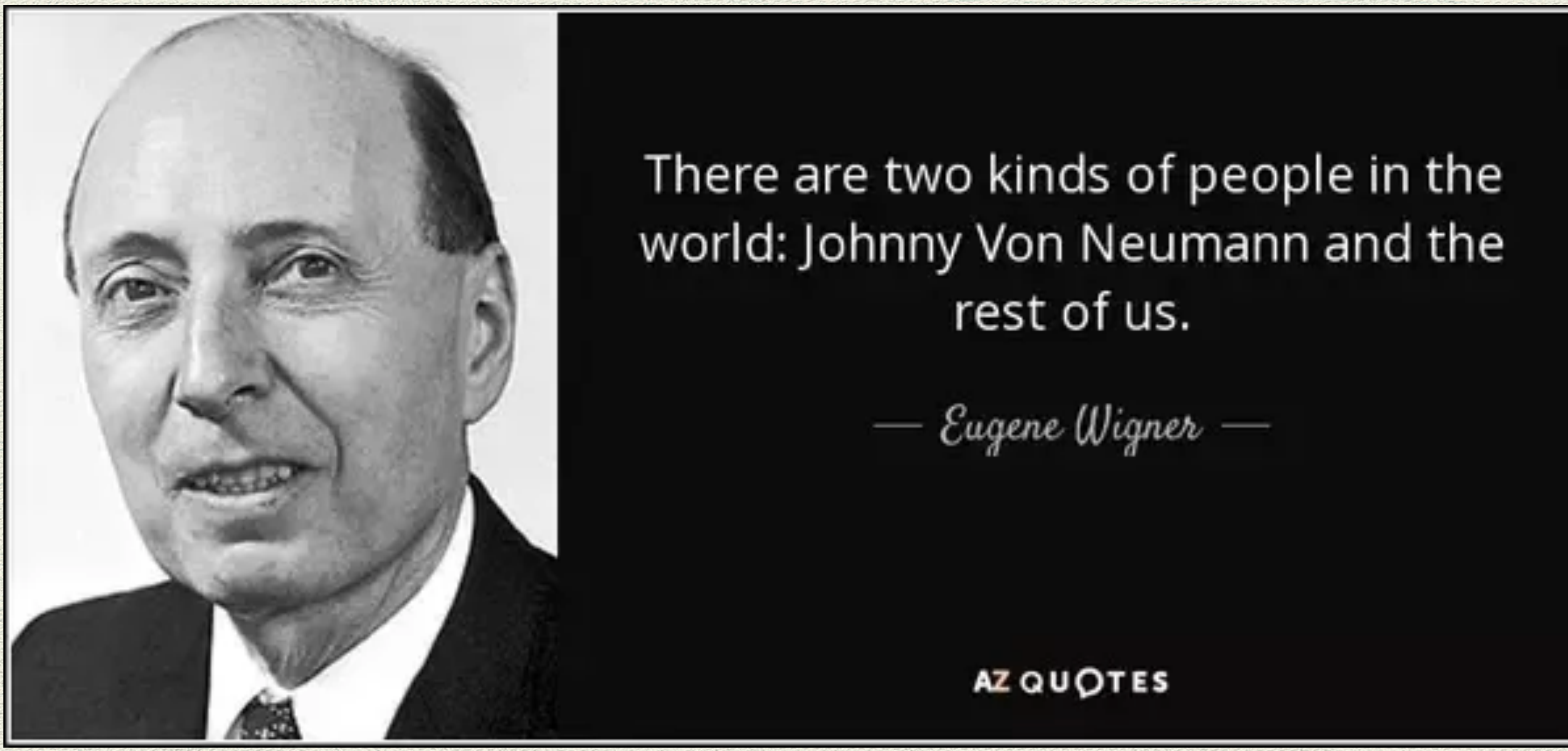
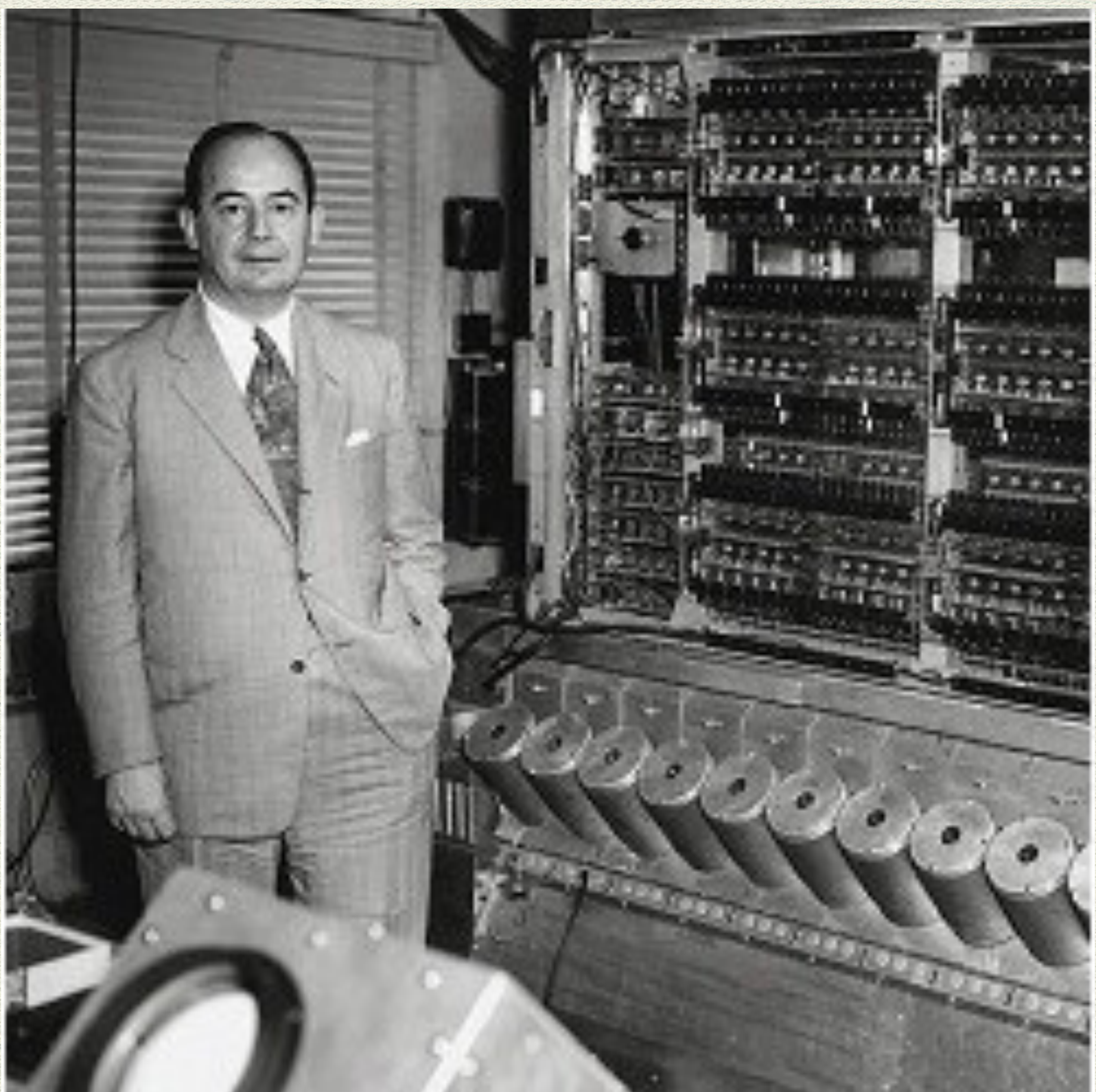
Introduction to Quantum Information-II

**Information is not Quantum,
Information is Classical**




John von Neumann


There are two kinds of people in the world: Johnny Von Neumann and the rest of us.
— Eugene Wigner —
AZ QUOTES



انریکو فرمی خطاب به یکی از همکارانش:
« می دانی، جانی می توانست ده برابر سریع تر از من محاسبه ذهنی انجام دهد و
من می توانم ده برابر سریع تر از تو محاسبه ذهنی انجام دهم.
حالا می توانی بفهمی که جانی چقدر شگفت انگیز بود.

یوجین ویگنر: در باره قوای ذهنی فون نویمان احساس می کردی که با یک ماشین فوق العاده
کامل روبرو هستی که همه اجزایش با دقت یک هزارم اینچ در حال کار کردن اند.

هانس بته: من بعضی وقت ها فکر می کنم که مغزی مثل مغز فون نویمان احتمالا
نشان دهنده این است که او به گونه ای فراتر از انسان تعلق داشته است.

