Information gain and approximate reversibility of quantum measurements

[see arXive:quant-ph/0702166v3]

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Overview

* Definition of information gain
* Definition of disturbance
* Balance of information
* Tradeoff for general measurements
* (Single-outcome analysis)
* (Relation with previous proposals: Grönewold-Lindblad-Ozawa, Maccone)

The setting

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Let the measurement on $\mathscr{H}^Q$
be described by the POVM $\mathbf{P}^Q := \{P_m^Q\}_{m\in\mathcal{X}}$

Information gain

The measurement on Q determines an **ensemble decomposition on R**: with probability $p(m) := \mathrm{Tr}[\rho^Q \ P_m^Q]$ we observe on R the conditional state $\rho_m^{R'} := \mathrm{Tr}_Q[\Psi^{RQ} \ (\mathbb{1}^R \otimes P_m^Q)]$.

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$$= S(\rho^R) - \sum_m p(m) S(\rho_m^{R'})$$

Why such a choice?

It is a natural choice for many reasons. In particular:

- it depends only on the input state and the POVM
- it is by construction positive definite
- it is a natural upper bound to the classical information gain, defined as the (classical) mutual information $I(X:\mathcal{X})$ between the measurement outcomes $\mathcal{X}$ and the alphabet X which is eventually encoded in the input state as

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...exactly as Holevo quantity is considered a natural upper bound to the accessible information.

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In order to analyze the disturbance, the description of the measurement by means of the POVM only is no more sufficient. We have to introduce a **state reduction recipe**, which takes into account the whole statistical description of a quantum measurement, that is, its outcome probability distribution (POVM) as well as its dynamics (state reduction).

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We are led to the notion of **quantum instrument**.

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* the probability of obtaining the m-th outcome is $p(m) := \mathrm{Tr}[\mathcal{E}_m(\rho^Q)]$
* the "a posteriori" state, given the m-th outcome, is $\rho_m^{Q'} := \mathcal{E}_m(\rho^Q)/p(m)$

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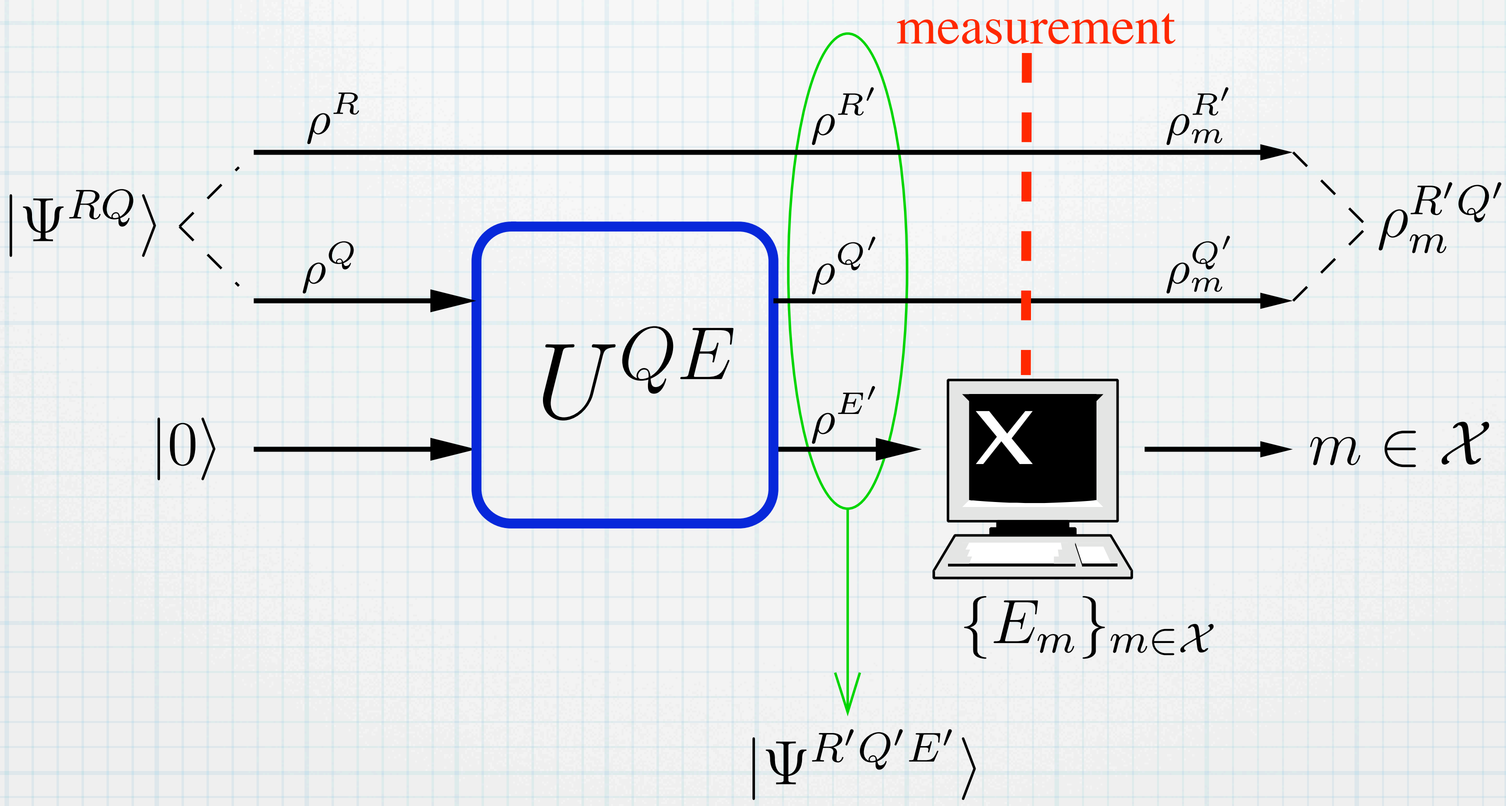

