Multipartite entangled states, orthogonal arrays & Hadamard matrices

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Maximally entangled pure quantum states

Bipartite systems $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}_d \otimes \mathcal{H}_d$

Example: **generalized** **Bell** states for 2 qu*d*its:

$$|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \tag{1}$$

distinguished by the fact that reduced states are **maximally mixed**,
e.g. $\rho_A = Tr_B|\psi_+\rangle\langle\psi_+| = \mathbb{1}_d$.
This property holds for all locally equivalent states, $(U_A \otimes U_B)|\psi_+\rangle$.

Three qubits, $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}_2^{\otimes 3}$

GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ has a similar property: all three one-partite reductions are **maximally mixed**,
$\rho_A = Tr_{BC}|GHZ\rangle\langle GHZ| = \mathbb{1}_2 = \rho_B = Tr_{AC}|GHZ\rangle\langle GHZ|$.
(what is **not** the case e.g. for $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$

Genuinely multipartite entangled states

k-uniform states of N qu$d$its

Definition. State $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ is called ***k*-uniform**
if for all possible splittings of the system into k and $N-k$ parts the reduced states are maximally mixed (**Scott 2001**),
(also called **MM**-states (maximally multipartite entangled)
Facchi et al. (2008,2010), **Arnaud & Cerf** (2012)

Applications: quantum error correction codes, ...

Example: 1–uniform states of N qu$d$its

Observation. A generalized, N–qu$d$it **GHZ** state,

$$|GHZ_N^d\rangle := \frac{1}{\sqrt{d}}\left[|1,1,...,1\rangle + |2,2,....,2\rangle + \cdots + |d,d,...,d\rangle\right]$$

is **1–uniform** (but not 2–uniform!)

Examples of k–uniform states

Observation: k–uniform states may exist if $N \geq 2k$ (**Scott 2001**) (traced out ancilla of size $(N-k)$ cannot be smaller than the principal k–partite system).

Hence there are no 2-uniform states of 3 **qubits**.

However, there exist no 2-uniform state of 4 qubits either!

Higuchi & Sudbery (2000) - **frustration** like in spin systems –
Facchi, Florio, Marzolino, Parisi, Pascazio (2010) –
it is not possible to satisfy simultaneously so many constraints...

2-uniform state of 5 and 6 qubits

$$|\Phi_5\rangle = |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle + \\ + |10000\rangle + |00101\rangle - |00011\rangle - |10110\rangle,$$

related to 5–qubit error correction code by **Laflamme et al.** (1996)

$$|\Phi_6\rangle = |111111\rangle + |101010\rangle + |001100\rangle + |011001\rangle + \\ + |110000\rangle + |100101\rangle + |000011\rangle + |010110\rangle.$$

The goal of this project is to:

1. Construct 2–**uniform** states of N qubits,
2. Discuss the question:
   For what N and k the k–**uniform** states of N qubits do exist,
3. Analyze a more general problem of k–**uniform** states of N **qudits**,
4. Show links to the problems of Mutually unbiased bases (MUB), and quantum error correction codes
5. Analyze properties of typical pure states of N qudits for large dimensions - are they **approximately** k-**uniform?**

Hadamard matrices (real)

Definition

matrix of order N with mutually orthogonal row and columns,

$$HH^* = N\mathbb{1} \ , \qquad H_{ij} = \pm 1. \tag{2}$$

given by

Hadamard matrices (real)

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given by **Sylvester** (1867)

The simplest example: one qubit, $N = 2$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{3}$$

m–qubit case, $N = 2^m$

$$H_{2^m} = H_2^{\otimes m}, \ . \tag{4}$$

works e.g. for $N = 2, 4, 8, 16, 32, ...$
Furthermore, there exist such matrices for $N = 12, 20, 24, 28, 36, ...$

