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Tehran notes on decay and quantum Zeno effect (DRAFT)

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Abstract. We give an elementary introduction to nonexponential decay and the quantum Zeno effect. The introduction is addressed to students and researchers with no previous knowledge on the subject. The prerequisites are the Schrödinger equation and the notion of Von Neumann projective measurement.

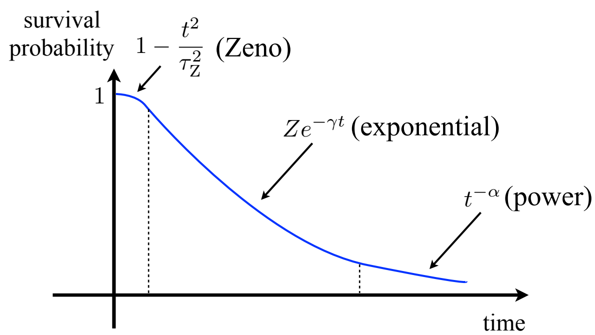

Fig. 1: Survival probability of a quantum decaying system.

1. Introduction

Fig. 1.

2. The quantum mechanical evolution

2.1. Evolution engendered by a Hermitian Hamiltonian

We start off by scrutinizing the quantum-mechanical evolution law, focusing on its short-time features. Let H be the Hamiltonian of a quantum system and $|\psi_0\rangle = |\psi(t=0)\rangle$ its initial state. We shall set henceforth $\hbar = 1$ and assume that all functions to be dealt with are sufficiently regular to admit series expansions. We shall focus on the "survival" amplitude $\mathcal{A}$ and probability p of the system in state $|\psi_0\rangle$ at time t:

$$\mathcal{A}(t) = \langle\psi_0|\psi(t)\rangle = \langle\psi_0|e^{-iHt}|\psi_0\rangle, \tag{2.1}$$
$$p(t) = |\mathcal{A}(t)|^2 = |\langle\psi_0|e^{-iHt}|\psi_0\rangle|^2. \tag{2.2}$$

Let the system evolve for a short time δt. The Schrödinger equation yields

$$\begin{aligned} |\psi(\delta t)\rangle = e^{-iH\delta t}|\psi_0\rangle &= |\psi_0\rangle - iH|\psi_0\rangle\delta t - \frac{1}{2}H^2|\psi_0\rangle(\delta t)^2 + \mathrm{O}((\delta t)^3) \\ &=: |\psi_0\rangle + |\delta\psi\rangle. \end{aligned} \tag{2.3}$$

The short-time expansion (2.3) yields

$$\mathcal{A}(\delta t) = 1 - i\langle H\rangle_0\delta t - \frac{1}{2}\langle H^2\rangle_0(\delta t)^2, \tag{2.4}$$
$$p(\delta t) = 1 - \frac{(\delta t)^2}{\tau_Z^2} + \mathrm{O}((\delta t)^4), \tag{2.5}$$

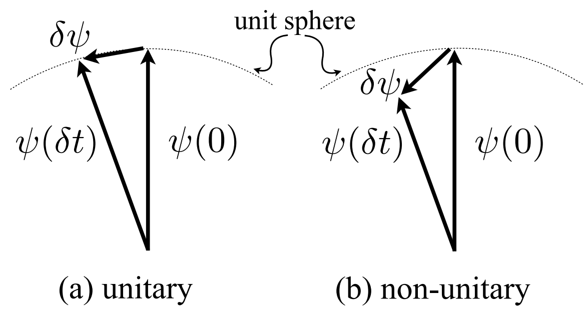

Fig. 2: (a) Unitary evolution engendered by a Hermitian Hamiltonian. The evolution takes place on the unit sphere: $||\psi(\delta t)|| = ||\psi(0)|| = 1$. (b) Non-unitary evolution engendered by a non-Hermitian Hamiltonian. The tip of the state vector can leave the unit sphere (and enter the unit ball): $||\psi(\delta t)|| \leq ||\psi(0)|| = 1$. In both cases, $\delta\psi$ is linear in δt.

where $\langle \cdots \rangle_0 := \langle \psi_0 | \cdots | \psi_0 \rangle$ and

$$\tau_{\rm Z}^{-2} := \langle H^2 \rangle_0 - \langle H \rangle_0^2, \tag{2.6}$$