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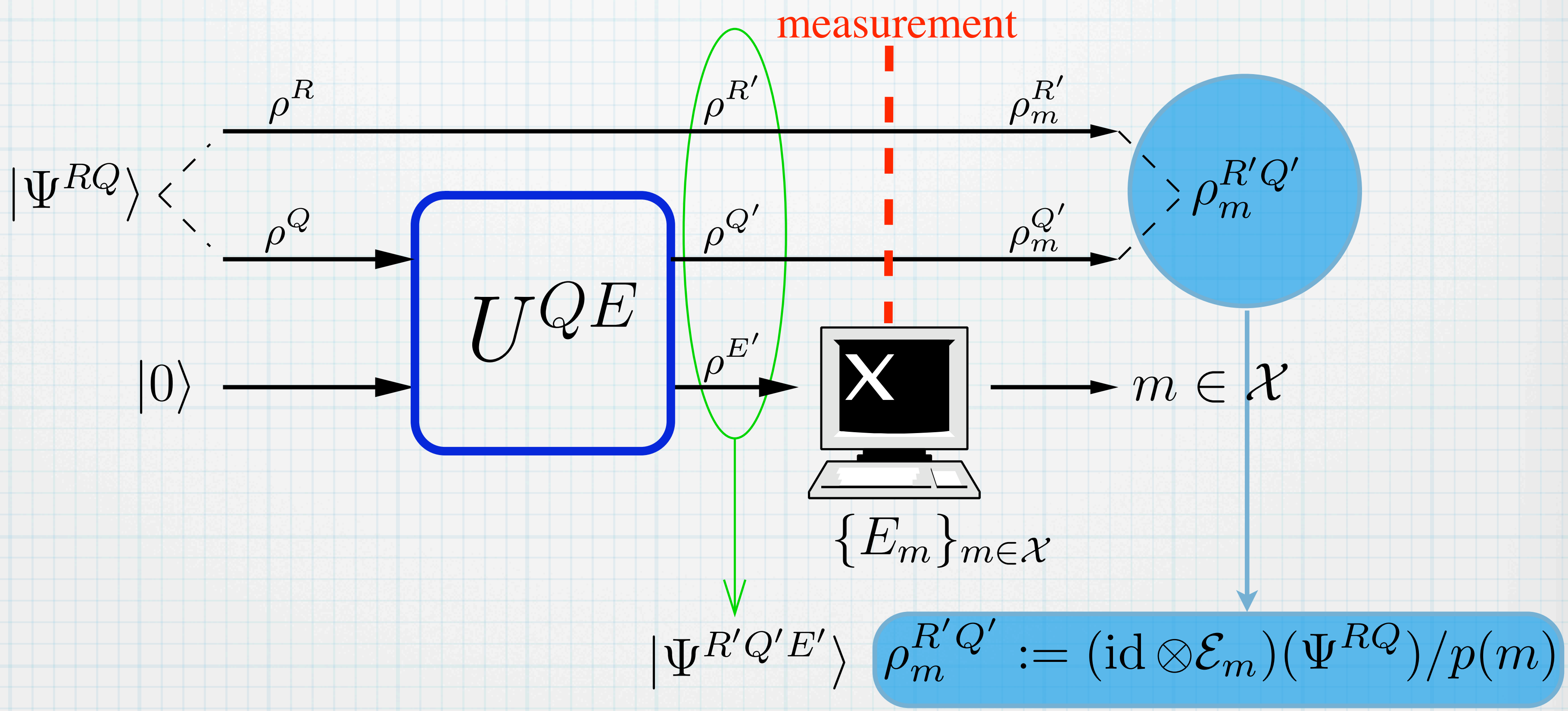