Hadamard matrices II

Hadamard conjecture

Hadamard matrices do exist for $N = 2$ and $N = 4n$ for any $n = 1, 2, \ldots$

After a discovery of $N = 428$ **Hadamard matrix**
(**Hadi Kharaghani** and **Tayfeh-Razaie**, 2005)
this conjecture is known to hold up to $N = 664$

see: Catalogue of Hadamard matrices of **Sloane**
http://neilsloane.com/hadamard

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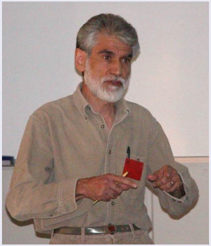

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Great challenge in combinatorics

Prove the **Hadamard conjecture**:

Construct Hadamard matrices for every $N = 4n$! !

Equivalent Hadamard matrices

$$H_1 \sim H_2$$

iff there exist permutation matrices P_1 and P_2 and diagonal sign matrices D_1 and D_2 containing ± 1 such that

$$H_1 = D_1 P_1 \; H_2 \; P_2 D_2 \; . \qquad (5)$$

$N \leq 12$

For $N = 2, 4, 8, 12$ all (real) Hadamard matrices are equivalent

higher dimensions

The number E of **equivalence classes** of real Hadamard matrices of order n reads

$n =$	2	4	8	12	16	20	24	28	32
$E =$	1	1	1	1	5	3	60	487	13 710 027

Fang and Ge 2004, **Orrick** 2005, **Tayfeh-Razaie** 2014.

Orthogonal Arrays

Combinatorial arrangements introduced by **Rao** in 1946 used in statistics and design of experiments, OA(r, N, d, k)

0	0
1	1

0	0	0
0	1	1
1	0	1
1	1	0

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Orthogonal arrays OA(2,2,2,1), OA(4,3,2,2) and OA(8,4,2,3).

Definition of an **Orthogonal Array**

An array A of size $r \times N$ with entries taken from a d–element set S is called **Orthogonal array** $\mathrm{OA}(r, N, d, k)$ with r runs, N factors, d levels, strength k and index λ **if** every $r \times k$ subarray of A contains each $k-$tuple of symbols from S exactly λ times as a row.

Each OA is determined by 4 independent parameters r, N, d, k satisfying **Rao bounds**

$$r \geq \sum_{i=0}^{k/2} \binom{N}{i} (d-1)^i \quad \text{if } k \text{ is even,} \tag{6}$$

$$r \geq \sum_{i=0}^{\frac{k-1}{2}} \binom{N}{i} (d-1)^i + \binom{N-1}{\frac{k-1}{2}} (d-1)^{\frac{k-1}{2}} \quad \text{if } k \text{ is odd.} \tag{7}$$

The **index** λ satisfies relation $r = \lambda d^k$ see **Hedayat, Sloane, Stufken** *Orthogonal Arrays: Theory and Applications* (1999)

Orthogonal Arrays & k-uniform states

A link between them

	orthogonal arrays	multipartite quantum state $\|\Phi\rangle$
r	Runs	Number of terms in the state
N	Factors	Number of qudits
d	Levels	dimension d of the subsystem
k	Strength	class of entanglement (k–uniform)

holds

provided an **orthogonal array** OA(r, N, d, k)

satisfies additional constraints !

(this relation is NOT one–to–one)

k–uniform states and Orthogonal Arrays I

Consider a **pure state** $|\Phi\rangle$ of N qudits,

$$|\Phi\rangle = \sum_{s_1,\ldots,s_N} a_{s_1,\ldots,s_N} |s_1,\ldots,s_N\rangle,$$

where $a_{s_1,\ldots,s_N} \in \mathbb{C}$, $s_1,\ldots,s_N \in S$ and $S = \{0,\ldots,d-1\}$. Vectors $\{|s_1,\ldots,s_N\rangle\}$ form an orthonormal basis.