Von Neumann Entropy and its Properties

$$S(\rho) = -Tr(\rho \log \rho)$$

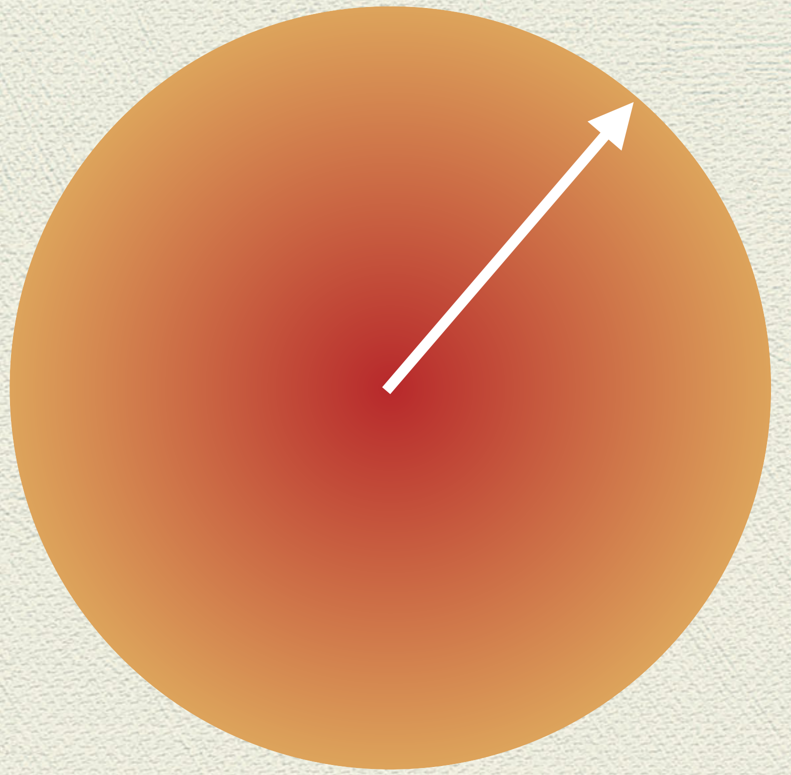
ظرفیت کانال کوانتومی

میزان فشرده سازی اطلاعات

میزان درهم تنیدگی

$$S(\rho) = -\,Tr(\rho \log \rho)$$

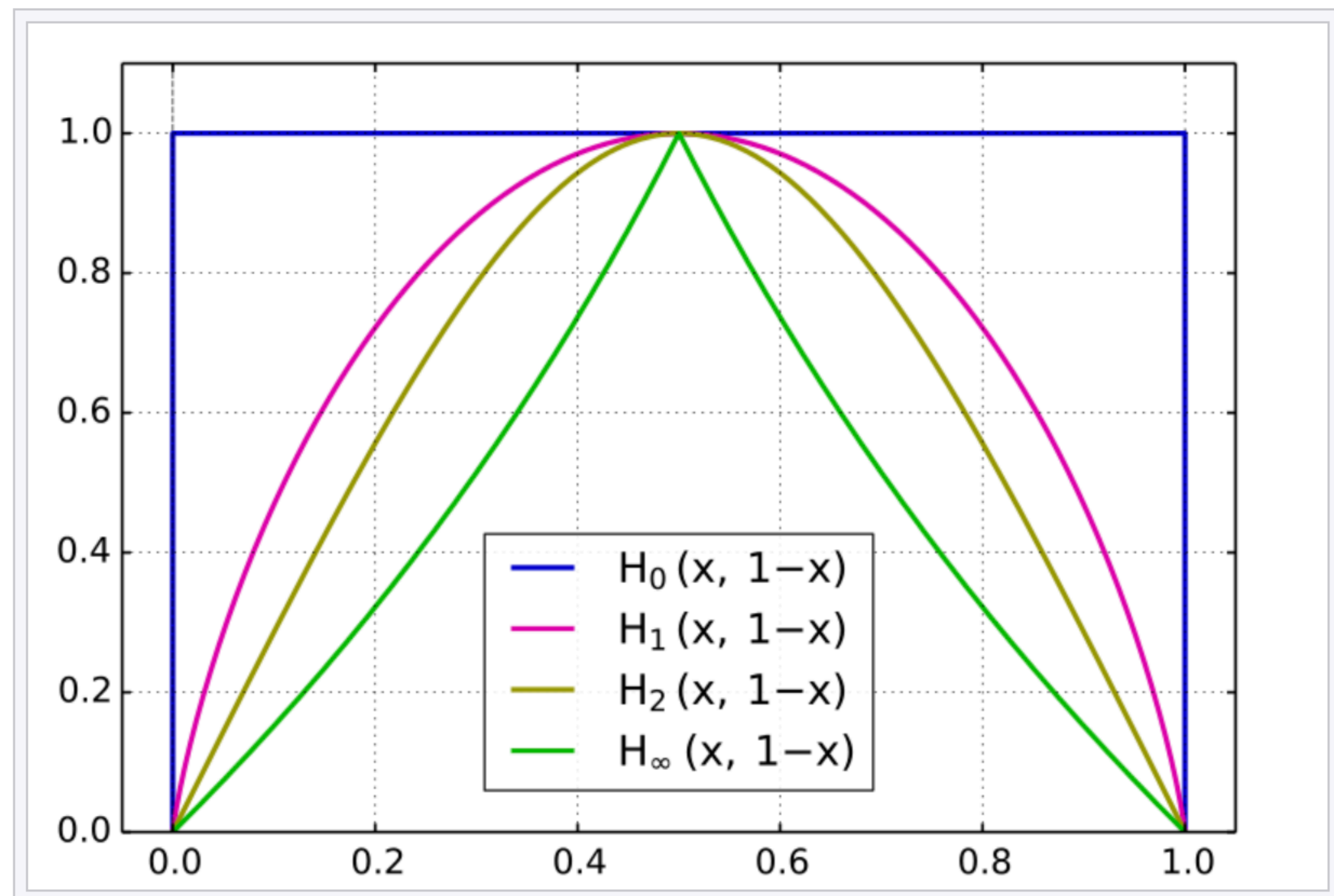
$$S(\rho) = -\sum_i \lambda_i \log \lambda_i$$



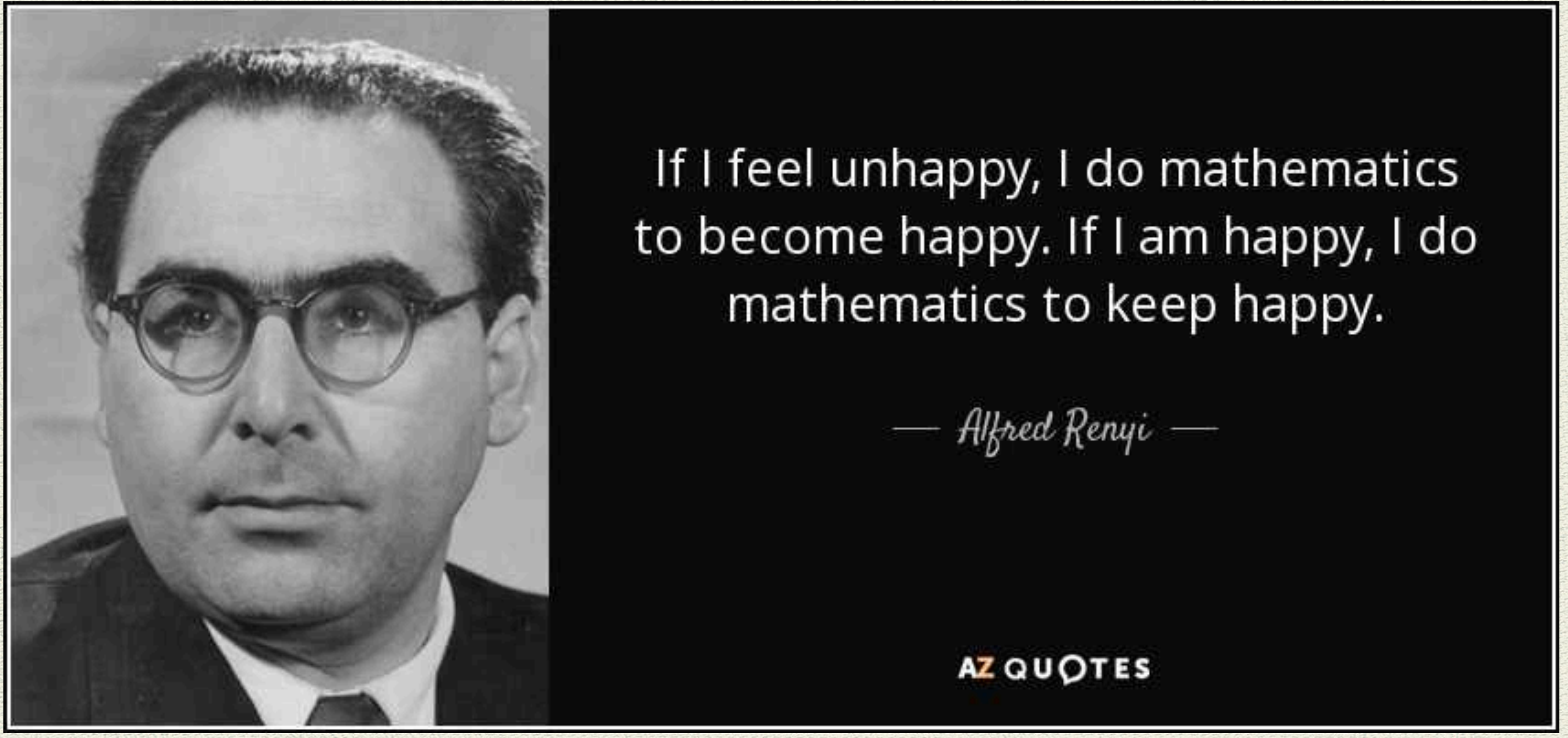
$$S(\rho) = -\left[(\frac{1+r}{2})\log(\frac{1+r}{2}) + (\frac{1-r}{2})\log(\frac{1-r}{2})\right]$$

$$S_\alpha(\rho) = \frac{\log \ (Tr\ \rho^\alpha)}{1-\alpha}$$

Renyi Entropy



Renyi Entropy



$$S_\alpha(\rho) = \frac{\log \ (Tr\ \rho^\alpha)}{1 - \alpha}$$

Properties of von-Neumann Entropy

$$S(\rho) = S(U\rho U^\dagger)$$

$$S(|\psi\rangle\langle\psi|) = 0$$

$$S(\frac{I}{d}) = \log_2 d$$

Relative Entropy

$$S(\rho\|\sigma) := Tr(\rho \log \rho - \rho \log \sigma)$$

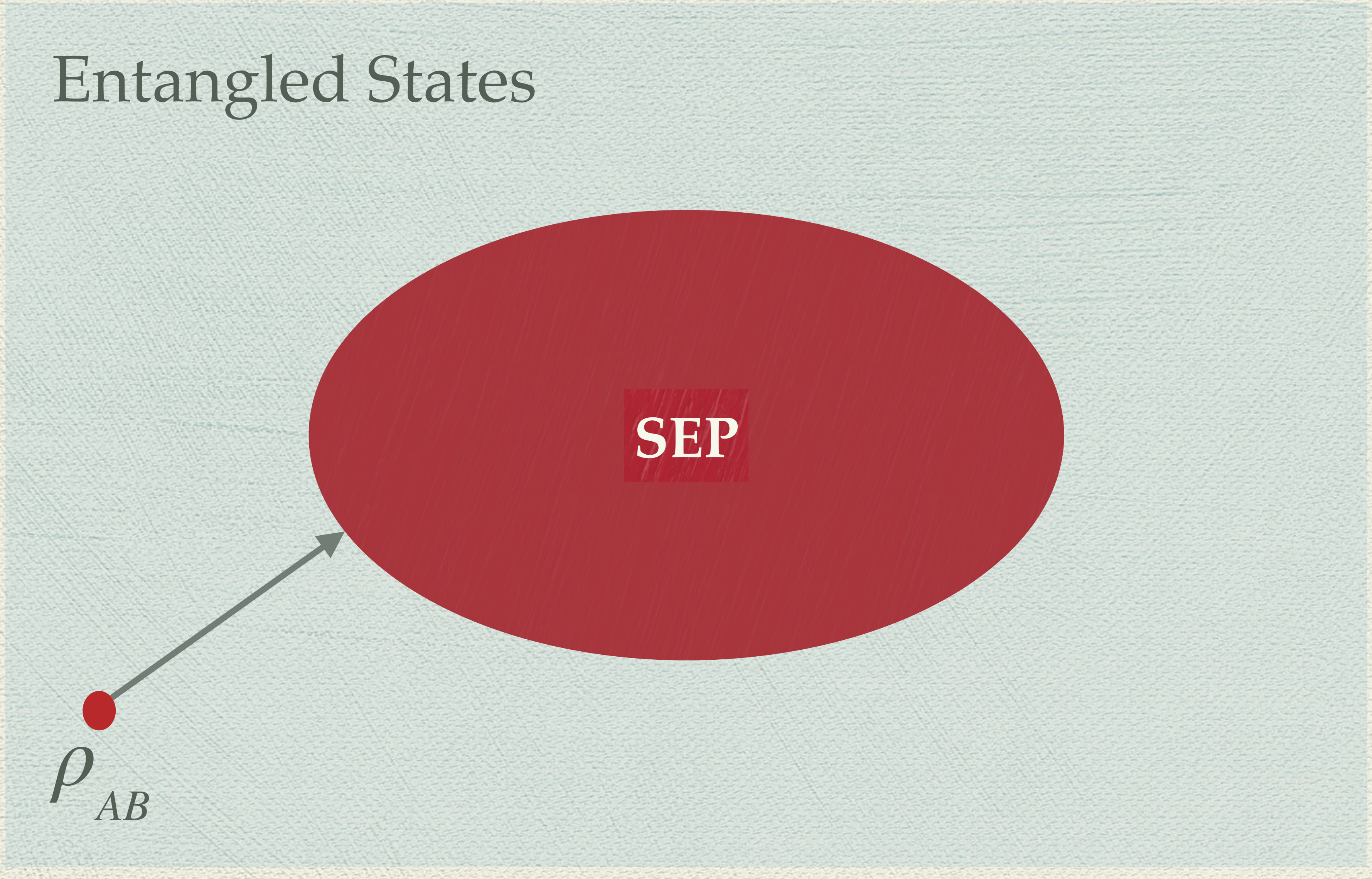
$$S(\rho\|\sigma) \neq S(\sigma\|\rho)$$

Klein Inequality

$$S(\rho\|\sigma) \geq 0 \qquad\qquad S(\rho\|\sigma) = 0 \quad \text{if} \quad \rho = \sigma$$

Relative entropy is a distance.

Separable States



$$E(\rho_{AB}) = min \quad S(\rho_{AB} \| \sigma_{sep})$$

Another version:

$$\langle \log \rho \rangle_\rho \geq \langle \log \sigma \rangle_\rho$$

Idea of the proof:

Expand $\log \sigma$ in the basis of ρ and use the convexity of $\log$.

نتیجه یک:

$$0 \leq S(\rho) \leq \log_2 d$$

Proof:

$$\langle \log \rho \rangle_\rho \geq \langle \log \sigma \rangle_\rho$$

Take $\sigma = \frac{I}{d}$.