$\tau_{\rm Z}$ being the the so-called Zeno time [44]. In deriving (2.5) from (2.4) the Hermitianity of H, ensuring the reality of $\langle H \rangle_0$, played a primary role. Notice that according to (2.4) the wave function evolves linearly away from the initial state, but the survival probability (of remaining in the initial state) evolves *quadratically* away from 1, due to (2.5). Recall that due to the unitarity of the evolution, wave functions are always normalized to unity: $||\psi(t)|| = ||\psi(0)|| = 1, \forall t$: the tip of the state vector never leaves the unit sphere. The features of the short time evolution are pictorially displayed in Fig. 2(a).

2.2. Evolution engendered by a non-Hermitian Hamiltonian

Let us add a non-Hermitian part to the Hamiltonian:

$$H' = H - iV, \tag{2.7}$$

where V is a real "optical" potential (taken to be position-independent for simplicity). The new survival amplitude and probability read

$$\mathcal{A}'(t) = \langle \psi_0 | \psi(t) \rangle = e^{-Vt} \langle \psi_0 | e^{-iHt} | \psi_0 \rangle, \tag{2.8}$$

$$p'(t) = e^{-2Vt} |\langle \psi_0 | e^{-iHt} | \psi_0 \rangle|^2. \tag{2.9}$$

A short-time expansion yields a *linear* behavior both for amplitude and probability

$$\mathcal{A}'(\delta t) = 1 - (V + i\langle H \rangle_0)\delta t - \frac{1}{2}(\langle H^2 \rangle_0 - V^2 - 2iV\langle H \rangle_0)(\delta t)^2 + \mathrm{O}((\delta t)^3),$$

$$p'(\delta t) \quad = \quad 1 - 2V\delta t + \mathrm{O}((\delta t)^2) \tag{2.11}$$

(2.10)

Optical potentials were frequently used in nuclear physics and quantum optics [REFS]. They "eat up" probability and describe decay channels. See Fig. 2(b). The tip of the state vector can leave the unit sphere and enter the unit ball: $||\psi(t)|| \leq ||\psi(0)|| = 1$. [It would leave the unit ball if the optical potential $-iV$ in (2.7) had the opposite sign.]

In physics, one tends to regards property (2.5) as more "fundamental", as it ensues from the Hermitianity of the Hamiltonian and the unitarity of the evolution, that are regarded as very general principles. Yet optical potentials have their own charm and play an important role in effective descriptions of decaying and dissipative systems. Nowadays they have been overcome by the rigorous mathematical framework of Gorini, Kossakowski, Sudarshan and Lindblad [19] that describes the physics of dissipative quantum systems [4, 5, 6].

2.3. Interaction Hamiltonian

Let the Hamiltonian be composed of a free and an interaction parts

$$H = H_0 + H_{\mathrm{int}}. \tag{2.12}$$

We also require that the initial state be an eigenstate of the free Hamiltonian and (as it is customary in quantum field theory) that the interaction be off-diagonal:

$$H_0|\psi_0\rangle = \omega_0|\psi_0\rangle, \qquad \langle H_{\mathrm{int}}\rangle_0 = 0. \tag{2.13}$$

In this interesting case the Zeno time reads

$$\tau_{\mathrm{Z}}^{-2} = \langle H_{\mathrm{int}}^2\rangle_0 = \sum_n \langle\psi_0|H_{\mathrm{int}}|\psi_n\rangle\langle\psi_n|H_{\mathrm{int}}|\psi_0\rangle \tag{2.14}$$

and depends only on the interaction Hamiltonian. In the last expression $|\psi_n\rangle$ are the eigenstates of the free Hamiltonian and form a complete set

$$H_0|\psi_n\rangle = \omega_n|\psi_n\rangle. \tag{2.15}$$

Formula (2.14) should be compared to the Fermi "Golden Rule" [7][1], yielding the inverse lifetime γ of a decaying quantum system:

$$\gamma = 2\pi \sum_f |\langle\psi_f|H_{\mathrm{int}}|\psi_0\rangle|^2\, \delta(\omega_f - \omega_0), \tag{2.16}$$

[1] Fermi considered (2.16) the *second* golden rule. If you are curious about the first one, see pages 136, 148 of *Nuclear Physics*.

