Local vs. Global Approach in Open Quantum Chains

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Main Question

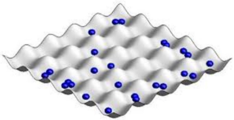
- Quantum Markov Process

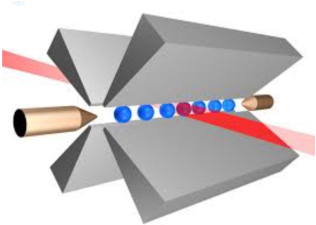
Local Approach

Global Approach

Motivation

- Ion Trap
- Ultra Cold Atom



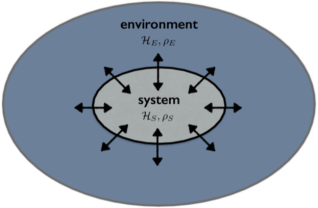


Introduction to QMP

- For quantum systems and environment, time-evolution is proved to be unitary

$$\rho_{SE}(t) = U_{SE}(t)\rho_{SE}(0)U^{\dagger}_{SE}(t),$$

$$U_{SE}(t) \neq U_S(t) \otimes U_E(t).$$



- The dynamic of system is described by a completely positive trace preserving (CPTP) map

$$\Phi : \rho_s(0) \longrightarrow \rho_s(t),$$

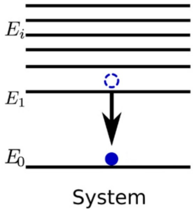
$$\Phi[\bullet] = e^{t\mathcal{L}}[\bullet].$$

Markovian Master Equation

$$\dot{\rho}_S(t) = -i[H_S, \rho_S(t)] + \mathcal{D}[\rho_S(t)] = \mathcal{L}[\rho_S(t)],$$

$$\mathcal{D}[\bullet] = \sum_{i,j} \sum_m \gamma_{ji}(\omega_m)(A_i(\omega_m) \bullet A_j^\dagger(\omega_m) - \frac{1}{2}\{A_i^\dagger(\omega_m) A_j(\omega_m), \bullet\}).$$

$$A(\omega_m) |E_i\rangle = \sum_{E_i - E_j = \omega_m} C_{ij} |E_j\rangle .$$

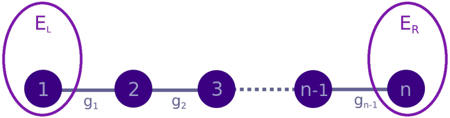


V. Gorini,A. Kossakowski and A.Sudarshan, Math. Phys. 17 (5): 821 (1976)

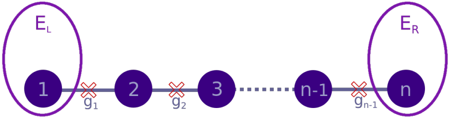
G. Lindblad. Communications in Math. Phys. 48(2):119–130 (1976)

Introduction to Global and Local Approach

Global approach

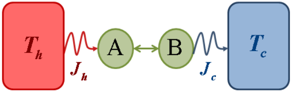


Local approach

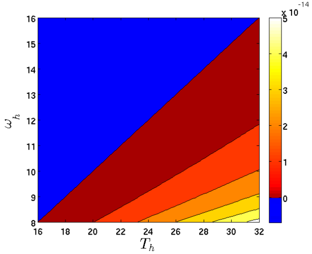


$G > L$

- Negative entropy production rate, a clear violation of the second law by using local approach



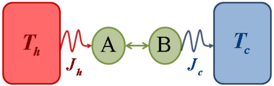
	local	global
second law	X	✓



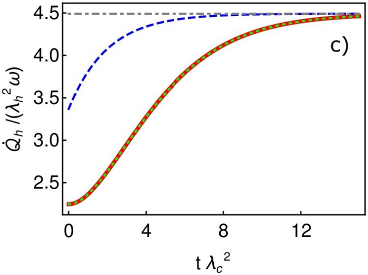
A. Levy and R. Kosloff, Europhysics Letters, 107(2):20004 (2014)

