Combinatorial Optimization

Solving Traveling Salesman Problem with D-Wave Quantum Computer

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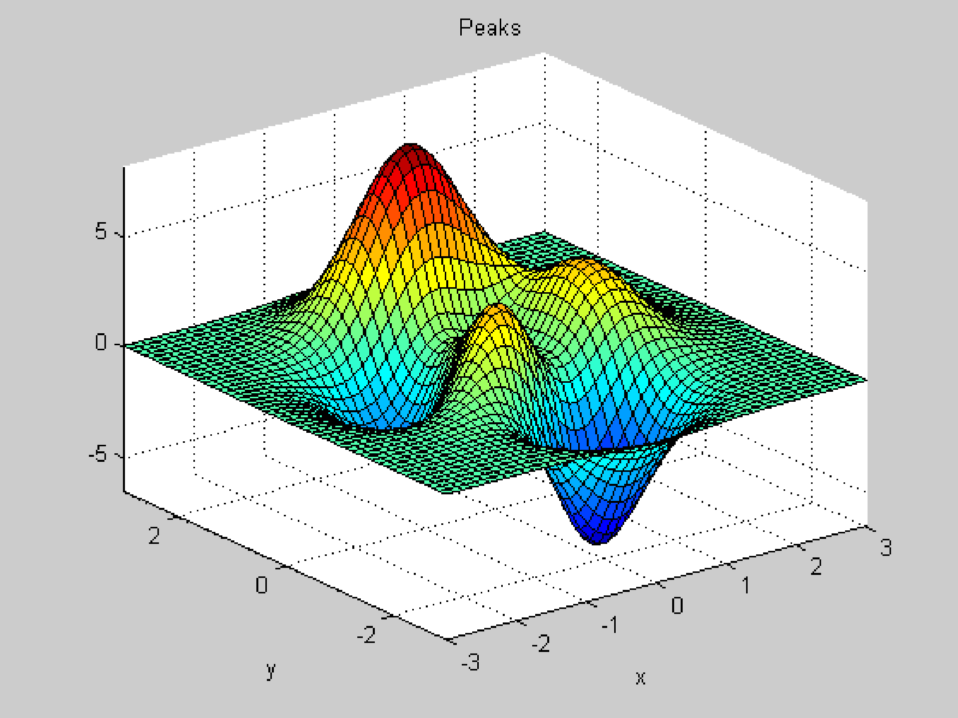
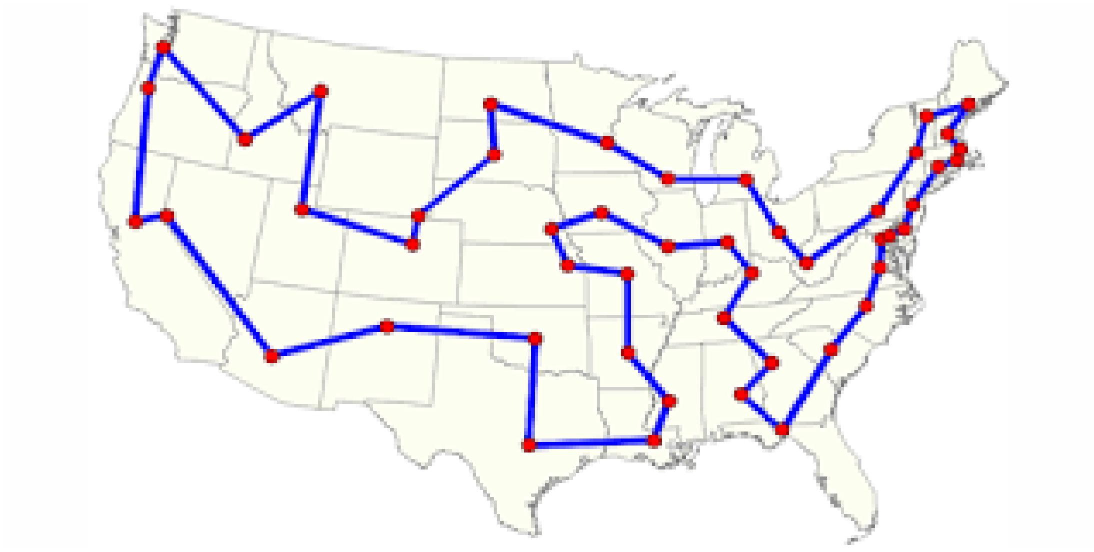
Outlines