نتیجه دو:

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

Proof:

Take $\rho = \rho_{AB}$

Take $\sigma = \rho_A \otimes \rho_B$

نتیجه سه:

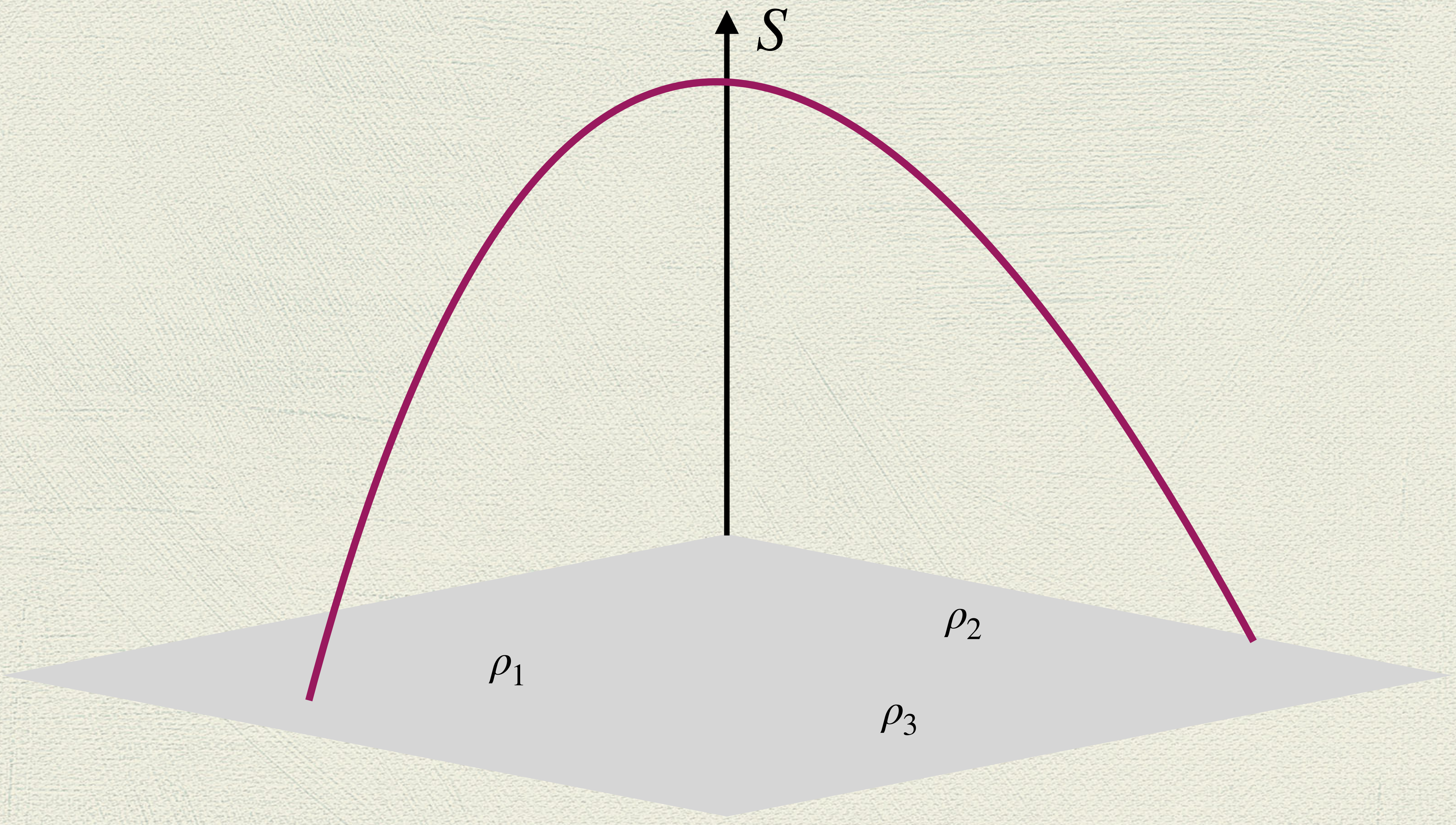
$$S(\lambda_1\rho_1 + \lambda_2\rho_2) \geq \lambda_1 S(\rho_1) + \lambda_2 S(\rho_2)$$

Proof:

Take $\sigma = \lambda_1\rho_1 + \lambda_2\rho_2$

and use Klein inequality for ρ_1 and ρ_2

$$S(\lambda_1\rho_1 + \cdots \lambda_n\rho_n) \geq \lambda_1 S(\rho_1) + \cdots \lambda_n S(\rho_n)\,.$$



If you want the absolute maximum, go up.

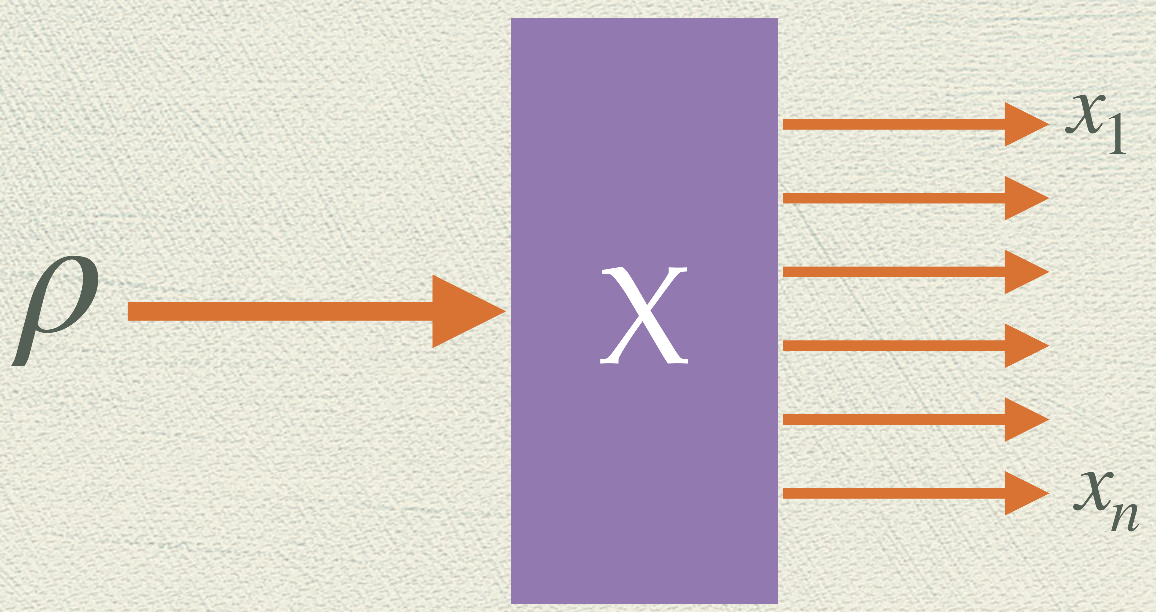
If you want a minimum, go down.

یک قضیه مهم: در هر کانال کوانتومی، آن حالت ورودی
که آنتروپی خروجی را می نیمم می کند،
یک حالت خالص است.

Idea of the proof:

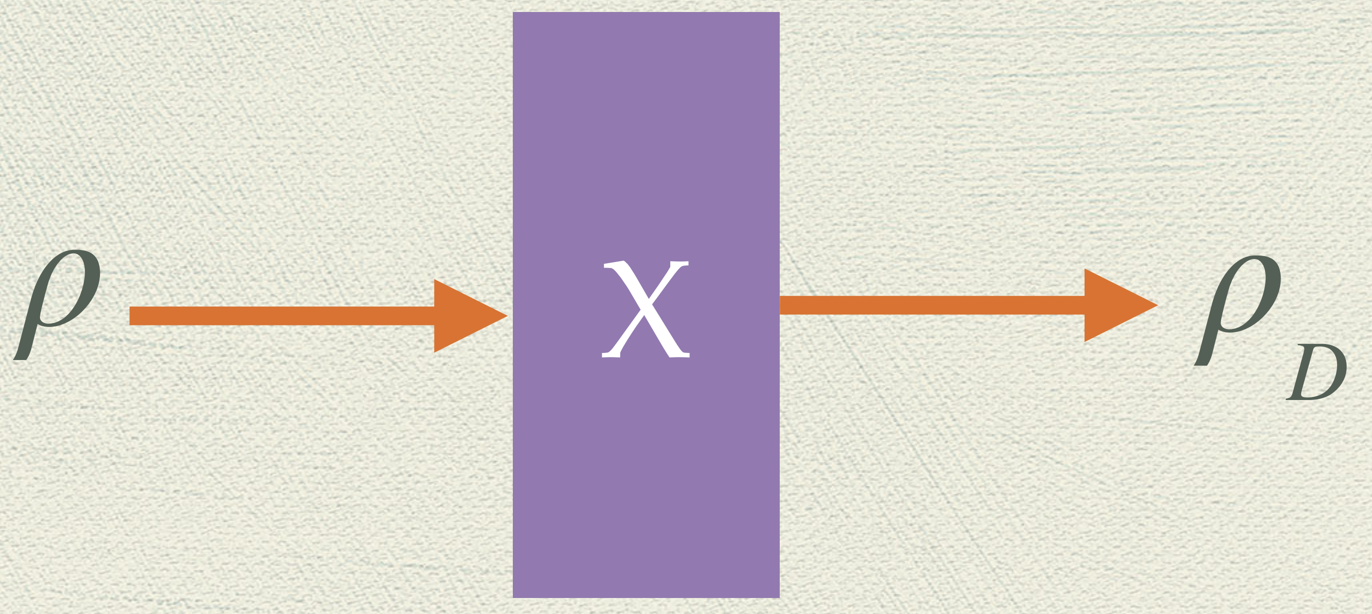
Decompose the output state and use the concavity of the entropy.

نتیجه چهار: در اثر اندازه گیری آنتروپی همیشه زیاد می شود.



$$S(\rho) \leq H(X)$$

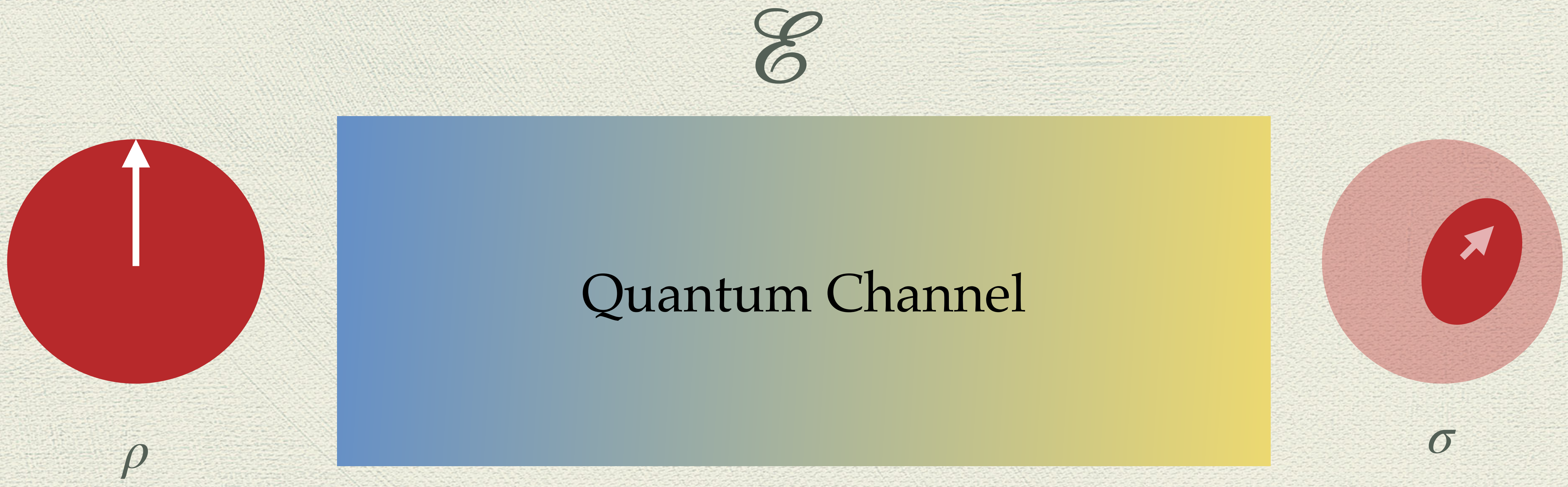
Idea of the proof:



ρ_D = diagonal matrix in a given basis

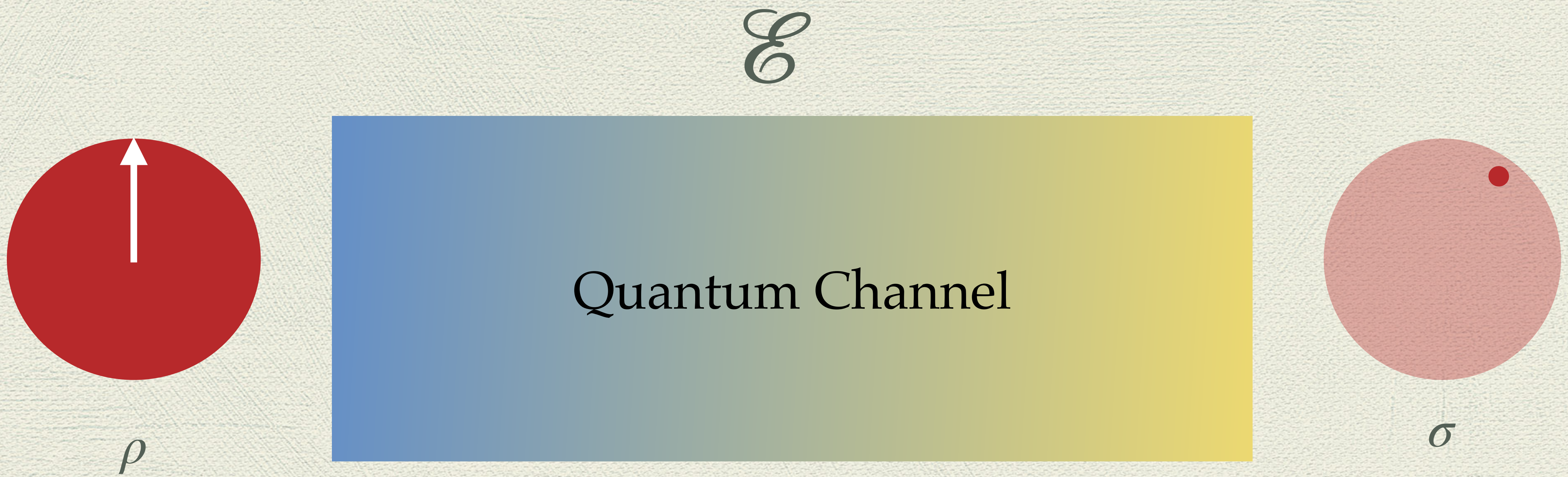
Use Klein inequality with $\sigma = \rho_D$