$L > G$

- Heat currents at an exceptional point



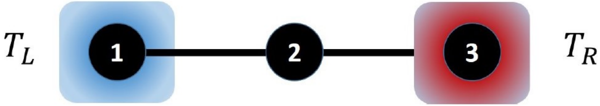
	local	global
experiment	✓	X



S. Scali, J. Anders, and L. A. Correa,Quantum 5, 451 (2021)

L or G

- sink/source term in spin-flow continuity equation



$$\frac{d}{dt}\langle\sigma_z^{(2)}(t)\rangle = \frac{d}{dt}Tr[\sigma_z^{(2)}\rho(t)].$$

	local approach	global approach
sink and source	X	✓

F. Benatti, R. Floreanini, and L. Memarzadeh. Physical Review A, 102(4):042219 (2020)

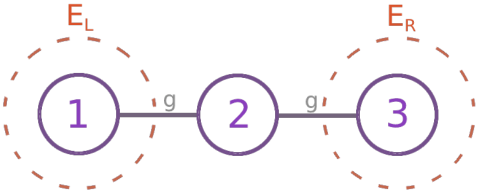
F. Benatti, R. Floreanini, and L. Memarzadeh, PRX Quantum, vol.2, p.030344 (2021)

Approach

- Compare with the experimental results.
- Compare with the exact solution, ✓
  - compare local with exact.
  - compare global with exact.

Model

Schematic representation of three-partite chain ρ_S, where first sub-system and third sub-system interacts with a thermal bath E_L and E_R respectively



Model

The total Hamiltonian of this model is

$$H = H_S + H_E + \lambda H_{int}, \tag{1}$$

where the Hamiltonian of system is

$$H_S = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + g \sum_{i=1}^{2} (a_i a_{i+1}^\dagger + a_i^\dagger a_{i+1}), \tag{2}$$

and the Hamiltonian of two thermal baths are

$$H_E = \sum_{\alpha=L,R} H_E^{(\alpha)} = \sum_{\alpha=L,R} \sum_{k=1}^{M} \omega_k^{(\alpha)} (b_k^{(\alpha)})^\dagger b_k^{(\alpha)}, \tag{3}$$

and the interaction Hamiltonian of two baths with system

$$H_{int} = \sum_{\alpha=L,R} H_{int}^{(\alpha)} = \sum_{\alpha=L,R} \sum_{k=1}^{M} \gamma_k^{(\alpha)} (a_\alpha^\dagger b_k^{(\alpha)} + a_\alpha (b_k^{(\alpha)})^\dagger). \tag{4}$$

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Result: Derivation of local master equation

The Master equation for the system in the local approach is

$$\dot{\rho}_s(t) = -i[H_S + H_{LS}^{(loc)}, \rho_S(t)] + \sum_{i=1,3} \mathcal{D}_i^{(loc)}[\rho_S(t)] = \mathcal{L}^{(loc)}[\rho_S(t)], \tag{5}$$

where $\mathcal{D}_i^{(loc)}[\bullet]$s are

$$\mathcal{D}_i^{(loc)}[\bullet] = J(\omega_i)\bar{n}^{(i)}(\omega_i)(a_i^\dagger \bullet a_i - \frac{1}{2}\{a_i a_i^\dagger, \bullet\}) + J(\omega_i)(\bar{n}^{(i)}(\omega_i) + 1)(a_i \bullet a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \bullet\}). \tag{6}$$

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Result: Diagonalize the Hamiltonian of system

$$H_S = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + g \sum_{i=1}^{2} (a_i a_{i+1}^\dagger + a_i^\dagger a_{i+1}).$$

$$\Downarrow$$

$$H_S = \sum_{i=1}^{3} \epsilon_i c_i^\dagger c_i.$$

$$c_1 = \frac{1}{\sqrt{2}}(a_1 - a_3), \qquad \epsilon_1 = \omega_0,$$

$$c_2 = \frac{1}{2}(a_1 - \sqrt{2}a_2 + a_3), \qquad \epsilon_2 = \omega_0 - \sqrt{2}g,$$

$$c_3 = \frac{1}{2}(a_1 + \sqrt{2}a_2 + a_3), \qquad \epsilon_3 = \omega_0 + \sqrt{2}g.$$