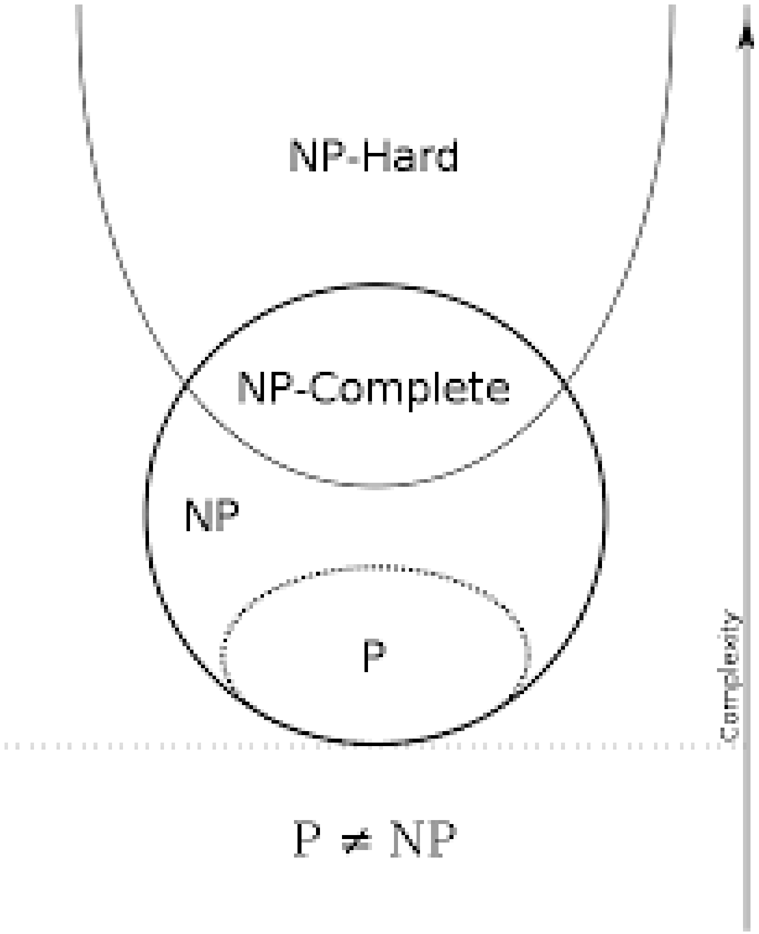
- Combinatorial Optimization
- Quantum Approximate Optimization Algorithm (QAOA)
- Binary Quadratic Model (BQM)
- D-Wave Quantum Computer
- Travelling Salesman Problem
- Related Works