آیا یک کانال آنتروپی حالت ها را افزایش می دهد؟



$$S(\sigma) \geq S(\rho) \quad ?$$

نه الزاما: مثال نقض.



$$S(\sigma) \leq S(\rho)$$

$$\mathcal{E}(\rho) = Tr(\rho)\,|\phi_0\rangle\langle\phi_0|$$

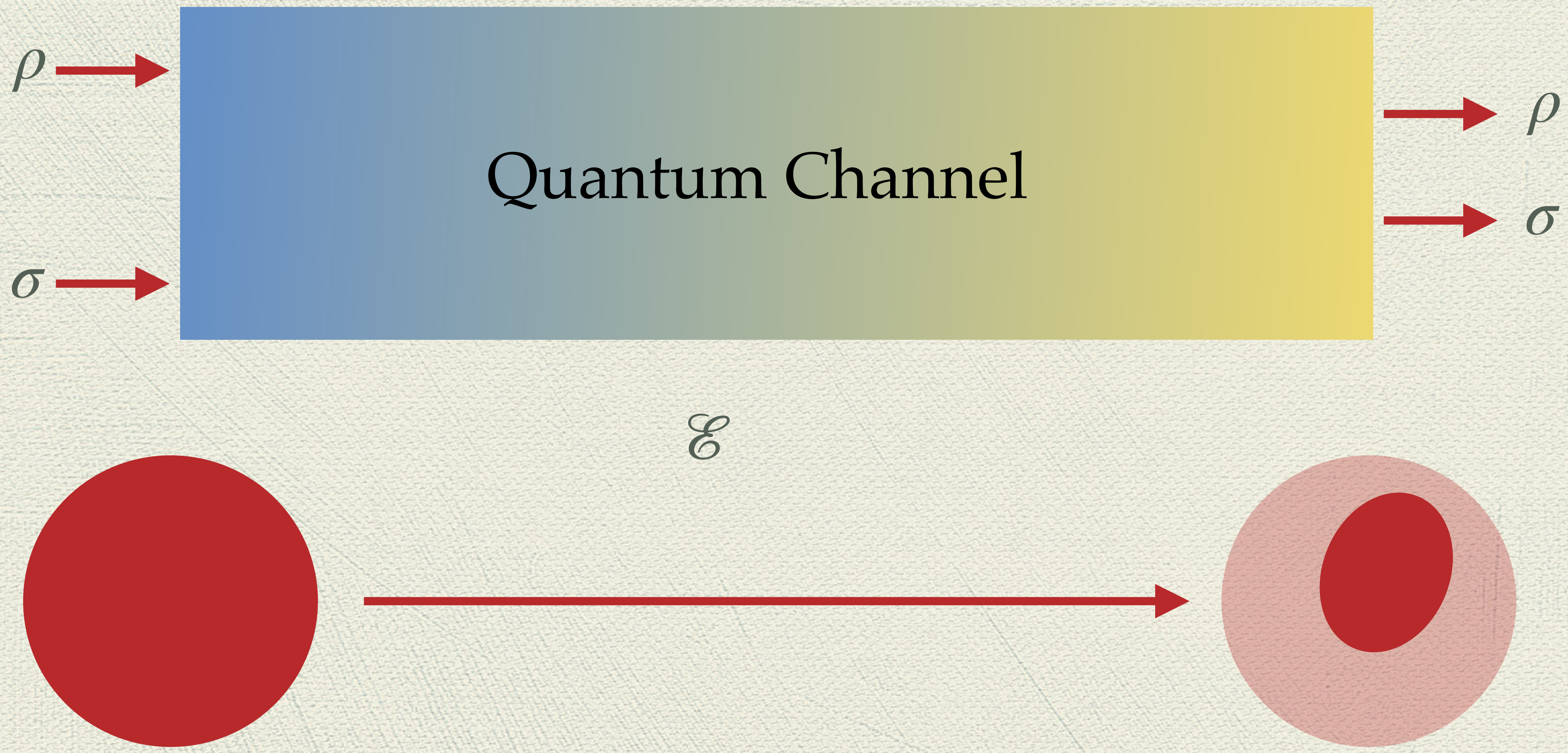
Classical Analog

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

آیا (آنتروپی نسبی) حالت ها پس از عبور از کانال کاهش می یابد؟

بله:

$$S(\mathscr{E}(\rho)\|\mathscr{E}(\sigma)) \leq S(\rho\|\sigma)$$



Idea of the proof:

Classical Sub-additivity

$$H(X,Y) \leq H(X) + H(Y)$$

But

$$H(X,Y) \geq H(X)\ ,\ H(Y)$$

$$H(Y|X) \geq 0$$

Quantum Sub-additivity

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

But

$$S(\rho_{AB}) \ngeq S(\rho_A)\ ,\ S(\rho_B)$$

با این وجود!

Araki-Lieb Inequality

$$S(\rho_{AB}) \geq |\, S(\rho_A) - S(\rho_B)\, |$$

When the two sides are equal?

Entropy of Mixing

$$p_1 S(\sigma_1) + p_2 S(\rho_2) \leq S(p_1\sigma_1 + p_2\sigma_2)$$

$$S(p_1\sigma_1 + p_2\sigma_2) - p_1 S(\sigma_1) + p_2 S(\rho_2) \leq H(p_1, p_2)$$

$$S(p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2|) \leq H(p_1, p_2)$$

When the two sides are equal?

$$\sum_i p_i S(\sigma_i) \leq S(\sum_i p_i \sigma_i) \leq H(\{p_i\}) + \sum_i p_i S(\sigma_i)$$

$$p_1 S(\sigma_1) + p_2 S(\rho_2) \leq S(p_1 \sigma_1 + p_2 \sigma_2)$$

Strong Sub-Additivity

Classical

$$H(X|Y,Z) \leq H(X|Y)$$

Quantum

$$S(A|B,C) \leq S(A|B)$$

$$S(A,B,C) - S(B,C) \leq S(A,B) - S(B)$$

Idea of the proof:

Strong Sub-Additivity

$$S(AB) \leq S(A) + S(B)$$

Strong Sub-Additivity

$$S(AB) \leq S(A) + S(B)$$

$$S(A) + S(B) \leq S(A, C) + S(B, C)$$

Strong Sub-Additivity

$$S(AB) \leq S(A) + S(B)$$

$$I(A : B) \leq I(A : B, C)$$

$$S(A) + S(B) \leq S(A, C) + S(B, C)$$

Strong Sub-Additivity

$$S(AB) \leq S(A) + S(B)$$

$$S(A) + S(B) \leq S(A, C) + S(B, C)$$

$$I(A : B) \leq I(A : B, C)$$

$$I(A : \mathcal{E}B) \leq I(A : B)$$


Compression of Quantum Information

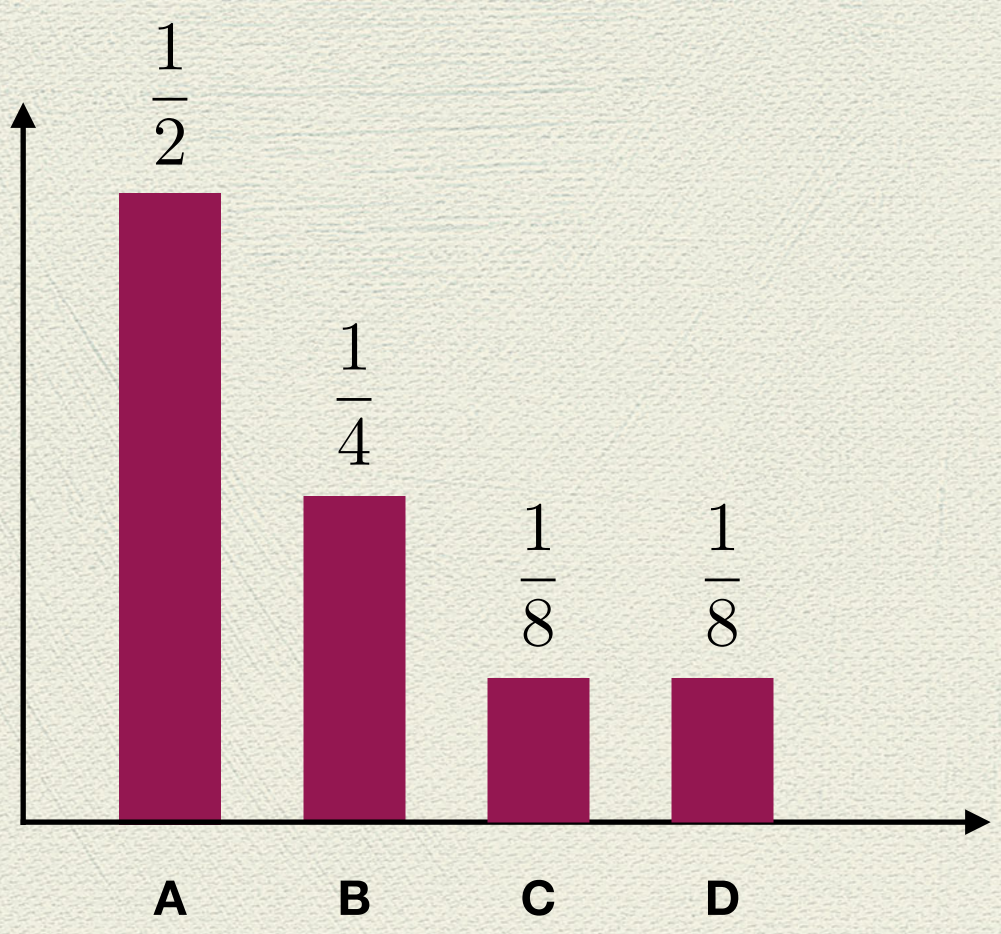
فشرده سازی اطلاعات کوانتومی

A ⟶ 0

B ⟶ 10

C ⟶ 110

D ⟶ 111



$$H(X) = -\sum_i P_i \log P_i$$

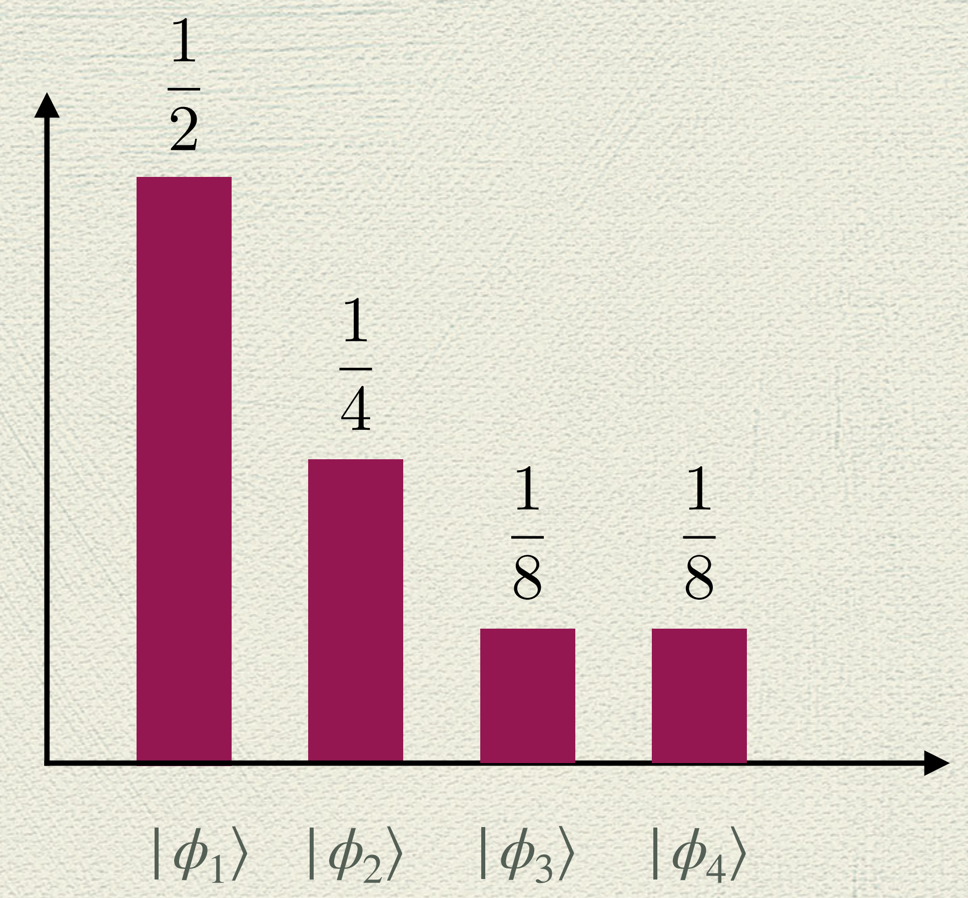
فشرده سازی اطلاعات کوانتومی

$|\phi_1\rangle \longrightarrow |0\rangle$

$|\phi_2\rangle \longrightarrow |10\rangle$

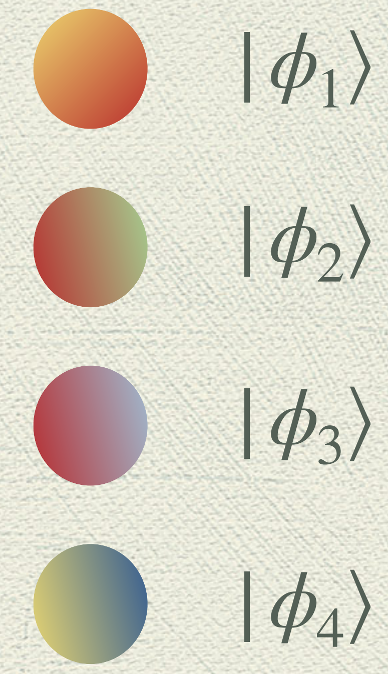
$|\phi_3\rangle \longrightarrow |110\rangle$