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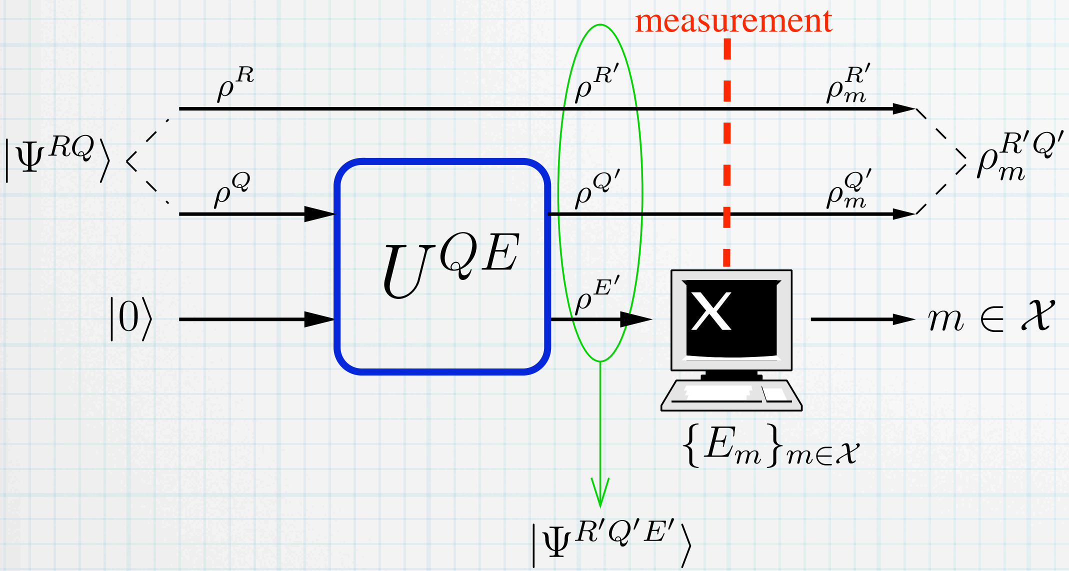

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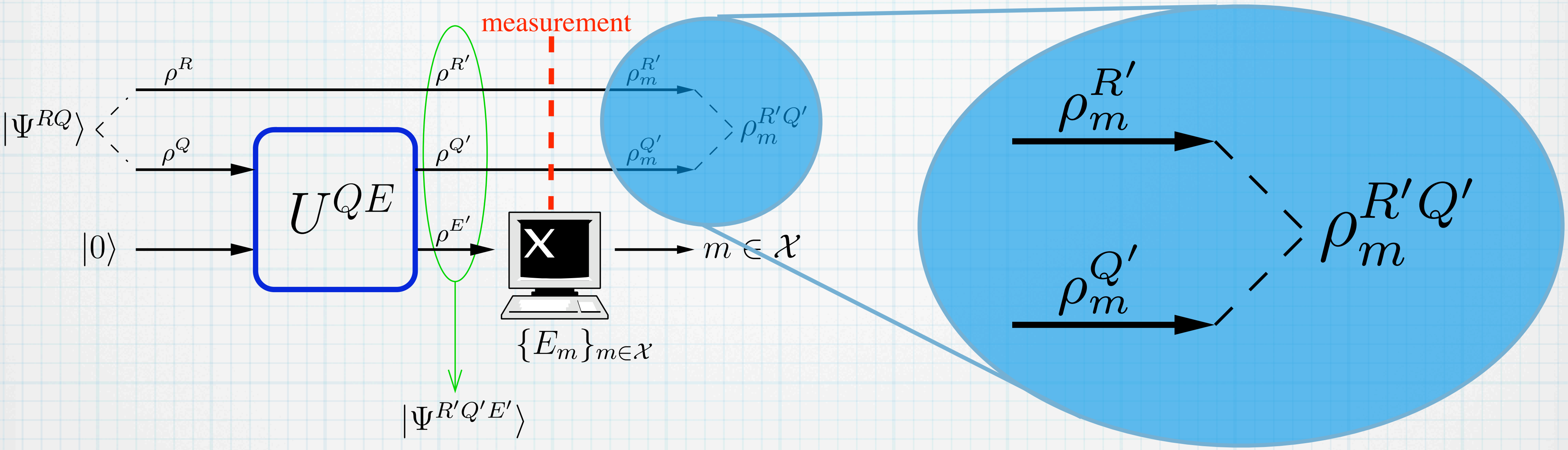