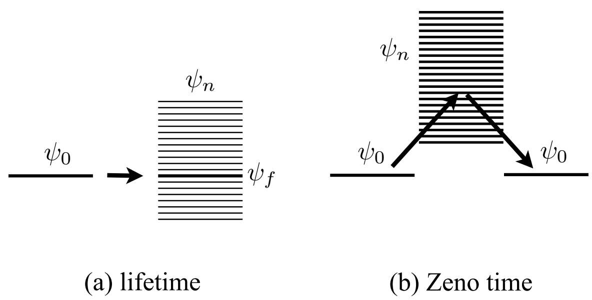

Fig. 3: (a) The lifetime γ in Eq. (2.16) contains only "on-shell" contributions: the delta function entails energy conservation $\omega_f = \omega_0$; ψ_f is in general (very) degenerate. (b) The Zeno time τ_Z in Eq. (2.14) explores the whole Hilbert space.

where the summation (integral) is over the final states and the continuum limit is implied.

One comment. While (2.16) contains only "on-shell" contributions (because the delta function ensures energy conservation), the expression (2.14) explores the *whole* Hilbert space. See Fig. 3. This is, I believe, the most remarkable difference between the lifetime and the Zeno time.

3. Quantum Zeno effect with Von Neumann measurements

The most familiar formulation of the QZE makes use of Von Neumann measurements, represented by one-dimensional projectors. Perform N measurements at time intervals $\tau = t/N$, in order to check whether the system is still in its initial state $|\psi_0\rangle$. After each measurement the system's state is "projected" back onto its initial state $|\psi_0\rangle$ and the evolution starts anew according to Schrödinger's equation with initial condition $|\psi_0\rangle$. [The system can also be projected onto an orthogonal state $|\psi_0^\perp\rangle$, with (quadratic) probability $1 - p(\tau) = \tau^2/\tau_Z^2$, according to Eq. (2.5). As we shall see, such an event becomes increasingly unlikely as N increases.]

The survival probability $p^{(N)}(t)$ at the final time $t = N\tau$ reads

$$\begin{aligned} p^{(N)}(t) &= p(\tau)^N = p(t/N)^N \\ &\simeq \left[1 - (t/N\tau_Z)^2\right]^N \stackrel{N \text{ large}}{\longrightarrow} \exp(-t^2/N\tau_Z^2) \stackrel{N\to\infty}{\longrightarrow} 1, \end{aligned} \tag{3.17}$$

where we made use of the property (2.5). For large N the quantum mechanical evolution is slowed down and in the $N \to \infty$ limit (infinitely frequent measurements) it is halted, so that the state of the system is "frozen" in its initial state. This is the QZE. It is a consequence of the short-time behavior (2.5).

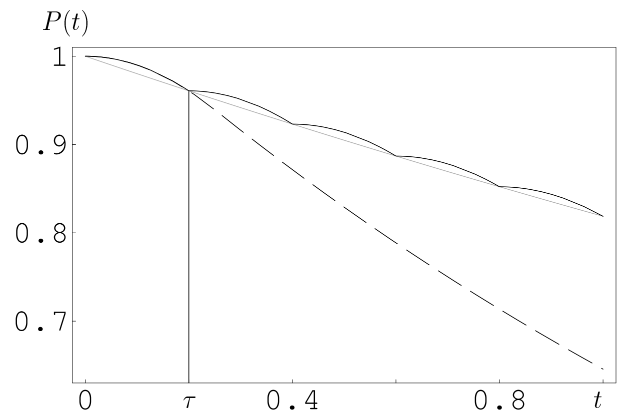

Fig. 4: Quantum Zeno effect for $N = 5$ "pulsed" Von Neumann measurements. The dashed (full) line is the survival probability without (with) measurements. The gray line is the interpolating exponential (3.18). As N increases, $p^{(N)}(t) \to 1$ uniformly in $[0, t]$. The units on the abscissae are arbitrarily chosen for illustrative purposes.