Result: Derivation of global master equation

The master equation in global approach would be

$$\dot{\rho}_s(t) = -i[H_S + \lambda^2 H_{LS}^{(glb)}, \rho_s(t)] + \lambda^2 \sum_{i=1}^{3} \mathcal{D}_i^{(glb)}[\rho_S(t)] = \mathcal{L}^{(glb)}[\rho_s(t)], \tag{7}$$

where $\mathcal{D}_i^{(glb)}[\bullet]$s are

$$\mathcal{D}_i^{(glb)}[\bullet] = \sum_{\alpha=L,R} J(\epsilon_i)\bar{n}^{(\alpha)}(\epsilon_i)(c_i^\dagger \bullet c_i - \frac{1}{2}\{c_i c_i^\dagger, \bullet\})$$

$$+ J(\epsilon_i)(\bar{n}^{(\alpha)}(\epsilon_i) + 1)(c_i \bullet c_i^\dagger - \frac{1}{2}\{c_i^\dagger c_i, \bullet\}). \tag{8}$$

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g=0

$$\mathcal{D}^{(loc)}[\bullet] = \sum_{i=1,3} J(\omega)\bar{n}(\omega)(a_i^\dagger \bullet a_i - \frac{1}{2}\{a_i a_i^\dagger, \bullet\})$$
$$+ J(\omega)(\bar{n}(\omega)+1)(a_i \bullet a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \bullet\}).$$

$$\mathcal{D}^{(glb)}[\bullet] = 2\sum_{i=1}^{3} J(\omega)\bar{n}(\omega)(a_i^\dagger \bullet a_i - \frac{1}{2}\{a_i a_i^\dagger, \bullet\})$$
$$+ J(\omega)(\bar{n}(\omega)+1)(a_i \bullet a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \bullet\}).$$

$$\Downarrow$$

$$\mathcal{L}^{(glb)}[\rho_S(t)] \neq \mathcal{L}^{(loc)}[\rho_S(t)]$$

Exact Solution

In the exact solution we have

$$\dot{\rho}_{sb}(t) = -i[H, \rho_{sb}(t)], \tag{9}$$

the density matrix of system is

$$\dot{\rho}_s(t) = tr_b(\rho_{sb}(t)). \tag{10}$$

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Till now

- Local

$$\dot{\rho}_s(t) = \mathcal{L}^{(loc)}[\rho_s(t)].$$

- Global

$$\dot{\rho}_s(t) = \mathcal{L}^{(glb)}[\rho_s(t)].$$

- Exact

$$\dot{\rho}_{sb}(t) = -i[H, \rho_{sb}(t)].$$

Method

- Local and Exact $\longrightarrow \mathcal{F}^{(loc)}(t)$.
- Global and Exact $\longrightarrow \mathcal{F}^{(glb)}(t)$.
- $\mathcal{F}^{(glb)}(t) \leq \mathcal{F}^{(loc)}(t)$ or $\mathcal{F}^{(glb)}(t) \geq \mathcal{F}^{(loc)}(t)$.
- Steady State $\longrightarrow \mathcal{F}_{\infty}^{(glb)}$ and $\mathcal{F}_{\infty}^{(loc)}$.
- Continuity Equation of $\left\langle a_2^{\dagger} a_2(t) \right\rangle$.

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- Continuity Equation of $\left\langle a_2^{\dagger} a_2(t) \right\rangle$.

Fidelity

We are not able to use this equation for an infinite dimensional continuous variable system

$$\mathcal{F}(\rho_1, \rho_2) = tr(\sqrt{\rho_1 \sqrt{\rho_2} \rho_1}).$$

We use a subset of these states which are Gaussian states

$$\mathcal{F}(\rho_1, \rho_2) = f(C_1, C_2).$$

Is it Gaussian?

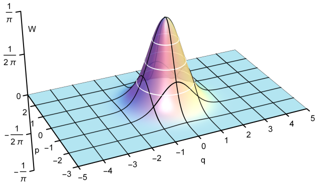
The state is Gaussian if its characteristic function is Gaussian:

$$\chi(\rho) = tr(\mathcal{D}(z)\rho) = e^{(\bar{z},z)C(z,\bar{z})^t},$$

where C is a covariance matrix:

$$C_{ij} = \langle r_i r_j + r_j r_i \rangle - \langle r_i \rangle \langle r_j \rangle .$$

- Initial Gaussian state,
  - Thermal state.
  - Squeezed state.
- Final Gaussian state,
  - Quadratic Hamiltonian.