Density matrix ρ reads

$$\rho_{AB} = |\Phi\rangle\langle\Phi| = \sum_{\substack{s_1,\ldots,s_N \\ s'_1,\ldots,s'_N}} a_{s_1,\ldots,s_N} a^*_{s'_1,\ldots,s'_N} \, |s_1,\ldots,s_N\rangle\langle s'_1,\ldots,s'_N|.$$

We split the system into **two** parts $\mathcal{S}_A$ and $\mathcal{S}_B$ containing N_A and N_B qudits, respectively, $N_A + N_B = N$. and obtain the **reduced state**
$\rho_A = \mathrm{Tr}_B(\rho_{AB})$
$= \sum_{\substack{s_1\cdots s_N \\ s'_1\cdots s'_N}} a_{s_1\ldots s_N} a^*_{s'_1\ldots s'_N} \langle s'_{N_A+1},\ldots,s'_N|s_{N_A+1}\ldots s_N\rangle \, |s_1\ldots s_{N_A}\rangle\langle s'_1\ldots s'_{N_A}|.$

k–uniform states and Orthogonal Arrays II

A simple, **special case**: coefficients $a_{s_1,\ldots,s_N}$ are *zero* or *one*. Then

$$|\Phi\rangle = |s_1^1, s_2^1, \ldots, s_N^1\rangle + |s_1^2, s_2^2, \ldots, s_N^2\rangle + \cdots + |s_1^r, s_2^r, \ldots, s_N^r\rangle,$$

upper index i on s denotes the $i-th$ term in $|\Phi\rangle$. These coefficients can be arranged in an **array**

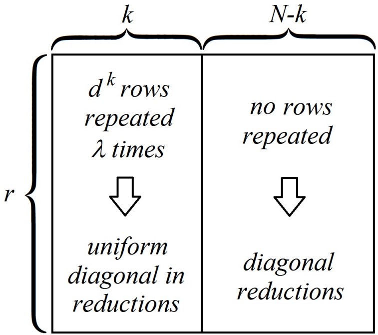
$$A = \begin{array}{cccc} s_1^1 & s_2^1 & \ldots & s_N^1 \\ s_1^2 & s_2^2 & \ldots & s_N^2 \\ \vdots & \vdots & \ldots & \vdots \\ s_1^r & s_2^r & \ldots & s_N^r \end{array}.$$

i). If A forms an **orthogonal array** for any partition the diagonal elements of the reduced state ρ_A are equal.

ii). If the sequence of N_B symbols appearing in every row of a subset of N_B columns **is not repeated** along the r rows (**irredundant OA**), the reduced density matrix ρ_A becomes diagonal.

How to construct a k–uniform state of N qudits ?

a) Take an **orthogonal array** OA(r, N, d, k) of **strength** k.



b) check if condition **ii)** is satisfied, so the array is **irredundant**.

c) If **yes**, write the corresponding k–uniform state!

Very simple examples

a) Two qubit, 1–uniform state

Orthogonal array

$$OA(2,2,2,1) = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

leads to the **Bell state** $|\Psi_2^+\rangle = |01\rangle + |10\rangle$, which is 1–uniform

b) Three–qubit, 1–uniform state

Orthogonal array

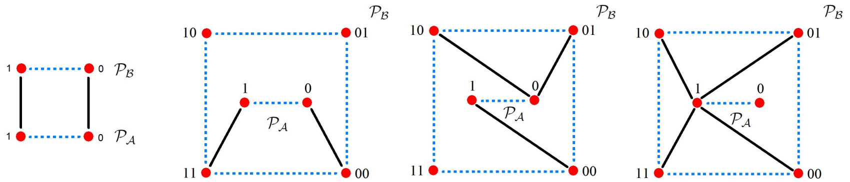
$$OA(4,3,2,{\color{red}2}) = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

leads to the **balanced, 1–uniform state**,

$$|\Phi_3\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle.$$

Graph representation of k-uniform states

*) Form polygons $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ representing principal and ancillary systems, respectively.
**) Connect vertex s_A^i to s_B^i for every $i = 0, \ldots, r-1$.



a) Bell $|\phi_+\rangle$ b) $|GHZ\rangle$ c) $|W\rangle$
d) separable state $|1\rangle\,(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$,

Criterion: A state $|\Psi\rangle$ is k–uniform if
i). every vertex of $\mathcal{P}_B$ is connected, at most, to one edge.
ii). every vertex of $\mathcal{P}_A$ is connected to the same number of edges.