Observe that the survival probability after N pulsed measurements ($t = N\tau$) is interpolated by an exponential law [37]

$$p^{(N)}(t) = p(\tau)^N = \exp(N \log p(\tau)) = \exp(-\gamma_{\rm eff}(\tau)t), \tag{3.18}$$

with an effective decay rate

$$\gamma_{\rm eff}(\tau) := -\frac{1}{\tau} \log p(\tau). \tag{3.19}$$

For $\tau \to 0$ ($N \to \infty$) one gets from (2.5) $p(\tau) \simeq \exp(-\tau^2/\tau_{\rm Z}^2)$, so that

$$\gamma_{\rm eff}(\tau) \simeq \tau/\tau_{\rm Z}^2, \qquad \tau \to 0. \tag{3.20}$$

The Zeno evolution for "pulsed" Von Neumann measurements is pictorially represented in Figure 4.

4. The simplest example: two-level system

Consider a two-level system undergoing Rabi oscillations. This is the simplest nontrivial quantum mechanical example, for it involves 2×2 matrices and very simple algebra. One can think of an atom shined by a laser field whose frequency resonates with one of the atomic transitions, or a neutron spin in a magnetic field. The (interaction) Hamiltonian reads

$$H = H_{\rm int} = \Omega\sigma_1 = \Omega(|+\rangle\langle-| + |-\rangle\langle+|) = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}, \tag{4.1}$$

where Ω is a real number, σ_j ($j = 1, 2, 3$) the Pauli matrices and

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.2}$$

are eigenstates of σ_3. We are neglecting the energy difference between the two states $|\pm\rangle$. Let the initial state be

$$|\psi_0\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.3}$$

so that the evolution yields

$$|\psi(t)\rangle = e^{-iH_{\rm int}t}|\psi_0\rangle = \cos(\Omega t)|+\rangle - i\sin(\Omega t)|-\rangle = \begin{pmatrix} \cos\Omega t \\ -i\sin\Omega t \end{pmatrix}. \tag{4.4}$$

The survival amplitude (2.1) and probability (2.2) and the Zeno time (2.6) or (2.14) read

$$\begin{aligned} \mathcal{A}(t) &= \cos\Omega t, & (4.5)\\ P(t) &= \cos^2\Omega t, & (4.6)\\ \tau_{\rm Z} &= \Omega^{-1}, & (4.7) \end{aligned}$$

respectively. The effective decay rate (3.19) if N measurements are performed in time t reads

$$\gamma_{\rm eff}(\tau) = \tau\Omega^2. \tag{4.8}$$

In this simple case, the approximation (3.20) is exact. See Figure 4.

5. Comments.

The QZE is ascribable to the following mathematical properties of the Schrödinger equation: in a short time $\delta\tau \sim 1/N$, the phase of the wave function evolves like $\mathrm{O}(\delta\tau)$, while the probability changes by $\mathrm{O}(\delta\tau^2)$, so that

$$P^{(N)}(t) \simeq \left[1 - \mathrm{O}(1/N^2)\right]^N \stackrel{N\to\infty}{\longrightarrow} 1. \tag{5.1}$$

Stated differently, the projection onto the inital state evolves "slowly" away from unity. This is sketched in Fig. 5 and is a very general feature of the Schrödinger equation, as well as of other "fundamental" evolution equations in physics. Equations that do not have this feature (e.g. dissipative equations) tend to be regarded as less fundamental, the consequence of approximations of some sort.

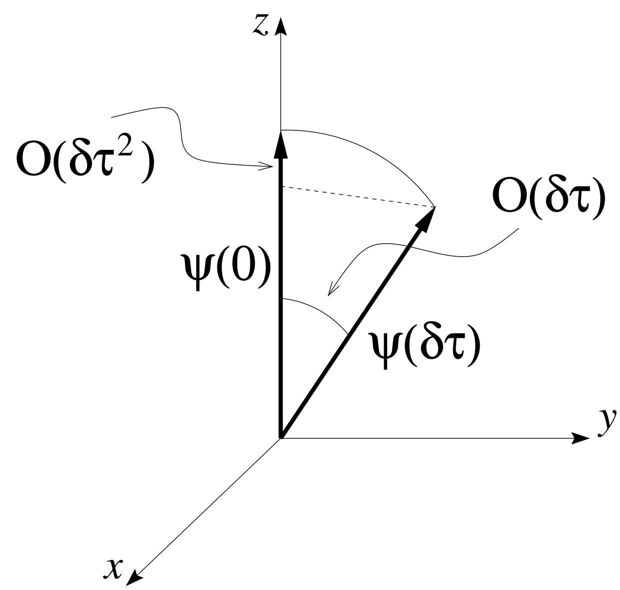

Fig. 5: Short-time evolution of phase and probability: $\delta\tau \sim 1/N$.