Combinatorial Optimization

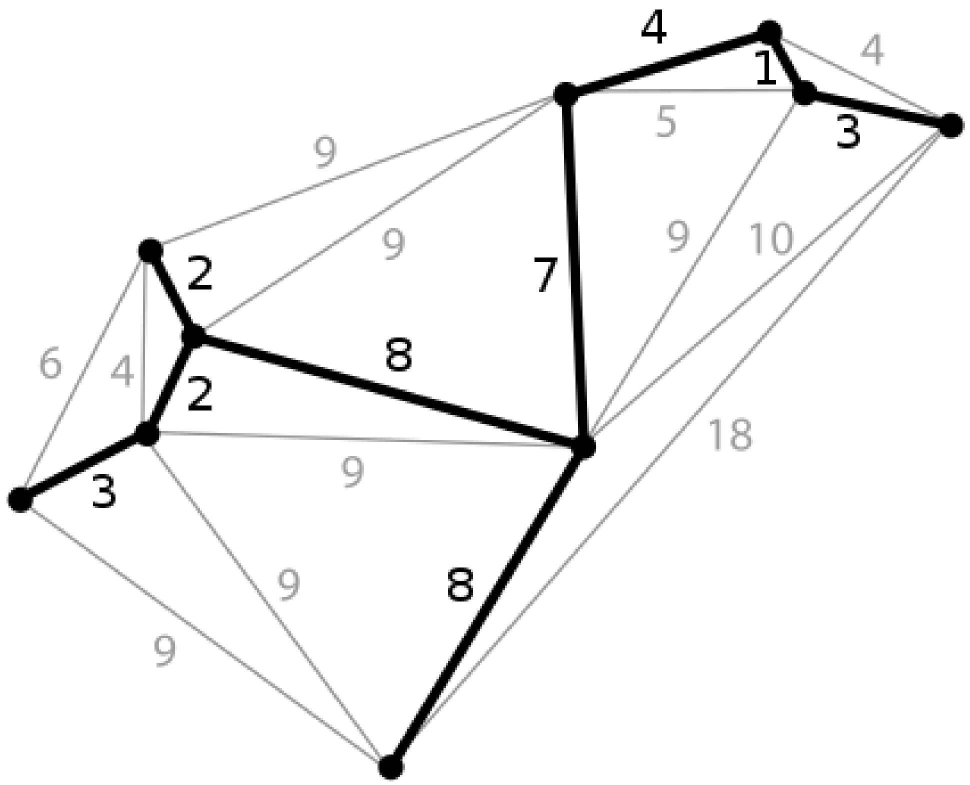




Combinatorial Optimization



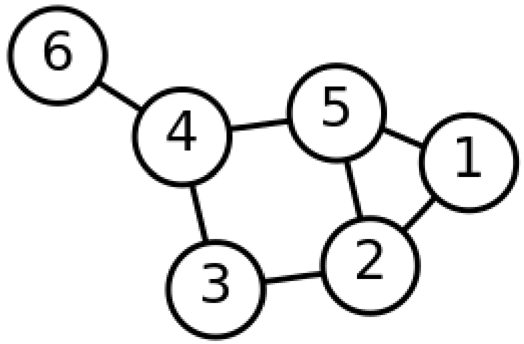
Minimum Spanning Tree (MST)



Minimum Spanning Tree (MST)

Given a graph G with n vertices $v_1, v_2, \ldots, v_n$ the Laplacian matrix $L_{n\times n}$ is defined as:

$$L_{i,j} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Minimum Spanning Tree (MST)

Kirchhoff's theorem: For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ be the non-zero eigenvalues of its Laplacian matrix. Then the number of spanning trees of G is:

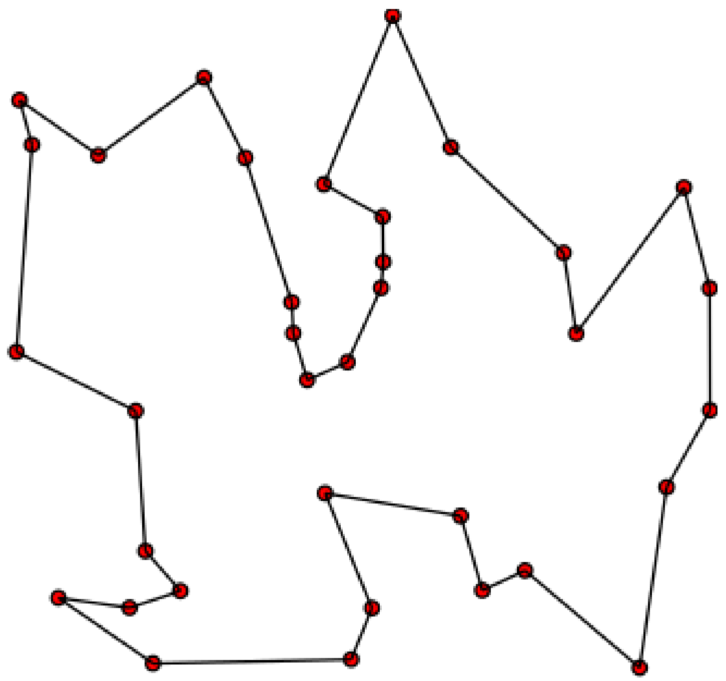
$$t(G) = \frac{1}{n}\lambda_1\lambda_2 \dots \lambda_{n-1}$$