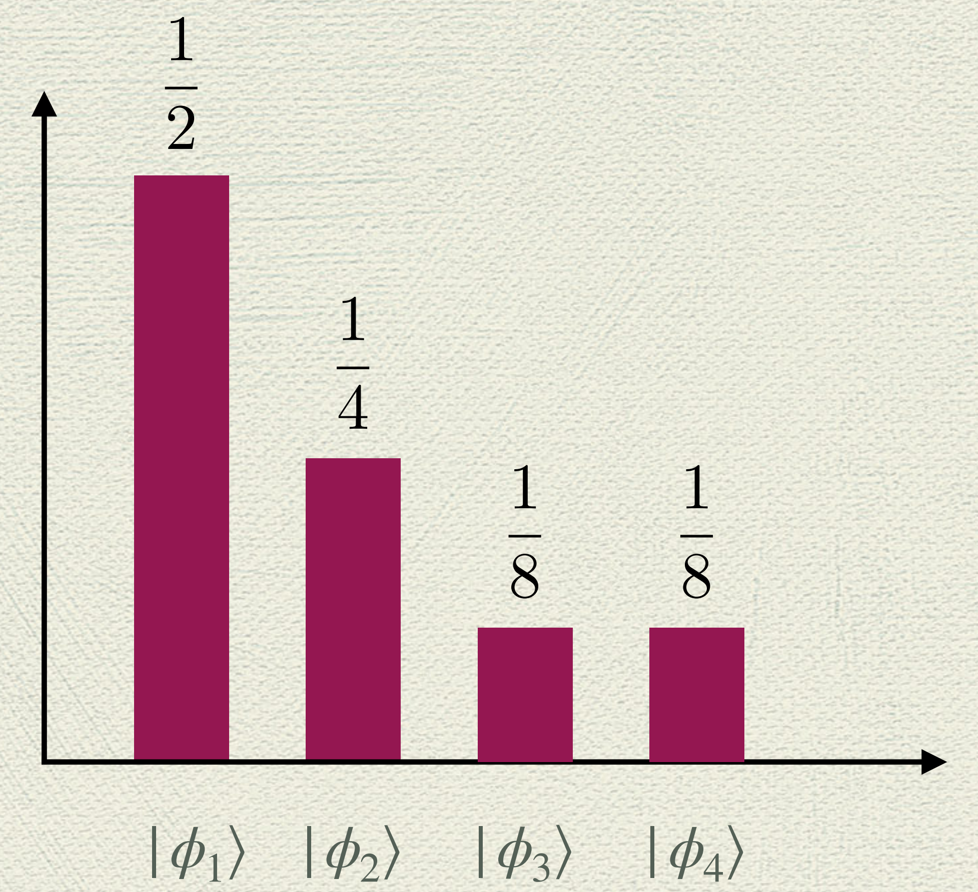
$|\phi_4\rangle \longrightarrow |111\rangle$



$S(\rho) = 1.75$ qubits

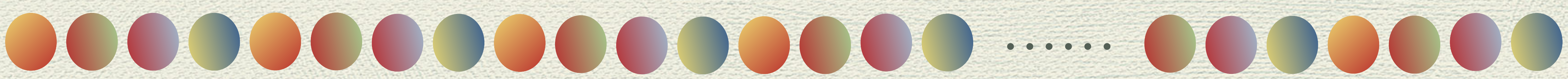
فشرده سازی اطلاعات کوانتومی





$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

$$S(\rho) = -Tr(\rho \log \rho) = 1.5$$



100 states $\longrightarrow$ 200 qubits

$$S(\rho) = -Tr(\rho \log \rho) = 1.5$$

100 states $\longrightarrow$ 150 qubits

Compression of Quantum Information

Compression of Quantum States

$$|\Phi_1\rangle = |000101011110\rangle$$

$$|\Phi_2\rangle = |01110100110\rangle$$

$$|\Phi_n\rangle = |10001100111\rangle$$

Compression of Quantum Information

A simple example

$$|\Phi_1\rangle = |000+0+0++++0\rangle$$

$$|\Phi_2\rangle = |0+++0+000++0\rangle$$

$$|\Phi_n\rangle = |+000++000+++\rangle$$

$$p(0) = \frac{1}{2}, \quad p(+) = \frac{1}{2}$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

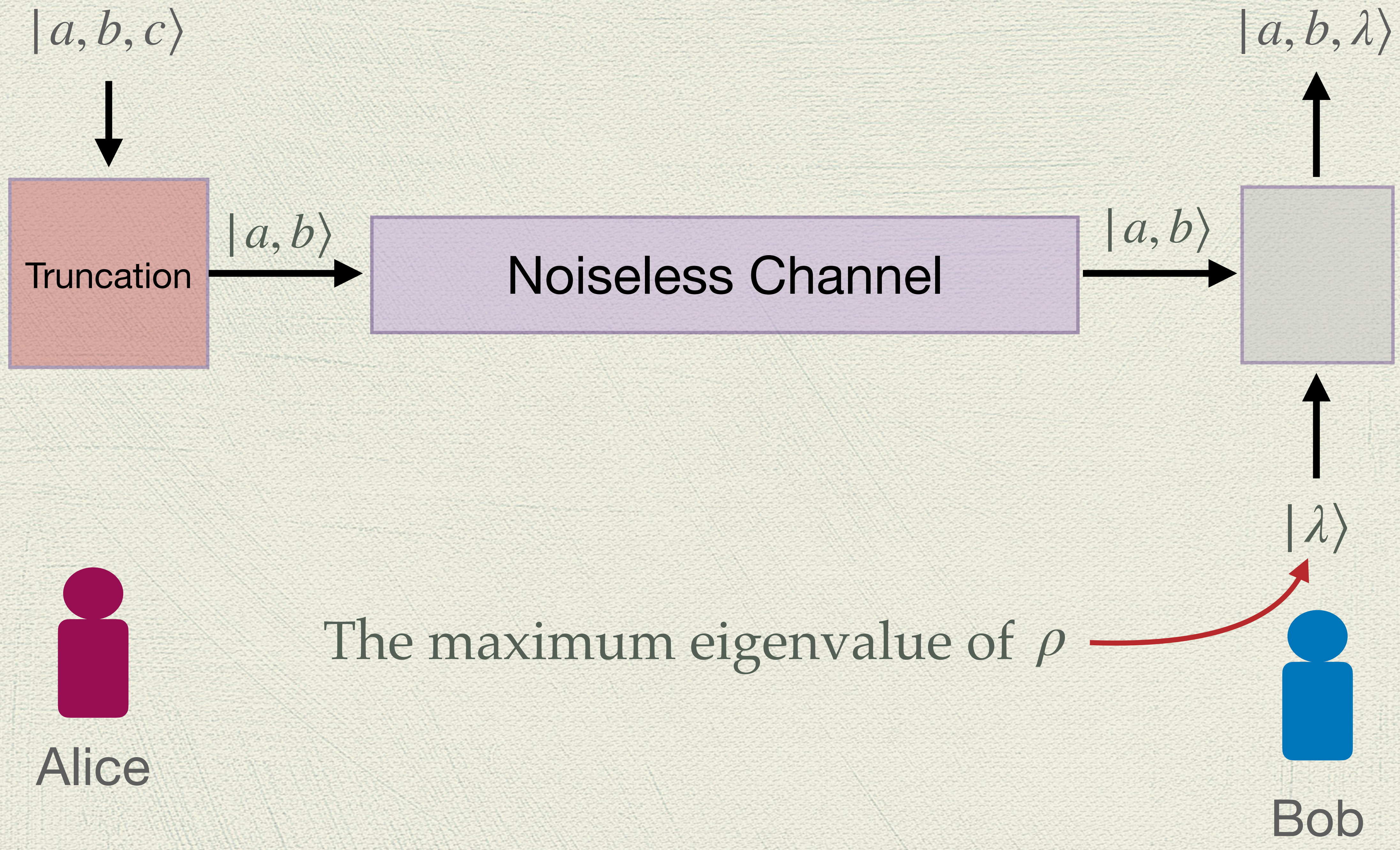
$$\rho^{(M)} = \rho^{\otimes M}$$

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\lambda_0 = \cos^2 \frac{\pi}{8} \qquad \lambda_1 = \sin^2 \frac{\pi}{8}$$

$$|0\rangle = \begin{pmatrix} \cos \frac{\pi}{8} \\ \sin \frac{\pi}{8} \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} -\sin \frac{\pi}{8} \\ \cos \frac{\pi}{8} \end{pmatrix}$$

A very simple strategy



$$\overline{F} = 0.853$$

Problem:

ثابت کنید که در این حالت تشابه متوسط برابر است با:

0.853

Schumacher Compression

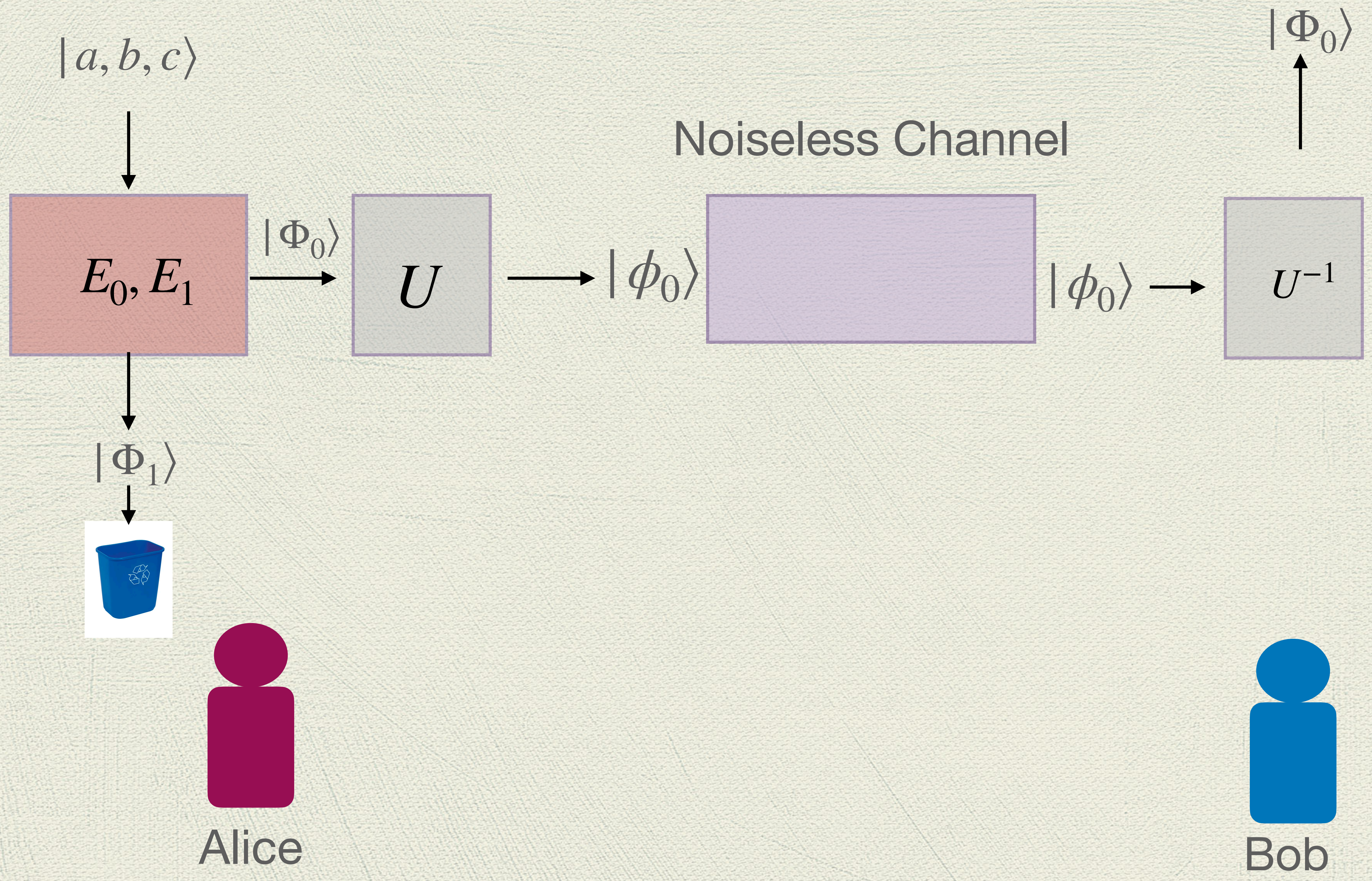
Typical Subspace

$$\Lambda_{typ} \equiv \Lambda_0 = Span\{\,|000\rangle, |001\rangle, |010\rangle, |100\rangle\}$$