6. Unraveling a Von Neumann measurement

Von Neumann measurements
effective description of a measurement process.
external apparatus
mistery still there.

6.1. Mimicking the projection with a non-Hermitian Hamiltonian

Let us show that the action of a measuring apparatus (performing the Von Neumann measurement) can be mimicked by a non-Hermitian Hamiltonian. Consider the Hamiltonian (notation as in Sec. 4.)

$$H_{\rm int} = \begin{pmatrix} 0 & \Omega \\ \Omega & -i2V \end{pmatrix} = -iV\mathbf{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}, \qquad \boldsymbol{h} = (\Omega, 0, iV)^T, \tag{6.2}$$

that yields Rabi oscillations of frequency Ω, but at the same time absorbs away the $|-\rangle$ component of the state vector, performing in this way a "measurement." Due to the non-Hermitian features of this description, probabilities are not conserved: we are concentrating our attention only on the $|+\rangle$ component.

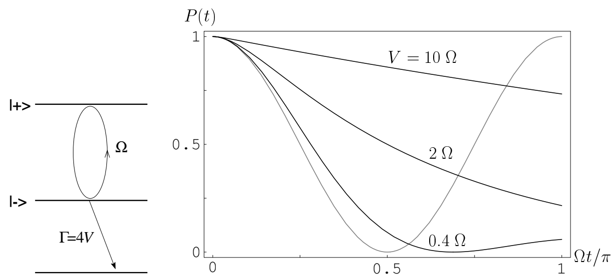

Fig. 6: Survival probability for a system undergoing Rabi oscillations in presence of absorption ($V = 0.4, 2, 10\Omega$). The gray line is the undisturbed evolution ($V = 0$).

An elementary SU(2) manipulation yields the following evolution operator

$$e^{-iH_{\rm int}t} = e^{-Vt}\left[\cosh(ht) - i\frac{\boldsymbol{h}\cdot\boldsymbol{\sigma}}{h}\sinh(ht)\right], \tag{6.3}$$

where $h = \sqrt{V^2 - \Omega^2}$ and we supposed $V > \Omega$. The survival amplitude in the initial state (4.3) reads

$$\begin{aligned}\mathcal{A}(t) &= \langle\psi_0|e^{-iH_{\rm I}t}|\psi_0\rangle \\ &= e^{-Vt}\left[\cosh(\sqrt{V^2-\Omega^2}t) + \frac{V}{\sqrt{V^2-\Omega^2}}\sinh(\sqrt{V^2-\Omega^2}t)\right] \\ &= \frac{1}{2}\left(1+\frac{V}{\sqrt{V^2-\Omega^2}}\right)e^{-(V-\sqrt{V^2-\Omega^2})t} \\ &\quad +\frac{1}{2}\left(1-\frac{V}{\sqrt{V^2-\Omega^2}}\right)e^{-(V+\sqrt{V^2-\Omega^2})t}.\end{aligned} \tag{6.4}$$

Notice the presence of a slow and a fast decay. The survival probability $P(t) = |\mathcal{A}(t)|^2$ is shown in fig. 6 for $V = 0.4, 2, 10\Omega$.

As expected, probability is (exponentially) absorbed away as $t \to \infty$. Moreover, as V increases, the survival probability reads

$$P(t) \simeq \left(1+\frac{\Omega^2}{2V}\right)\exp\left(-\frac{\Omega^2}{V}t\right), \tag{6.5}$$

the above expansion becoming valid very quickly, on a time scale of order V^{-1}. The effective decay rate $\gamma_{\rm eff}(V) = \Omega^2/V$ becomes smaller, eventually halting the "decay" (absorption) of the initial state in the $V \to \infty$ limit. This yields an interesting example of QZE: a larger V entails a more "effective" measurement of the initial state.