Solution

- Local

$$\dot{\rho}_s(t) = \mathcal{L}^{(loc)}[\rho_s(t)] \Longrightarrow \left\langle \dot{O}(t) \right\rangle = \left\langle (\mathcal{L}^{(loc)})^\dagger [O] \right\rangle .$$

- Global

$$\dot{\rho}_s(t) = \mathcal{L}^{(glb)}[\rho_s(t)] \Longrightarrow \left\langle \dot{O}(t) \right\rangle = \left\langle (\mathcal{L}^{(glb)})^\dagger [O] \right\rangle .$$

- Exact

$$\dot{\rho}_{sb}(t) = -i[H, \rho_{sb}(t)] \Longrightarrow \left\langle \dot{O}(t) \right\rangle = i[H, \langle O \rangle].$$

Complicated!

Solution

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$$\dot{\rho}_s(t) = \mathcal{L}^{(loc)}[\rho_s(t)] \Longrightarrow \left\langle \dot{O}(t) \right\rangle = \left\langle (\mathcal{L}^{(loc)})^\dagger [O] \right\rangle .$$

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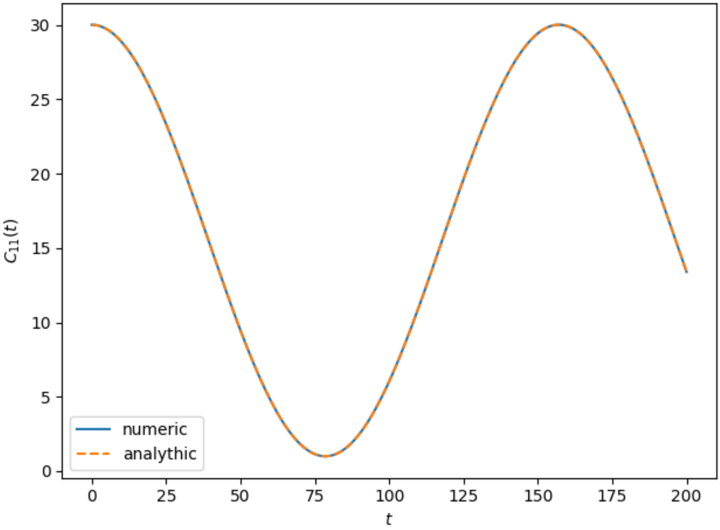
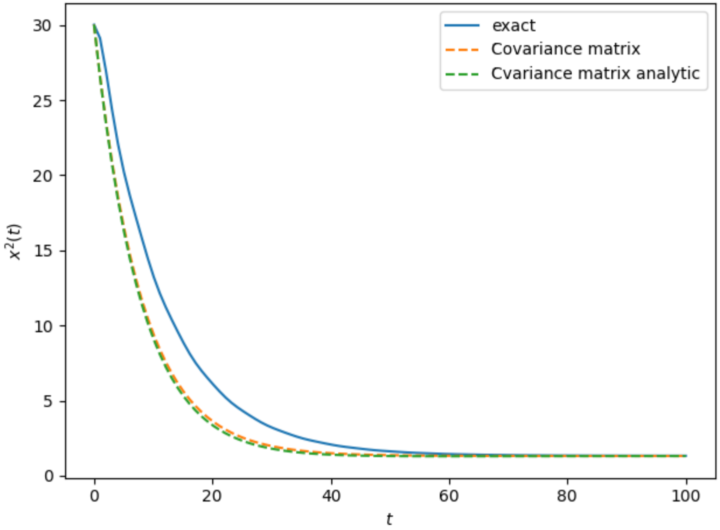
$$\dot{\rho}_{sb}(t) = -i[H, \rho_{sb}(t)] \Longrightarrow \left\langle \dot{O}(t) \right\rangle = i[H, \langle O \rangle].$$

Complicated!