Laplacian matrix of a fully connected graph

$$\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

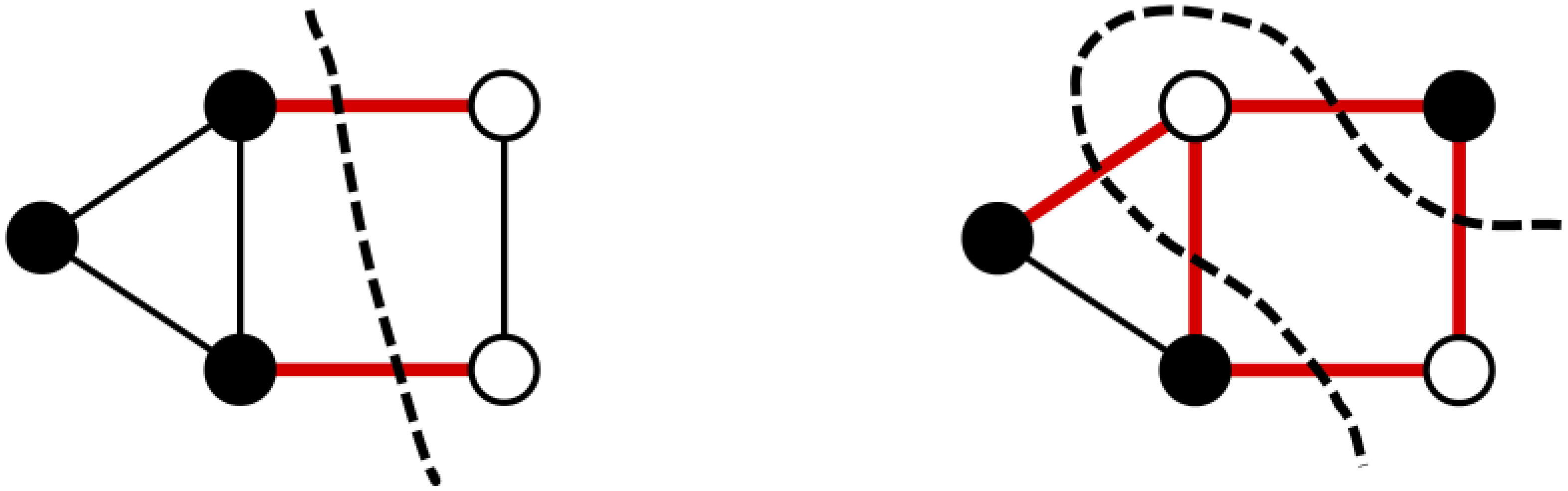
$$t(G) = n^{n-2}$$

Traveling Salesman Problem (TSP)



$$\text{Number of routes} = \frac{(n-1)!}{2}$$

Max-Cut



Number of cuts of a graph with n vertices $= 2^{n-1} - 1$

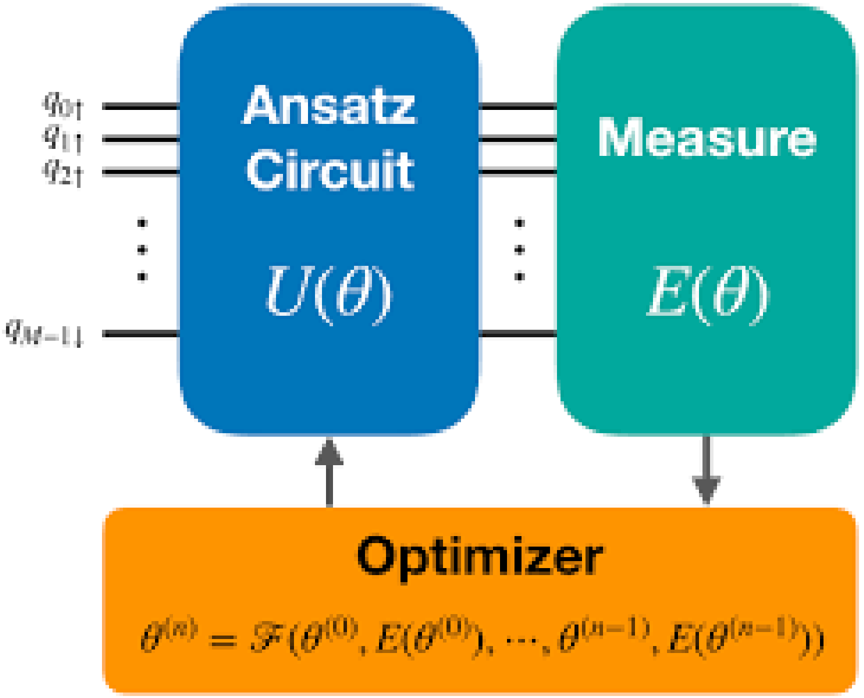
Quantum Approximate Optimization Algorithm (QAOA)