Hadamard matrices & Orthogonal Arrays

A Hadamard matrix $H_8 = H_2^{\otimes 3}$ of order $N = 8$ implies OA(8,7,2,2)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

This OA allows us to construct a **2–uniform state** of 7 qubits:

$$\begin{aligned} |\Phi_7\rangle \;\; = \;\; & |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + \\ & |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle. \end{aligned}$$

a **simplex** state $|\Phi_7\rangle$

Examples of 2–uniform states obtained form H_{12}

8 **qubits**

$$
\begin{aligned}
|\Phi_8\rangle = \; & |00000000\rangle + |00011101\rangle + |10001110\rangle + |01000111\rangle + \\
& |10100011\rangle + |11010001\rangle + |01101000\rangle + |10110100\rangle + \\
& |11011010\rangle + |11101101\rangle + |01110110\rangle + |00111011\rangle.
\end{aligned}
$$

9 **qubits**

$$
\begin{aligned}
|\Phi_9\rangle = \; & |000000000\rangle + |100011101\rangle + |010001110\rangle + |101000111\rangle + \\
& |110100011\rangle + |011010001\rangle + |101101000\rangle + |110110100\rangle + \\
& |111011010\rangle + |011101101\rangle + |001110110\rangle + |000111011\rangle.
\end{aligned}
$$

10 **qubits**

$$
\begin{aligned}
|\Phi_{10}\rangle = \; & |0000000000\rangle + |0100011101\rangle + |1010001110\rangle + |1101000111\rangle + \\
& |0110100011\rangle + |1011010001\rangle + |1101101000\rangle + |1110110100\rangle + \\
& |0111011010\rangle + |0011101101\rangle + |0001110110\rangle + |1000111011\rangle,
\end{aligned}
$$

Higher dimensions: uniform states of qutrits and ququarts

From OA(9,4,3,2) we get a 2–**uniform** state of **4 qutrits**:

$$
\begin{aligned}
|\Psi_3^4\rangle \;=\;& |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle + \\
& |1202\rangle + |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

From OA(16,5,4,2) we get the 2–**uniform** state of **5 ququarts**,

$$
\begin{aligned}
& |\Psi_4^5\rangle = \\
& |00000\rangle + |01111\rangle + |02222\rangle + |03333\rangle + |10123\rangle + |11032\rangle + \\
& |12301\rangle + |13210\rangle + |20231\rangle + |21320\rangle + |22013\rangle + |23102\rangle + \\
& |30312\rangle + |31203\rangle + |32130\rangle + |33021\rangle
\end{aligned}
$$

related to the **Reed–Solomon code** of length 5.

Six ququarts

From OA(64,6,4,3) we construct a 3–**uniform** state of **6 ququarts**:
$|\Psi_4^6\rangle =$

$$|000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle +$$
$$|012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle + |022013\rangle + |023102\rangle +$$
$$|030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle +$$
$$|102310\rangle + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle +$$
$$|120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle + |130220\rangle + |131331\rangle +$$
$$|132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle +$$
$$|210330\rangle + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle +$$
$$|222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle + |232323\rangle + |233232\rangle +$$
$$|300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle +$$
$$|312020\rangle + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle +$$
$$|330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle.$$

Low number of subsystems: qubit states

$k \setminus N$	2	3	4	5	6	7	8
1	p	p	p	p	p	p	p
2	-	-	0	n	p	p	p
3	-	-	-	-	n	?	p
4	-	-	-	-	-	-	0