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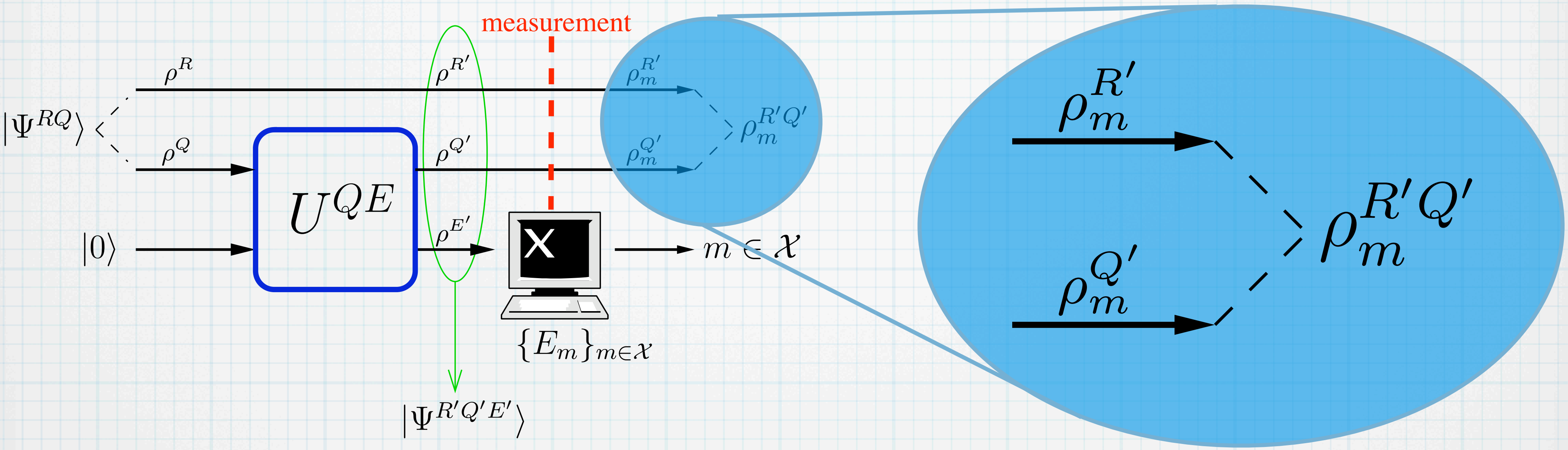

Given the input state and the instrument, we define the **disturbance** as the **conditional coherent information loss**

$$\delta(\rho^Q, \mathscr{I}^Q) := S(\rho^Q) - \sum_m p(m) I_c^{R'\to Q'}(\rho_m^{R'Q'})$$

where $I_c^{A\to B}(\sigma^{AB}) := S(\sigma^B) - S(\sigma^{AB})$

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$$\delta(\rho^Q, \mathscr{I}^Q) \leq \mathrm{h}\left(1 - F^e(\rho^Q, \sum_m \mathcal{R}_m \circ \mathcal{E}_m)\right)$$

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The global reference+system+apparatus state after the measurement can be written w.l.o.g. as follows

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which in turn implies that $\rho_m^{R'Q'}$ is pure, that means, reference+system, given the m-th outcome, is factorized from the rest of the universe

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In other words, the disturbance equals the **total correlations between the reference and the apparatus**:

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Usually, apparatus internal degrees of freedom are out of our control, and the information gain is strictly less than the disturbance introduced.