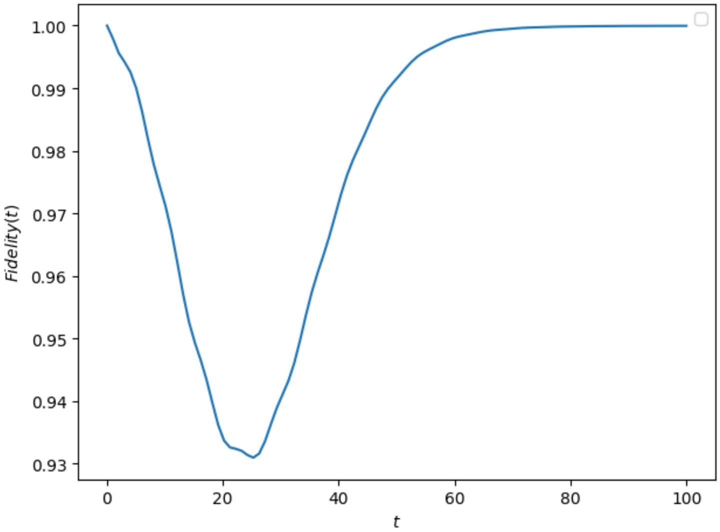
Validity of numerical results: One Harmonic Oscillator, One Bath



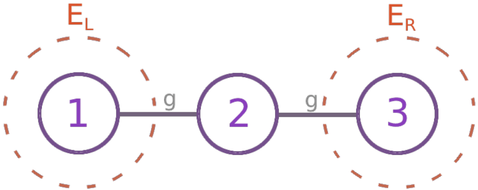
Increase the number of oscillator in bath



Fidelity: Master equation and Exact solution



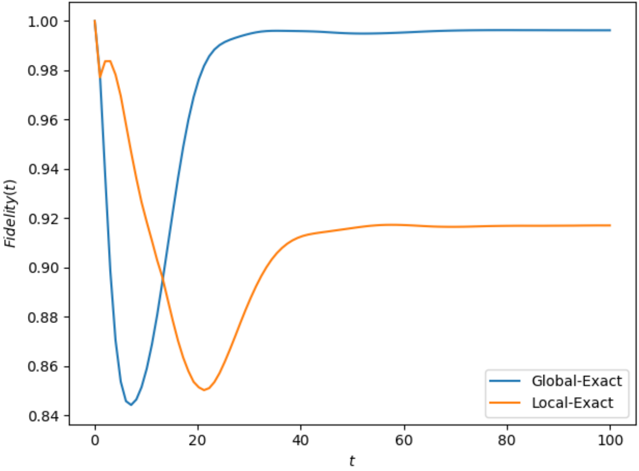
Back to our model



$$C_{ij} = \langle r_i r_j + r_j r_i \rangle - \langle r_i \rangle \langle r_j \rangle \,,$$

$$\Longrightarrow \begin{cases} \dot{C}_{ext}(t) = M_{ext} C_{ext}(t) + C_{ext}(t) M_{ext}^T. \\ \dot{C}_{loc}(t) = M_{loc} C_{loc}(t) + C_{loc}(t) M_{loc}^T + F_{loc}. \\ \dot{C}_{glb}(t) = M_{glb} C_{glb}(t) + C_{glb}(t) M_{glb}^T + F_{glb}. \end{cases}$$

Fidelity


1.00
0.98
0.96
0.94
0.92
0.90
0.88
0.86
0.84
Fidelity(t)
0
20
40
60
80
100
t
Global-Exact
Local-Exact

$g = 0$, $T_l = T_r$

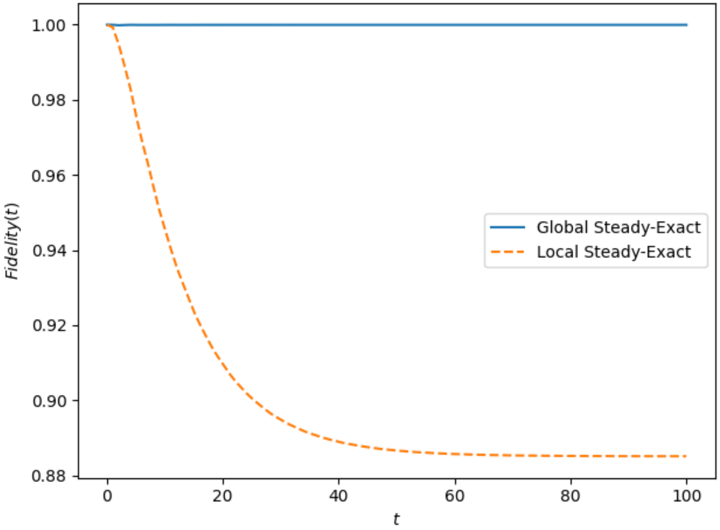
$$\dot{C}(t) = MC(t) + C(t)M^\dagger + F,$$