The global process described here can be viewed as "continuous" (negative result) measurements performed on the initial state $|+\rangle$. State $|-\rangle$ is continuously monitored with a response time $1/V$: as soon as it becomes populated, it is detected within a time $1/V$. The "strength" V of the observation can be compared to the frequency $\tau^{-1} = (t/N)^{-1}$ of measurements in the "pulsed" formulation. Indeed, for large values of V one gets from Eq. (6.5)

$$\gamma_{\rm eff}(V) = \frac{\Omega^2}{V} = \frac{1}{\tau_{\rm Z}^2 V}, \tag{6.6}$$

which, compared with Eq. (3.20), yields an interesting relation between continuous and pulsed measurements [14]

$$V \simeq 1/\tau. \tag{6.7}$$

6.2. Interaction with an external field yields a non-Hermitian Hamiltonian

We now show that the non-Hermitian Hamiltonian (6.2) can be obtained by considering the evolution engendered by a Hermitian Hamiltonian acting on a larger Hilbert space and then restricting the attention to the subspace spanned by $\{|+\rangle, |-\rangle\}$. Let

$$H = \Omega(|+\rangle\langle-| + |-\rangle\langle+|) + \int d\omega\ \omega|\omega\rangle\langle\omega| + \sqrt{\frac{\Gamma}{2\pi}} \int d\omega\ (|-\rangle\langle\omega| + |\omega\rangle\langle-|), \tag{6.8}$$

which describes a two level system coupled to a one-dimensional "photon" field in the rotating-wave approximation. Notice that the coupling is "flat": the two level system couples to all frequencies in the same way. The state of the system at time t can be written as

$$|\psi(t)\rangle = x(t)|+\rangle + y(t)|-\rangle + \int d\omega\ z(\omega,t)|\omega\rangle \tag{6.9}$$

and the Schrödinger equation reads

$$\begin{aligned} i\dot{x}(t) &= \Omega y(t), \\ i\dot{y}(t) &= \Omega x(t) + \sqrt{\frac{\Gamma}{2\pi}} \int d\omega\ z(\omega,t), \\ i\dot{z}(\omega,t) &= \omega z(\omega,t) + \sqrt{\frac{\Gamma}{2\pi}} y(t). \end{aligned} \tag{6.10}$$

By using the initial condition $x(0) = 1$ and $y(0) = z(\omega,0) = 0$ one obtains

$$z(\omega,t) = -i\sqrt{\frac{\Gamma}{2\pi}} \int_0^t d\tau\ e^{-i\omega(t-\tau)} y(\tau) \tag{6.11}$$

and

$$i\dot{y}(t) = \Omega x(t) - i\frac{\Gamma}{2\pi}\int d\omega \int_0^t d\tau\ e^{-i\omega(t-\tau)}y(\tau) = \Omega x(t) - i\frac{\Gamma}{2}y(t). \qquad (6.12)$$

Therefore $z(\omega,t)$ disappears from the equations and we get two first order differential equation for x and y. The only effect of the continuum is the appearance of the imaginary frequency $-i\Gamma/2$. This is ascribable to the "flatness" of the continuum [there is no form factor or frequency cutoff in the interaction term of eq. (6.8)], which yields a purely exponential (Markovian) decay of $y(t)$.

In conclusion, the (reduced) dynamics in the subspace spanned by $|+\rangle$ and $|-\rangle$ reads

$$\begin{aligned} i\dot{x}(t) &= \Omega y(t), \\ i\dot{y}(t) &= -i\frac{\Gamma}{2}y + \Omega x(t). \end{aligned} \qquad (6.13)$$

Of course, this dynamics is not unitary, for probability flows out of the subspace, and is generated by the non-Hermitian Hamiltonian

$$H = \Omega(|+\rangle\langle -| + |-\rangle\langle +|) - i\frac{\Gamma}{2}|-\rangle\langle -| = \begin{pmatrix} 0 & \Omega \\ \Omega & -i\Gamma/2 \end{pmatrix} \qquad (6.14)$$

This Hamiltonian is the same as (6.2) when one sets $\Gamma = 4V$. QZE is obtained by increasing Γ: a larger coupling to the environment leads to a more effective "continuous" observation on the system (quicker response of the measuring apparatus), and as a consequence to a slower decay (QZE).