$$\Lambda_{typ}^{\perp} \equiv \Lambda_1 = Span\{\,|111\rangle, |110\rangle, |101\rangle, |011\rangle\}$$

$$\{E_0\ , E_1\}$$

$$\overline{F} = 0.9419$$



$|a, b, c\rangle$
Noiseless Channel
$|\Phi_0\rangle$
E_0, E_1
$|\Phi_0\rangle$
U
$|\phi_0\rangle$
$|\phi_0\rangle$
U^{-1}
$|\Phi_1\rangle$
Alice
Bob

$$|\Phi_0\rangle = \frac{E_0|\Phi\rangle}{\sqrt{\langle\Phi|E_0|\Phi\rangle}}$$

$$P(0) = \langle\Phi|E_0|\Phi\rangle$$

$$\rho^B = p(0)|\Phi_0\rangle\langle\Phi_0| + p(1)(|\lambda_0\rangle\langle\lambda_0|)^{\otimes 3}$$

$$\rho^B = E_0|\Phi\rangle\langle\Phi|E_0 + \Big(1 - \langle\Phi|E_0|\Phi\rangle\Big) \ \ (|\lambda_0\rangle\langle\lambda_0|)^{\otimes 3}$$

Compression of Quantum Information

The general case

$$|\Phi_{\mathbf{x}}\rangle = |\phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \cdots\cdots, \phi_{x_M}\rangle$$

$$P_{\mathbf{x}} = p_{x_1} p_{x_2} p_{x_3} \cdots p_{x_M}$$

$$\rho^{(M)} = \sum_{\mathbf{x}} P_{\mathbf{x}} |\Phi_{\mathbf{x}}\rangle\langle\Phi_{\mathbf{x}}|$$

$$\rho = \sum_{x} p_x |\phi_x\rangle\langle\phi_x|$$

$$\rho^{(M)} = \rho^{\otimes M}$$

Typical Subspace

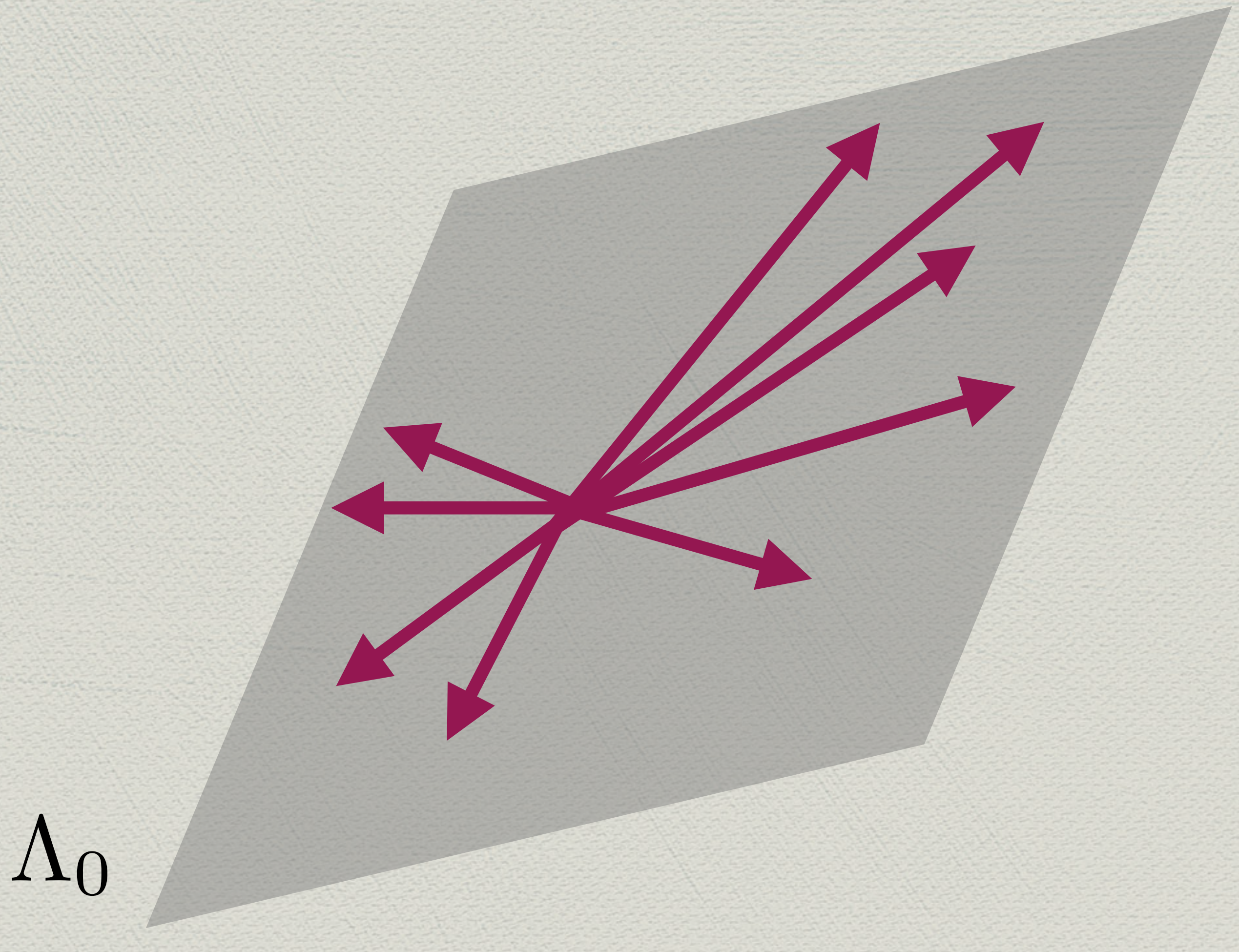
$$\Lambda_{typ} = \text{Span}\{\, |s_1, s_2, \cdots s_M\rangle, \ \ M_0 = M\lambda_0, \ \ M_1 = M\lambda_1\}$$

$$dim(\Lambda_{typ}) = \binom{M}{M\lambda_0} = 2^{MS(\rho)}$$

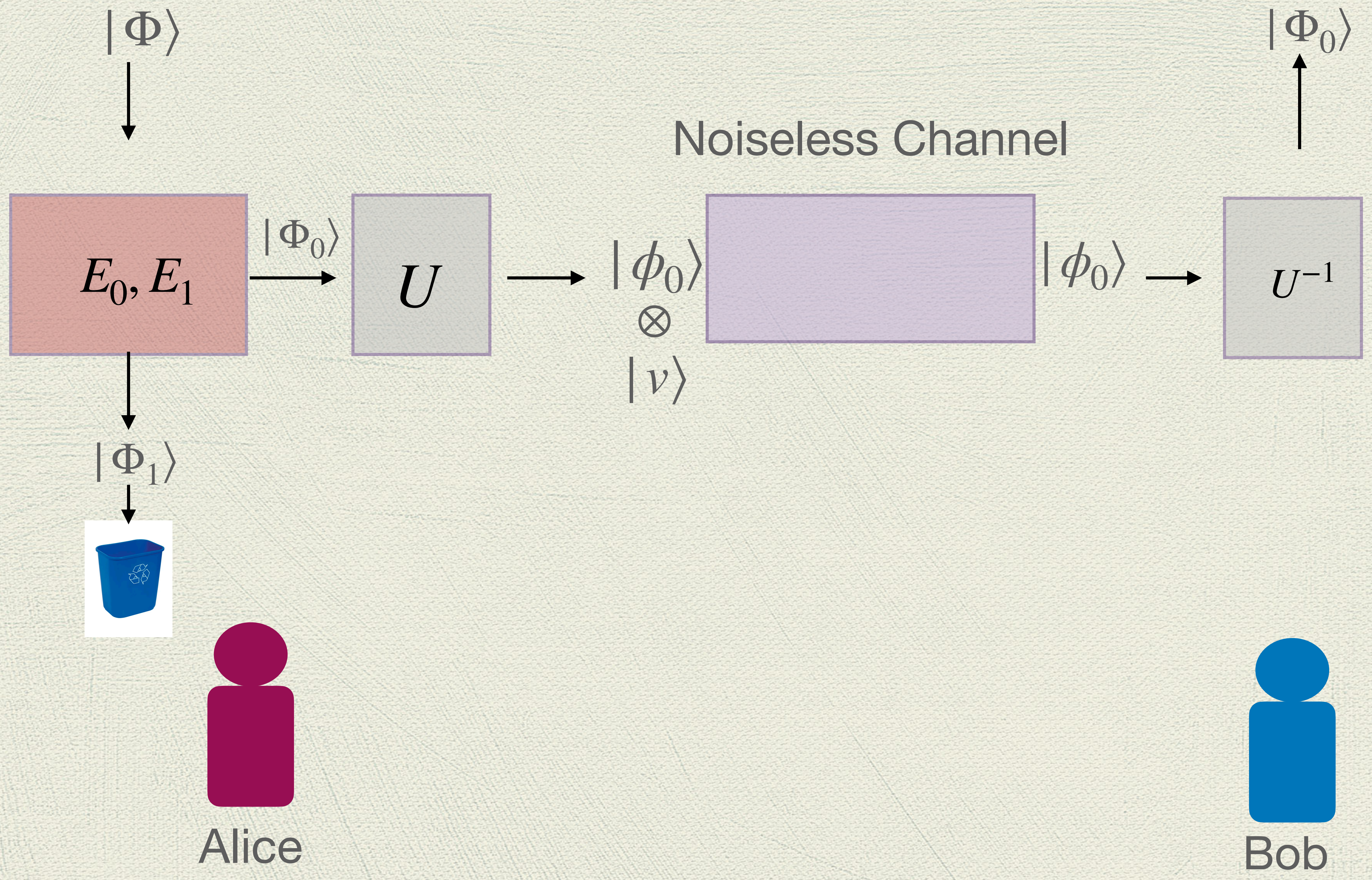
$$E_0 = \sum_{v \in \Lambda_0} |v\rangle\langle v| \qquad\qquad E_1 = I - E_0$$