$$\Downarrow$$

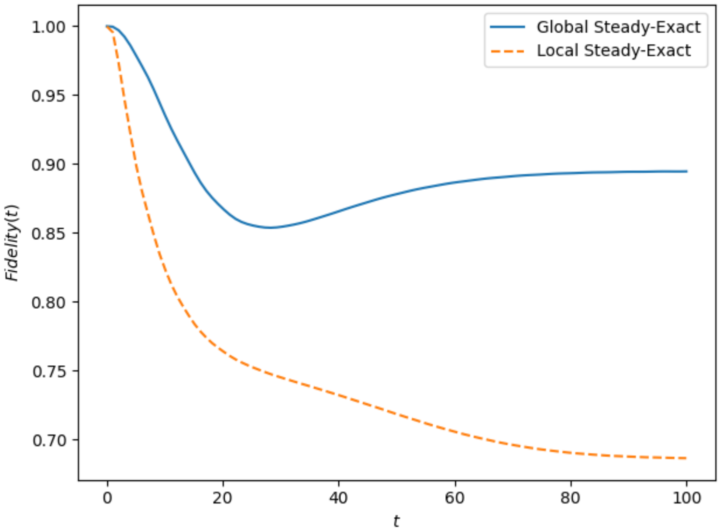
$$C_\infty^{(loc)} = (\bar{n}(\omega) + \frac{1}{2})\,\mathrm{id}(6).$$

$$C_\infty^{(glb)} = (\bar{n}(\omega) + 1)\,\mathrm{id}(6).$$

Steady state



Steady state



Continuity Equation

$$\frac{d}{dt}\left\langle a_2^\dagger a_2(t)\right\rangle = tr(\mathcal{L}^\dagger[a_2^\dagger a_2]\rho(t))$$

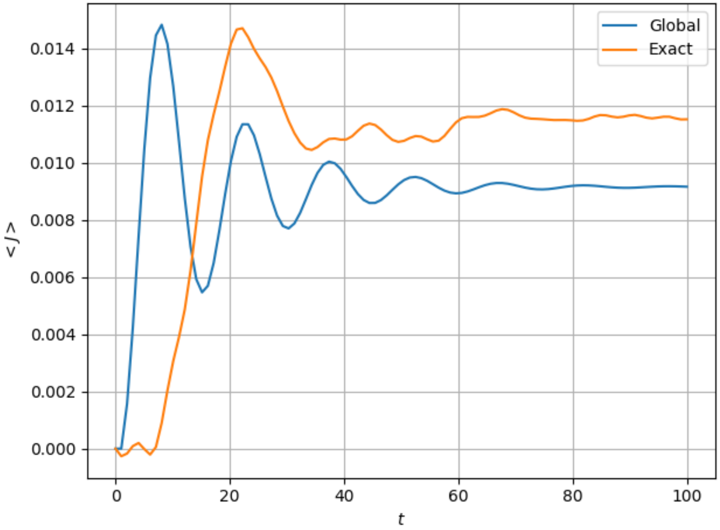
$$\Downarrow$$

$$\frac{d}{dt}\left\langle a_2^\dagger a_2\right\rangle_{glb} = (g - \frac{\sqrt{2}\lambda^2}{4}(\epsilon_1' - \epsilon_3'))\left\langle J_{12} - J_{23}\right\rangle + \sum_\alpha \left\langle \mathcal{Q}_\alpha\right\rangle$$

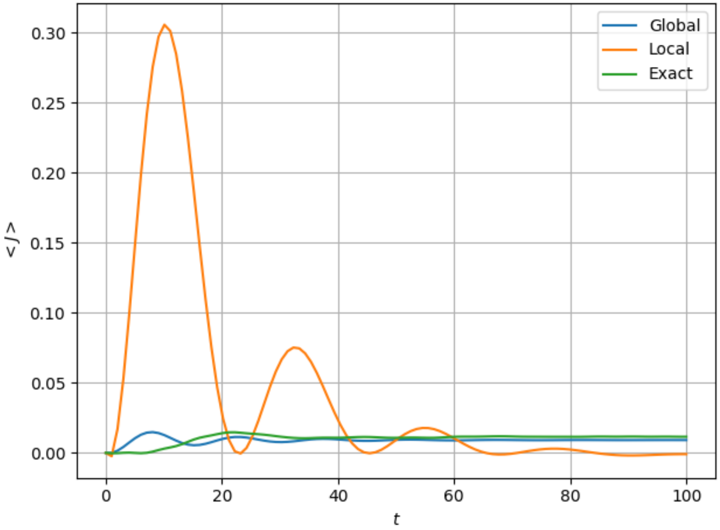
$$\frac{d}{dt}\left\langle a_2^\dagger a_2\right\rangle_{loc} = g\left\langle J_{12} - J_{23}\right\rangle$$