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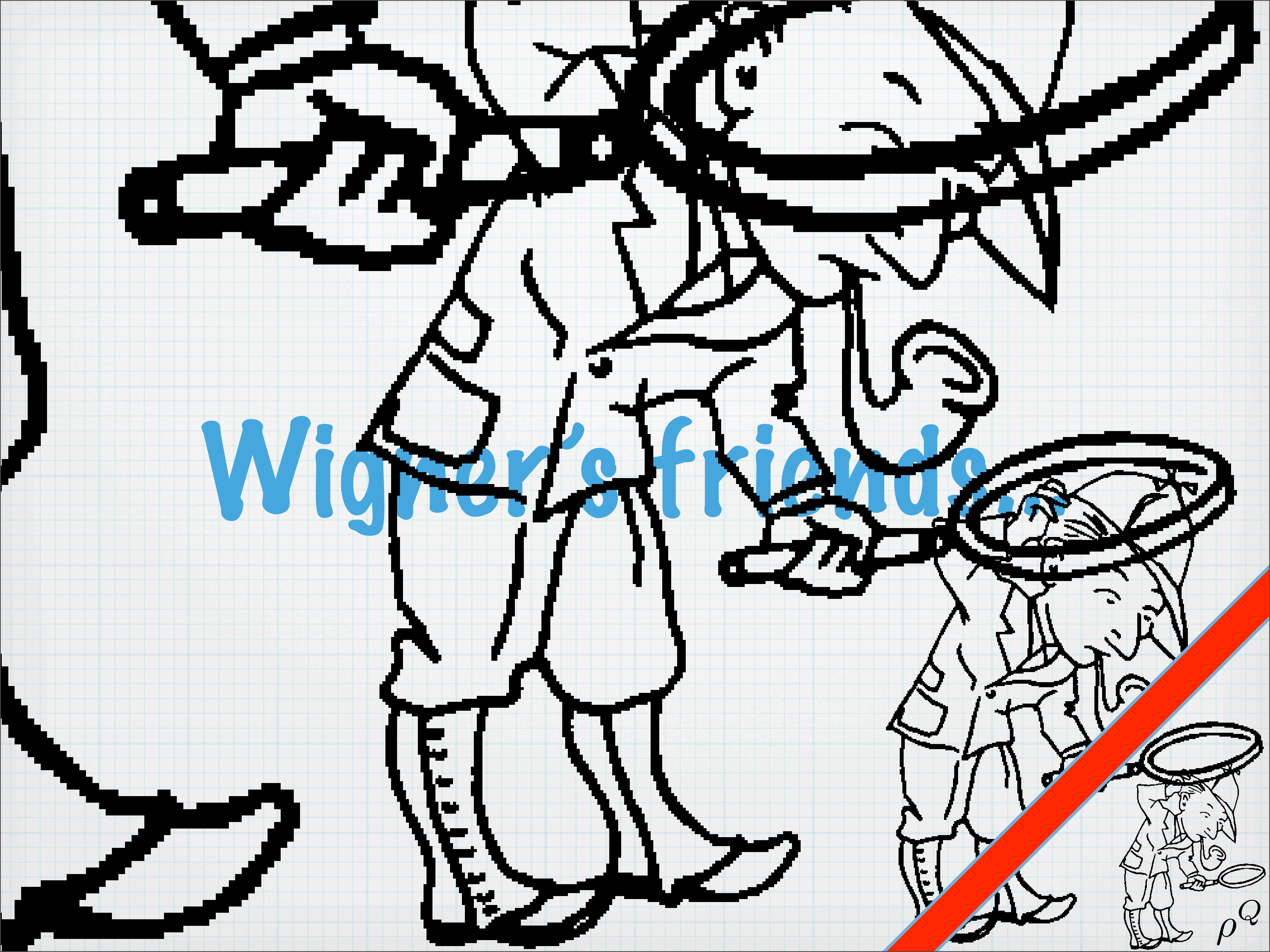
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