$$dim(V) = d^M = 2^{M \log_2 d}$$

$$\{E_0\ , E_1\}$$



$$dim(\Lambda_0) \approx 2^{MS(\rho)}$$

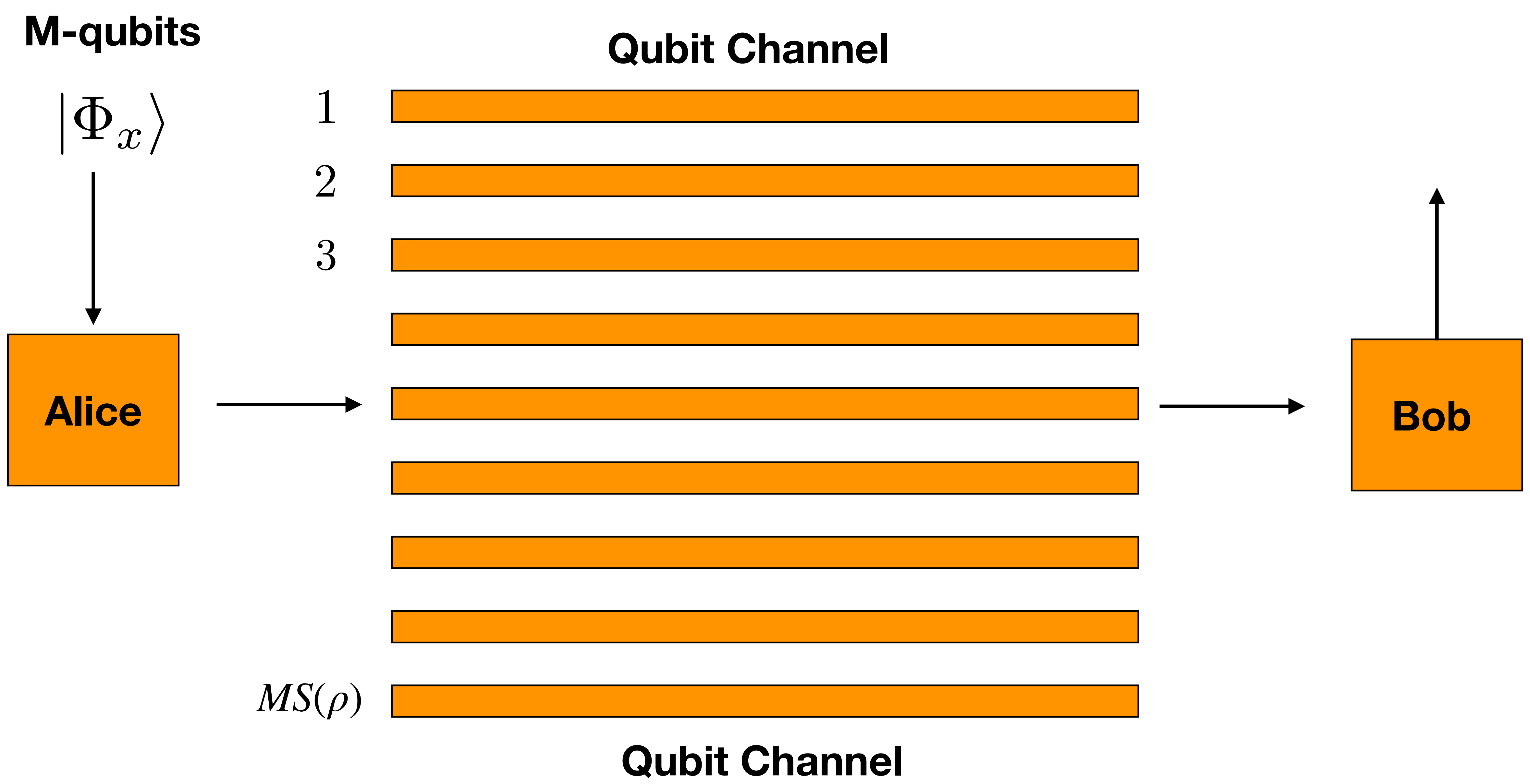

$|\Phi\rangle$
E_0, E_1
$|\Phi_0\rangle$
U
$|\phi_0\rangle$
$\otimes$
$|v\rangle$
Noiseless Channel
$|\phi_0\rangle$
U^{-1}
$|\Phi_0\rangle$
$|\Phi_1\rangle$
Alice
Bob

$$|\Phi_0\rangle = \frac{E_0|\Phi\rangle}{\sqrt{\langle\Phi|E_0|\Phi\rangle}}$$

$$P(0) = \langle\Phi|E_0|\Phi\rangle$$

$$\rho^B = p(0)|\Phi_0\rangle\langle\Phi_0| + p(1)(|\lambda_0\rangle\langle\lambda_0|)^{\otimes n}$$

$$\rho^B = E_0|\Phi\rangle\langle\Phi|E_0 + \left(1 - \langle\Phi|E_0|\Phi\rangle\right) \ (|\lambda_0\rangle\langle\lambda_0|)^{\otimes n}$$


M-qubits
$|\Phi_x\rangle$
Qubit Channel
1
2
3
Alice
Bob
$MS(\rho)$
Qubit Channel

Compression of mixed states?

Question: Can we use von-Neumann entropy for compression of mixed states?

$$\rho_{\mathbf{x}} = \rho_{x_1} \otimes \rho_{x_2} \otimes \rho_{x_2} \cdots\cdots \rho_{x_M}$$

$$\rho^{(M)} = \rho^{\otimes M}$$

$$\rho = \sum_x p_x \rho_x$$

Answer: NO.

$\sigma \otimes \sigma \otimes \sigma \otimes \sigma \cdots \otimes \sigma$

$\sigma \otimes \sigma \otimes \sigma \otimes \sigma \cdots \otimes \sigma$

$\sigma \otimes \sigma \otimes \sigma \otimes \sigma \cdots \otimes \sigma$

$\sigma \otimes \sigma \otimes \sigma \otimes \sigma \cdots \otimes \sigma$

Zero Information

$S(\sigma) \neq 0$

Conjecture

Consider the following special cases:

$|\phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \cdots\cdots, \phi_{x_M}\rangle$

Pure states

$\sigma_{x_1} \otimes \sigma_{x_2} \otimes \sigma_{x_3} \otimes \cdots\cdots \otimes \sigma_{x_M}$

Orthogonal mixed states

$$|\phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \cdots\cdots, \phi_{x_M}\rangle$$

$$\chi = S\left(\sum_x p_x \sigma_x\right) - \sum_x p_x S(\sigma_x)$$

$$\sigma_{x_1} \otimes \sigma_{x_2} \otimes \sigma_{x_3} \otimes \cdots\cdots \otimes \sigma_{x_M}$$

$$\rho = p_1\sigma_1 + p_2\sigma_2 \qquad\qquad S(\rho) = ?$$

$$\rho = \begin{pmatrix} p_1\sigma_1 & 0 \\ 0 & p_2\sigma_2 \end{pmatrix}$$

$$\rho \log \rho = \begin{pmatrix} p_1\sigma_1 \log p_1\sigma_1 & 0 \\ 0 & p_2\sigma_2 \log p_2\sigma_2 \end{pmatrix}$$

$$\rho \log \rho = \begin{pmatrix} p_1\sigma_1 \log p_1 & 0 \\ 0 & p_2\sigma_2 \log p_2 \end{pmatrix} + \begin{pmatrix} p_1\sigma_1 \log \sigma_1 & 0 \\ 0 & p_2\sigma_2 \log \sigma_2 \end{pmatrix}$$

$$\rho \log \rho = \begin{pmatrix} p_1\sigma_1 \log p_1 & 0 \\ 0 & p_2\sigma_2 \log p_2 \end{pmatrix} + \begin{pmatrix} p_1\sigma_1 \log \sigma_1 & 0 \\ 0 & p_2\sigma_2 \log \sigma_2 \end{pmatrix}$$

$$S(\rho) = H(p) + p_1 S(\sigma_1) + p_2 S(\sigma_2)$$

$$H(p) = S(\rho) - p_1 S(\sigma_1) - p_2 S(\sigma_2)$$

$$\chi = S(\rho) - p_1 S(\sigma_1) - p_2 S(\sigma_2)$$

Holevo Quantity

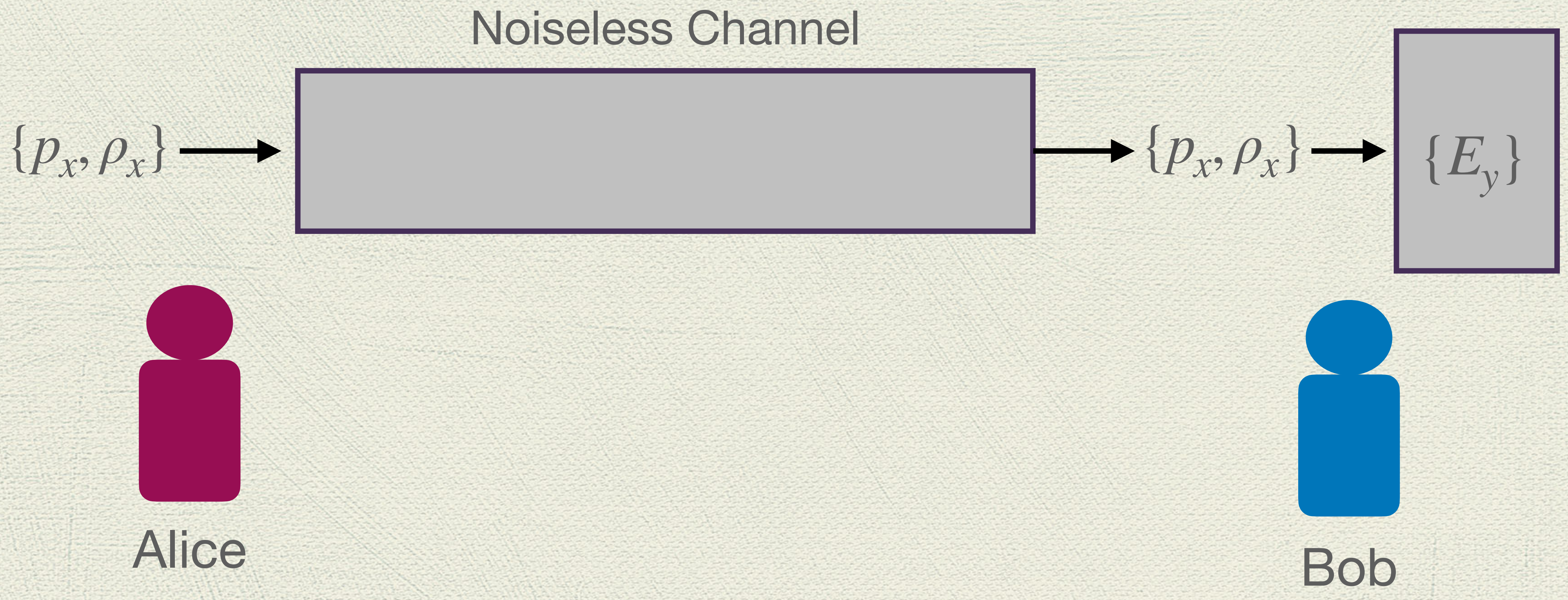
$$\chi = S(\sum_x p_x \sigma_x) - \sum_x p_x S(\sigma_x)$$

Properties of Holevo Quantity

$$\chi = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$\rho_{AB} = \sum_x p_x |x\rangle\langle x| \otimes \sigma_x \qquad \rho_A = \sum_x p_x |x\rangle\langle x| \qquad \rho_B = \sum_x p_x \sigma_x$$