Existence of k–**uniform states** for N **qubits**:

p – all positive coefficients

n – some of coefficients are negative

0 – no existence established

? - case open

Low number of subsystems: qudit states

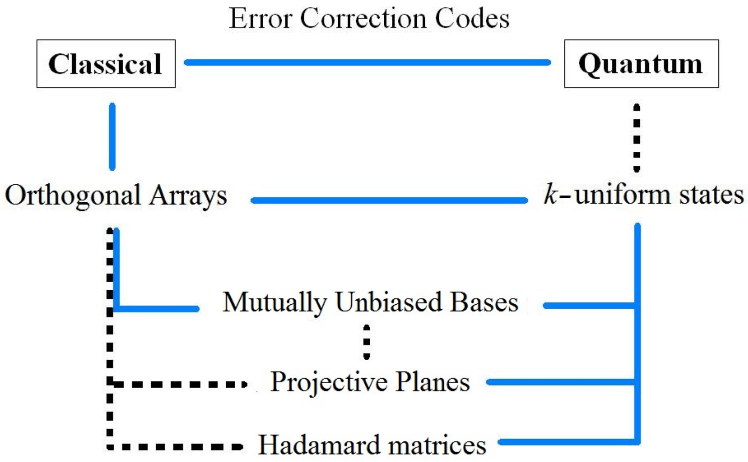
$N \setminus d$	2	3	4	5	6	7	8
2	✓	✓	✓	✓	✓	✓	✓
4	-	✓	✓	✓	**?**	✓	✓
6	✓	✓	✓	✓	✓	✓	✓
8	-	**?**	**?**	**?**	**?**	✓	✓

Existence of k-**uniform** states of d–**level** subsystems for the highest possible strength, $k = N/2$.

Constructive results

1. Basing on multi–qubit Hadamard matrices, $H_{2^m} = H_2^{\otimes m}$, we constructed 2–**uniform states** of N qubits for any $N \geq 6$.
2. Every orthogonal array of **index unity**, $\mathrm{OA}(d^k, N, d, k)$ allows us to generate a k–**uniform** state of N **qudits** of d levels if and only if $k \leq N/2$.
3. Making use of known results on **orthogonal matrices** we demonstrate existence of show following k–**uniform states**:
   - *(i)* k–uniform states of $d+1$ qudits with d levels, where $d \geq 2$ and $k \leq \frac{d+1}{2}$.
   - *(ii)* 3–uniform states of $2^m + 2$ qudits with 2^m levels, where $m \geq 2$.
   - *(iii)* $(2^m - 1)$–uniform states of $2^m + 2$ qudits with 2^m levels, where $m = 2, 4$.
4. From every k–**uniform** state generated from an OA we construct an entire **orbit** of **maximally entangled** states.
   Three–qubit example: a 3–parameter orbit of 1–uniform states
   $|\Phi_3\rangle(\alpha_1, \alpha_2, \alpha_3) = |000\rangle + e^{i\alpha_1}|011\rangle + e^{i\alpha_2}|101\rangle + e^{i\alpha_3}|110\rangle$,

Links explored



Relationship between ***k*–uniform states**, quantum (classical) error correction codes, mutually unbiased bases and **orthogonal arrays**.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Open questions

1. Solve the existence problem of 3–**uniform** states of 7 and 8 qubits.
2. Find for what N there exist 3–**uniform** states of N **qubits** and 2–**uniform** states of N **qutrits**.
3. Find how the maximal value k_{max}, for which k_{max}–**uniform** states of N-qubit exist, depends on N. Analyze the dependence $k_{max}(N)$ for qutrits and higher, d-dimensional systems
4. Investigate existence of the approximate, (ϵ, k)–uniform states,
5. Investigate the relation between **orthogonal arrays** corresponding to two **locally equivalent states**.