$$\frac{d}{dt}\left\langle a_2^\dagger a_2\right\rangle_{ext} = g\left\langle J_{12} - J_{23}\right\rangle$$

Continuity Equation



Continuity Equation



Conclusion

- Derived the master equation in local and global approach: $\mathcal{L}^{(glb)}[\rho_s(t)] \neq \mathcal{L}^{(loc)}[\rho_s(t)]$.
- Checked the validity of numerical result: Validity of master equation.
- Studied the evolution of Fidelity.
- Studied the fidelity in steady state: $C_\infty^{(loc)} \neq C_\infty^{(glb)}$.
- Studied continuity equation.

Thanks for your attention!

Exact Solution

If we define the $|Q(t)\rangle_{ext}$ as

$$|Q(t)\rangle_{ext} =: (x_1(t), x_2(t), x_3(t), \overrightarrow{x}_l(t), \overrightarrow{x}_r(t); p_1(t), p_2(t), p_3(t), \overrightarrow{p}_l(t), \overrightarrow{p}_r(t))^T$$

We would see the differential equation of this vector is

$$\left|\dot{Q}(t)\right\rangle_{ext} = i(\sigma_y \otimes W_{ext}) |Q(t)\rangle_{ext} := M_{ext} |Q(t)\rangle \tag{11}$$

And the differential equation of the covariance matrix is

$$\dot{C}_{ext}(t) = M_{ext} C_{ext}(t) + C_{ext}(t) M_{ext}^T \tag{12}$$

Exact Solution

Where W_{ext} is:

$$W_{ext} = \begin{pmatrix} \omega_1 & g & 0 & \lambda\gamma_1^{(l)} & \cdots & \lambda\gamma_M^{(l)} & 0 & \cdots & 0 \\ g & \omega_2 & g & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & g & \omega_3 & 0 & \cdots & 0 & \lambda\gamma_1^{(r)} & \cdots & \lambda\gamma_M^{(r)} \\ \lambda\gamma_1^{(l)} & 0 & 0 & \omega_1^{(l)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \lambda\gamma_M^{(l)} & 0 & 0 & 0 & 0 & \omega_M^{(l)} & 0 & \cdots & 0 \\ 0 & 0 & \lambda\gamma_1^{(r)} & 0 & 0 & 0 & \omega_1^{(r)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \lambda\gamma_M^{(r)} & 0 & \cdots & \cdots & \cdots & 0 & \omega_M^{(r)} \end{pmatrix}.$$

Local Solution

If we define $|Q(t)\rangle_{loc}$ as

$$|Q(t)\rangle_{loc} =: (x_1(t), x_2(t), x_3(t), p_1(t), p_2(t), p_3(t))^T \tag{13}$$

We would see the differential equation of this vector is

$$|Q(t)\rangle_{loc} = i(\sigma_y \otimes W_{app} - i\,\mathrm{id}_2 \otimes U_{loc})\,|Q(t)\rangle_{loc} := M_{loc}\,|Q(t)\rangle_{loc}$$

And the differential equation of the covariance matrix is

$$\dot{C}(t)_{loc} = M_{loc}C_{loc}(t) + C_{loc}(t)M_{loc}^T + F_{loc} \tag{14}$$

Local Solution

Where W_{app} is:

$$W_{app} = \begin{pmatrix} \omega_1 & g & 0 \\ g & \omega_2 & g \\ 0 & g & \omega_3 \end{pmatrix}$$

and U_{loc} is:

$$U_{loc} = -\frac{\lambda^2}{2} \begin{pmatrix} J(\omega_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J(\omega_3) \end{pmatrix}$$

Global Solution

If we define $|Q(t)\rangle_{glb}$ as

$$|Q(t)\rangle_{glb} =: (x_1(t), x_2(t), x_3(t), p_1(t), p_2(t), p_3(t))^T \tag{15}$$