7. Genuine unstable systems and Zeno effects

We shall now discuss the primary role played by the form factors of the interaction by making use of a quantum field theoretical framework. We start by generalizing the two-level Hamiltonian (4.1) to N states $|j\rangle$ ($j = 1,\dots,N$) with different energies

$$H_0 = \omega_0|+\rangle\langle +| + \sum_{j=1}^{N}\omega_j|j\rangle\langle j| = \begin{pmatrix} \omega_0 & 0 & \dots & 0 \\ 0 & \omega_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \omega_N \end{pmatrix}. \qquad (7.1)$$

and coupling

$$H_{\rm int} = \sum_{j=1}^{N} g_j(|+\rangle\langle j| + |j\rangle\langle +|) = \begin{pmatrix} 0 & g_1 & \dots & g_N \\ g_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_N & 0 & \dots & 0 \end{pmatrix} \qquad (7.2)$$

In order to obtain a truly unstable system we need a continuous spectrum, so we consider the continuum limit

$$H = H_0 + H_{\rm int} = \omega_0 |+\rangle\langle +| + \int d\omega\ \omega |\omega\rangle\langle\omega| + \int d\omega\ g(\omega)(|+\rangle\langle\omega| + |\omega\rangle\langle +|). \tag{7.3}$$

The transition to a quantum field theoretical framework is an important component of our analysis, as we shall see. As before, we take as initial state $|\psi_0\rangle = |+\rangle$. The interaction of this normalizable state with the continuum of states $|\omega\rangle$ is responsible for its decay and depends on the *form factor* $g(\omega)$. We reobtain the physics of two-level systems in the limit $g^2(\omega) = \Omega^2\delta(\omega)$.

The Fourier-Laplace transform of the survival amplitude for this model can be given a convenient analytic expression: notice that the transform of the survival amplitude is the expectation value of the resolvent

$$\mathcal{A}(E) = \int_0^\infty dt\ e^{iEt}\mathcal{A}(t) = \langle +| \int_0^\infty dt\ e^{iEt} e^{-iHt} |+\rangle = \langle +| \frac{i}{E-H} |+\rangle \tag{7.4}$$

and is defined for $\mathrm{Im}E > 0$. By using twice the operator identity

$$\frac{1}{E-H} = \frac{1}{E-H_0} + \frac{1}{E-H_0} H_{\rm int} \frac{1}{E-H} \tag{7.5}$$

one obtains

$$\begin{aligned} \mathcal{A}(E) &= \langle +| \left[\frac{i}{E-H_0} + \frac{1}{E-H_0} H_{\rm int} \frac{i}{E-H_0} + \frac{1}{E-H_0} H_{\rm int} \frac{1}{E-H_0} H_{\rm int} \frac{i}{E-H} \right] |+\rangle \\ &= \frac{i}{E-\omega_0} + \frac{1}{E-\omega_0} \int d\omega\ \frac{|\langle +|H_{\rm int}|\omega\rangle|^2}{E-\omega}\ \mathcal{A}(E). \end{aligned} \tag{7.6}$$

In the above derivation we used the fact that $H_{\rm int}$ is completely off-diagonal in the eigenbasis of H_0, $\{|+\rangle, |\omega\rangle\}$, which is a resolution of the identity

$$|+\rangle\langle +| + \int d\omega\ |\omega\rangle\langle\omega| = 1. \tag{7.7}$$

The algebraic equation (7.6) can be solved and gives

$$\mathcal{A}(E) = \frac{i}{E - \omega_0 - \Sigma(E)}, \tag{7.8}$$

where the self-energy function $\Sigma(E)$ is related to the form factor $g(\omega)$ by a simple integration

$$\Sigma(E) = \int d\omega\ \frac{|\langle +|H_{\rm int}|\omega\rangle|^2}{E-\omega} = \int d\omega\ \frac{g^2(\omega)}{E-\omega}. \tag{7.9}$$

By inverting eq. (7.4) we finally get

$$\mathcal{A}(t) = \int_{\mathrm{B}} \frac{dE}{2\pi}\, e^{-iEt} \mathcal{A}(E) = \frac{i}{2\pi} \int_{\mathrm{B}} dE\, \frac{e^{-iEt}}{E - \omega_0 - \Sigma(E)}, \tag{7.10}$$

the Bromwich path B being a horizontal line $\mathrm{Im}E$ =constant> 0 in the half plane of convergence of the Fourier-Laplace transform (upper half plane).