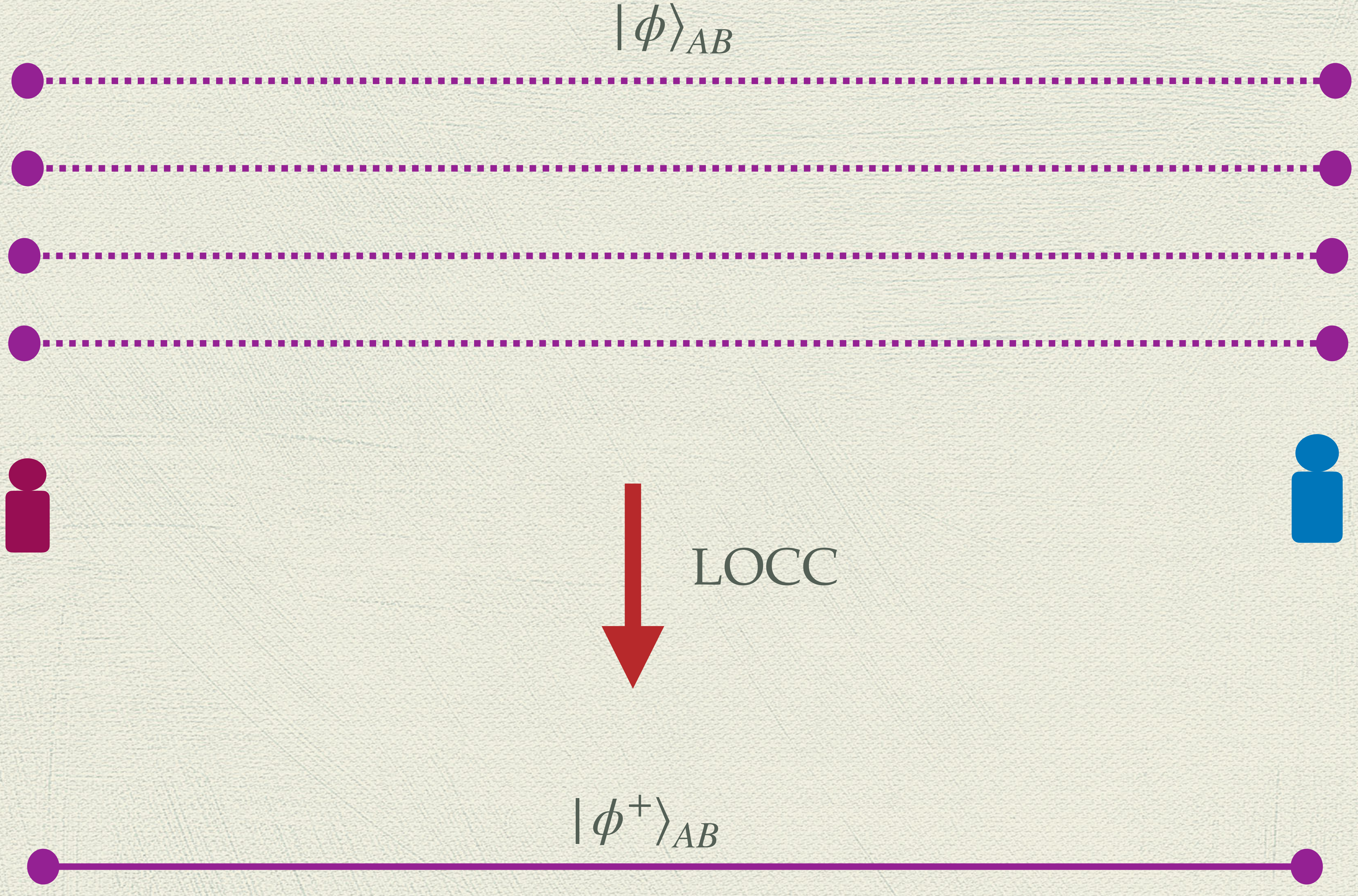
Accessible Information



$$I(X : Y) \leq \chi\{p_x, \rho_x\}$$


Measures of Entanglement


EXCHANGE
صرافة



$$E(|\phi\rangle_{AB}) = \frac{1}{4}$$

$$|\phi\rangle_{AB} = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$$

$$|\phi\rangle_{AB}$$

$$\frac{1}{2} \leq p \leq 1$$

$$|\phi\rangle_{AB}$$

$$|\phi\rangle_{A'AB} = \sqrt{p}\,|000\rangle + \sqrt{1-p}\,|011\rangle$$

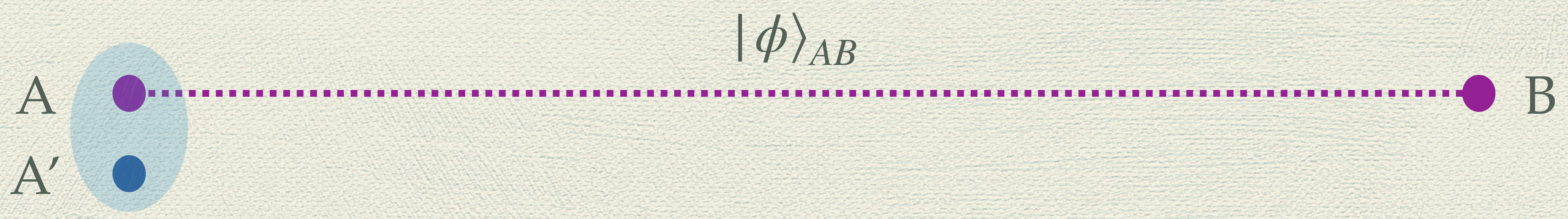
$$|\phi\rangle_{AB}$$

$$|\phi\rangle_{A'AB} = \sqrt{p}\,|000\rangle + \sqrt{1-p}\,|011\rangle$$

$$\Big\downarrow\ U_{A'A}$$

$$|\phi\rangle_{A'AB} = x\,|0\rangle_{A'}|\phi^{+}\rangle_{AB} + y\,|1\rangle_{A'}|00\rangle_{AB}$$

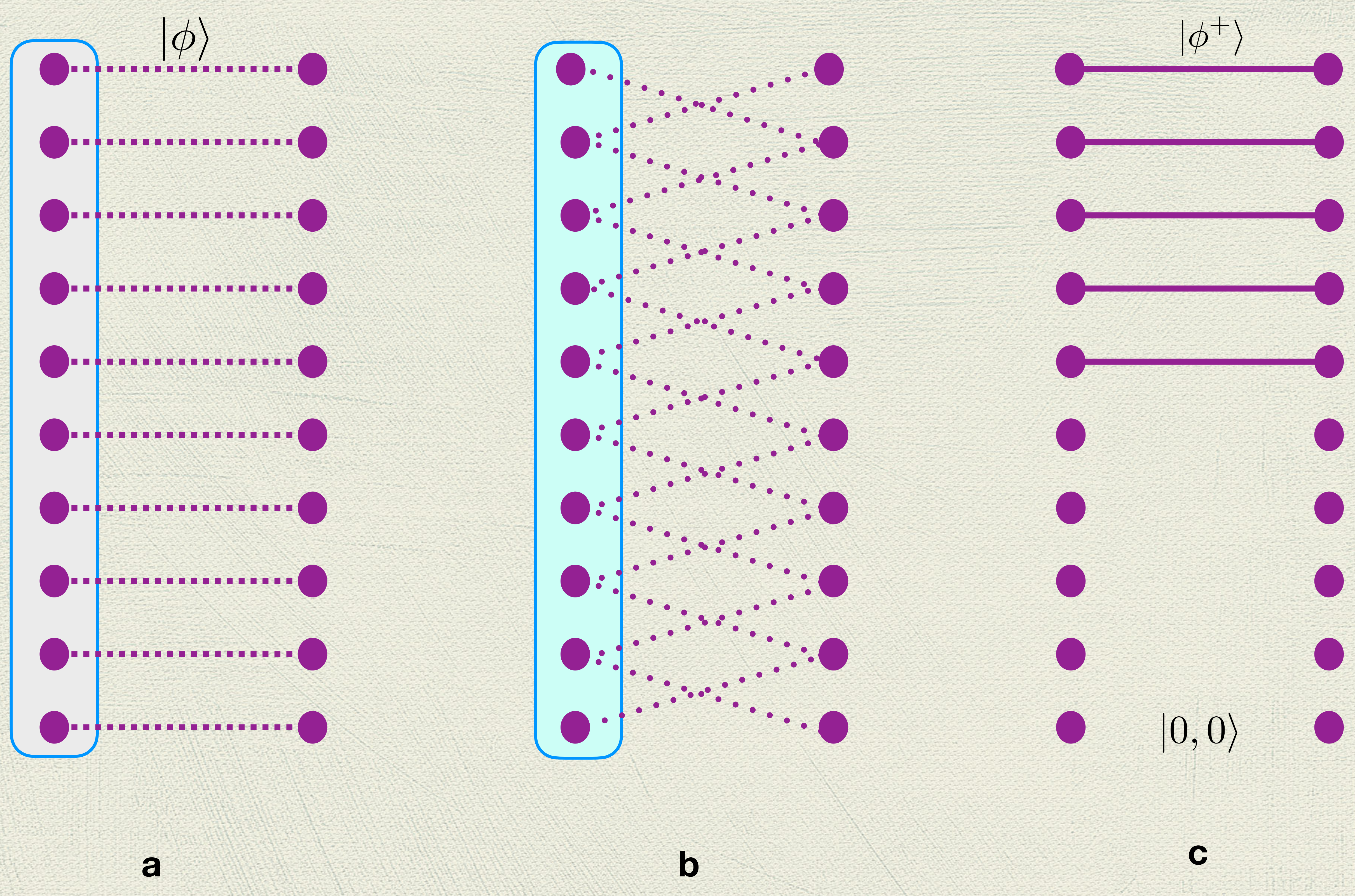
$$= \frac{x}{\sqrt{2}}\,|0\rangle_{A'}(|00\rangle + |11\rangle_{AB}) + y\,|1\rangle_{A'}|00\rangle_{AB}$$



$$|\phi\rangle_{A'AB} = \sqrt{p}\,|000\rangle + \sqrt{1-p}\,|011\rangle$$

$$|\phi\rangle_{A'AB} = \sqrt{2(1-p)}\,|0\rangle_{A'}|\phi^{+}\rangle_{AB} + \sqrt{2p-1}\,|1\rangle_{A'}|00\rangle_{AB}$$

$$\mathscr{P}(|\phi\rangle_{AB} \longrightarrow |\phi^{+}\rangle_{AB}) = 2(1-p)$$



$|\phi\rangle$
$|\phi^+\rangle$
$|0,0\rangle$
a
b
c

Example

$$|\Psi(3)\rangle_{a_1b_1,a_2b_2,a_3b_3} = (\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|11\rangle)^{\otimes 3}$$

$$= \sqrt{1-p}^{\,3}\,|00,00,00\rangle + \sqrt{1-p}^{\,2}\sqrt{p}\,\big(|00,00,11\rangle + |00,11,00\rangle + |11,00,00\rangle\big)$$

$$+\sqrt{1-p}\sqrt{p}^{\,2}\,\big(|11,11,00\rangle + |11,00,11\rangle + |00,11,11\rangle\big) + \sqrt{p}^{\,3}\,|11,11,11\rangle$$

Again, Typical Sequences

$$|\Psi(3)\rangle_{a_1a_2a_3;b_1b_2b_3} =$$

$$\sqrt{1-p}^3\,|000,000\rangle + \sqrt{1-p}^2\sqrt{p}\,\left(|001,001\rangle + |010,010\rangle + |100,100\rangle\right)$$

$$+\sqrt{1-p}\sqrt{p}^2\left(|110,110\rangle + |101,101\rangle + |011,011\rangle\right) + \sqrt{p}^3\,|111,111\rangle$$

$$|\Psi_0(3)\rangle = |000,000\rangle \qquad Pr(0) = (1-p)^3$$

$$|\Psi_1(3)\rangle = \frac{1}{\sqrt{3}}\big(|001,001\rangle + |010,010\rangle + |100,100\rangle\big) \qquad Pr(1) = (1-p)^2 p$$

$$|\Psi_2(3)\rangle = \frac{1}{\sqrt{3}}\big(|110,110\rangle + |101,101\rangle + |011,011\rangle\big) \qquad Pr(2) = (1-p)p^2$$

$$|\Psi_3(3)\rangle = |111,111\rangle \qquad Pr(3) = p^3$$

$$|\Psi_k(N)\rangle_{a_1a_2\cdots a_N;b_1b_2\cdots b_N} = \frac{1}{\sqrt{\binom{N}{k}}} \sum_{s_1,s_2\cdots s_N=0,1}^{\prime} |s_1s_2\cdots s_N; s_1s_2\cdots s_N\rangle$$

$$Pr(k) = \binom{N}{k} \times (1-p)^k p^{N-k}$$

$$|\Psi_k(N)\rangle_{a_1a_2\cdots a_N;b_1b_2\cdots b_N} = \frac{1}{\sqrt{\binom{N}{k}}} \sum_{s_1,s_2\cdots s_N=0,1}^{\prime} |s_1s_2\cdots s_N; s_1s_2\cdots s_N\rangle$$

$$Pr(k) = \binom{N}{k} \times (1-p)^k p^{N-k}$$

$$Pr(k=Np) = \binom{N}{Np} \times (1-p)^{Np} p^{N(1-p)}$$

$$Pr(k=Np) \approx 2^{NH(p)} \times 2^{-NH(p)} \approx 1$$

$$|\Psi_1(4)\rangle = \frac{1}{2}\left(|1000,1000\rangle + |0100,0100\rangle + |0010,0010\rangle + |0001,0001\rangle\right)$$

$$|1000\rangle \longrightarrow |00\rangle \otimes |00\rangle$$

$$|0100\rangle \longrightarrow |01\rangle \otimes |00\rangle$$

$$|0010\rangle \longrightarrow |10\rangle \otimes |00\rangle$$

$$|0001\rangle \longrightarrow |11\rangle \otimes |00\rangle$$

$$|\Psi_1(4)\rangle \longrightarrow |\phi^+\rangle \otimes |\phi^+\rangle \otimes |00\rangle \otimes |00\rangle$$

$$|\Psi_k(N)\rangle_{a_1a_2\cdots a_N;b_1b_2\cdots b_N} = \frac{1}{\sqrt{\binom{N}{k}}} \sum_{s_1,s_2\cdots s_N=0,1}^{\prime} |s_1s_2\cdots s_N; s_1s_2\cdots s_N\rangle$$

$$\downarrow$$

$$\underbrace{|\phi^+\rangle \otimes |\phi^+\rangle \otimes |\phi^+\rangle \cdots\cdots\cdots |\phi^+\rangle}_{NS(\rho)}$$

$$E(\psi_{AB}) = S(\rho_A)$$

End of part II