We would see the differential equation of this vector is

$$\left|\dot{Q(t)}\right\rangle_{glb} = (i\sigma_y \otimes W_{app} + \mathrm{id}_2 \otimes U_{glb}) |Q(t)\rangle_{glb} := M_{glb} |Q(t)\rangle_{glb}$$

And the differential equation of the covariance matrix is

$$\dot{C}_{glb}(t) = M_{glb} C_{glb}(t) + C_{glb}(t) M_{glb}^T + F_{glb} \tag{16}$$

Global Solution

Where U_{glb} is:

$$U_{glb} = \frac{\lambda^2}{8} \begin{pmatrix} u_1 & u_2 & u_3 \\ \sqrt{2}u_2 & u_4 & \sqrt{2}u_2 \\ u_3 & u_2 & u_1 \end{pmatrix} \tag{17}$$

that u_is are:

$$\begin{aligned} u_1 &= -(2J(\epsilon_1) + J(\epsilon_2) + J(\epsilon_3)) \\ u_2 &= J(\epsilon_2) - J(\epsilon_3) \\ u_3 &= 2J(\epsilon_1) - J(\epsilon_2) - J(\epsilon_3) \\ u_4 &= -2J(\epsilon_2) - 2J(\epsilon_3) \end{aligned} \tag{18}$$

Fidelity

The fidelity for Three mode Gaussian state:

$$\mathcal{F}(\rho_1, \rho_2) = \mathcal{F}_0(C_1, C_2) e^{-\frac{1}{4}\delta_u^T (C_1 + C_2)\delta_u}$$

$$\mathcal{F}_0(C_1, C_2) = \frac{F_{tot}}{\sqrt{det(C_1 + C_2)}}$$

$$F_{tot}^4 = det(2(\sqrt{id(6) + \frac{(C_{aux}\Omega)^{-2}}{4}} + id(6))C_{aux})$$

$$C_{aux} = \Omega^T (C_1 + C_2)^{-1} (\frac{\Omega}{4} + C_2 \Omega C_1)$$

$$\Omega = \bigoplus_1^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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Steady state

The local steady state is

$$C_1 = \frac{1}{4}(n_l + n_r + 1)\Gamma_1 + \frac{1}{2}(n_l - n_r)\frac{1}{1 + (\frac{2g}{\lambda^2 J(\omega)})^2}\Gamma_2$$

$$- \frac{g}{\lambda^2 J(\omega)}\frac{n_l - n_r}{1 + (\frac{2g}{\lambda^2 J(\omega)})}\Gamma_3 + \frac{1}{4}(n_l + n_r + 1)\frac{1 - (\frac{4g}{\lambda^2 J(\omega)})}{1 - 2(\frac{4g}{\lambda^2 J(\omega)})}\Gamma_4$$

$$+ \frac{1}{2}(n_l + n_r + 1)\Gamma_5$$

Steady state: Local

where Γ_is are

$$\Gamma_1 = \begin{pmatrix} 1 & & -1 \\ & 0 & \\ -1 & & 1 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} & i & \\ -i & & i \\ & -i & \end{pmatrix}, \qquad \Gamma_4 = \begin{pmatrix} 1 & & 1 \\ & 0 & \\ 1 & & 1 \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

Steady state: Global

$$C_1 = \frac{1}{4}\begin{pmatrix} A & B & C \\ B & D & B \\ C & B & A \end{pmatrix}$$

$$A = 2n_l(\epsilon_1) + 2n_r(\epsilon_1) + n_l(\epsilon_2) + n_r(\epsilon_2) + n_l(\epsilon_3) + n_r(\epsilon_3) + 4$$

$$B = \sqrt{2}(n_l(\epsilon_2) + n_r(\epsilon_2) - n_l(\epsilon_3) - n_r(\epsilon_3))$$

$$C = -2n_l(\epsilon_1) - 2n_r(\epsilon_1) + n_l(\epsilon_2) + n_r(\epsilon_2) + n_l(\epsilon_3) + n_r(\epsilon_3)$$

$$D = 2(n_l(\epsilon_2) + n_r(\epsilon_2) + n_l(\epsilon_3) + n_r(\epsilon_3) + 1)$$