We consider now the case of an unstable system. The initial state has energy $\omega_0 > \omega_g$ (ω_g being the lower bound of the continuous spectrum of the Hamiltonian H) and is therefore embedded in the continuous spectrum of H. If $-\Sigma(\omega_g) < \omega_0$ (which happens for sufficiently smooth form factors and small coupling), the resolvent is analytic in the whole complex plane cut along the real axis (continuous spectrum of H) [57, 13]. On the other hand, there exists a pole E_{pole} located just below the branch cut in the second Riemann sheet, solution of the equation

$$E_{\mathrm{pole}} - \omega_0 - \Sigma_{\mathrm{II}}(E_{\mathrm{pole}}) = 0, \tag{7.11}$$

Σ_{II} being the determination of the self-energy function in the second sheet. The pole has a real and imaginary part

$$E_{\mathrm{pole}} = \omega_0 + \delta\omega_0 - i\gamma/2 \tag{7.12}$$

given by

$$\begin{aligned} \delta\omega_0 &= \mathrm{Re}\,\Sigma_{\mathrm{II}}(E_{\mathrm{pole}}) \simeq \mathrm{Re}\,\Sigma(\omega_0 + i0^+) = \mathrm{P}\!\int d\omega \frac{g^2(\omega)}{\omega_0 - \omega}, && (7.13) \\ \gamma &= -2\,\mathrm{Im}\,\Sigma_{\mathrm{II}}(E_{\mathrm{pole}}) \simeq -2\,\mathrm{Im}\,\Sigma(\omega_0 + i0^+) = 2\pi g^2(\omega_0), && (7.14) \end{aligned}$$

up to fourth order in the coupling constant. One recognizes the second-order energy shift $\delta\omega_0$ and the celebrated Fermi “golden” rule γ [45]. The survival amplitude has the general form

$$\mathcal{A}(t) = \mathcal{A}_{\mathrm{pole}}(t) + \mathcal{A}_{\mathrm{cut}}(t), \tag{7.15}$$

where

$$\mathcal{A}_{\mathrm{pole}}(t) = \frac{e^{-i(\omega_0+\delta\omega_0)t-\gamma t/2}}{1 - \Sigma'_{\mathrm{II}}(E_{\mathrm{pole}})}, \tag{7.16}$$

and $\mathcal{A}_{\mathrm{cut}}$ is the branch-cut contribution.

$$\mathcal{A}_{\mathrm{cut}}(t) = \frac{i}{2\pi} \int_{\mathrm{cut}} dE\, \frac{e^{-iEt}}{E - \omega_0 - \Sigma(E)}, \tag{7.17}$$

At intermediate times, the pole contribution dominates the evolution and

$$P(t) \simeq |\mathcal{A}_{\mathrm{pole}}(t)|^2 = \mathcal{Z} e^{-\gamma t}, \qquad \mathcal{Z} = \left|1 - \Sigma'_{\mathrm{II}}(E_{\mathrm{pole}})\right|^{-2}, \tag{7.18}$$

where $\mathcal{Z}$, the intersection of the asymptotic exponential with the $t = 0$ axis, is the wave function renormalization.

Notice that, in order to obtain a purely exponential decay, one neglects all branch cut and/or other contributions from distant poles and considers only the contribution of the dominant pole. In other words, one does not look at the rich analytical structure of the propagator and retains only its dominant polar singularity. In this case the self-energy function becomes a constant (equal to its value at the pole), namely

$$\Sigma(E) \longrightarrow \Sigma^{\mathrm{WW}}(E) = \frac{1}{E - \omega_0 - \Sigma_{\mathrm{II}}(E_{\mathrm{pole}})} = \frac{1}{E - E_{\mathrm{pole}}}, \tag{7.19}$$

where in the last equality we used the pole equation (7.11). This is the celebrated Weisskopf-Wigner approximation [8] and yields a purely exponential behavior, $\mathcal{A}(t) = \exp(-iE_{\mathrm{pole}}t)$, without short- and long-time corrections.

8. Conclusions.

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