Idea

Cost Function $\rightarrow$ Hamiltonian

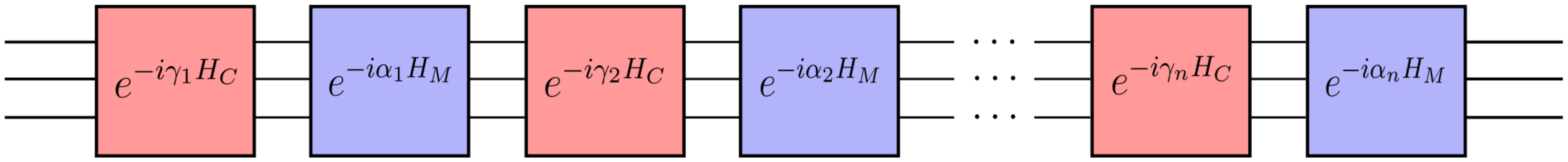
Prepare $|\psi(\theta)\rangle$

Minimize $\langle\psi(\theta)|H|\psi(\theta)\rangle$



Quantum Approximate Optimization Algorithm (QAOA)

Ansatz

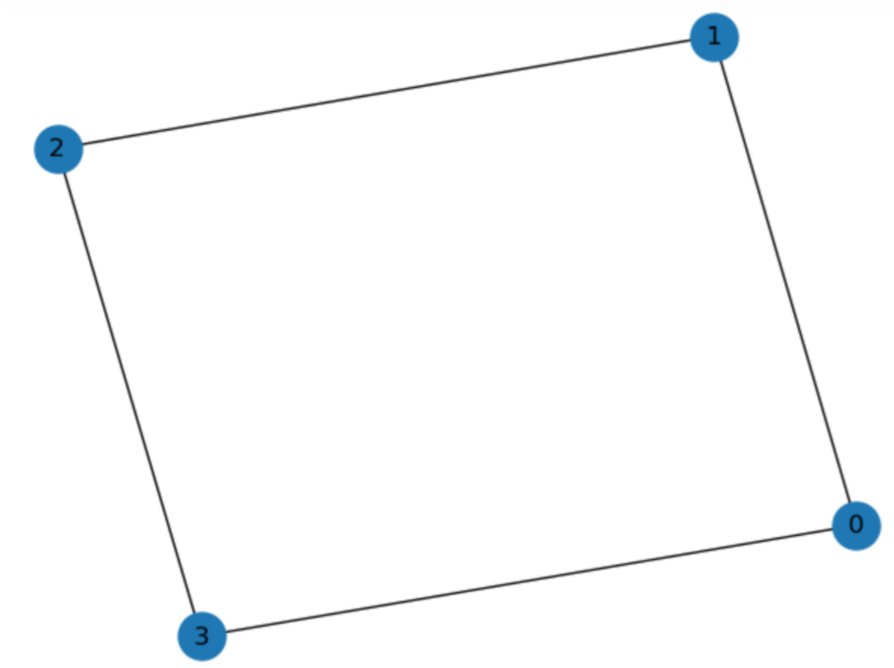


$$H_M = \sum_i \sigma_x^{(i)}$$

$$U_M = \exp(-i\alpha H_M) = \exp\left(-i\alpha \sum_i \sigma_x^{(i)}\right) = \prod_i \exp\left(-i\alpha\sigma_x^{(i)}\right) = \prod_i R_x^{(i)}(2\alpha)$$

Quantum Approximate Optimization Algorithm (QAOA)

Example: Max-Cut Problem

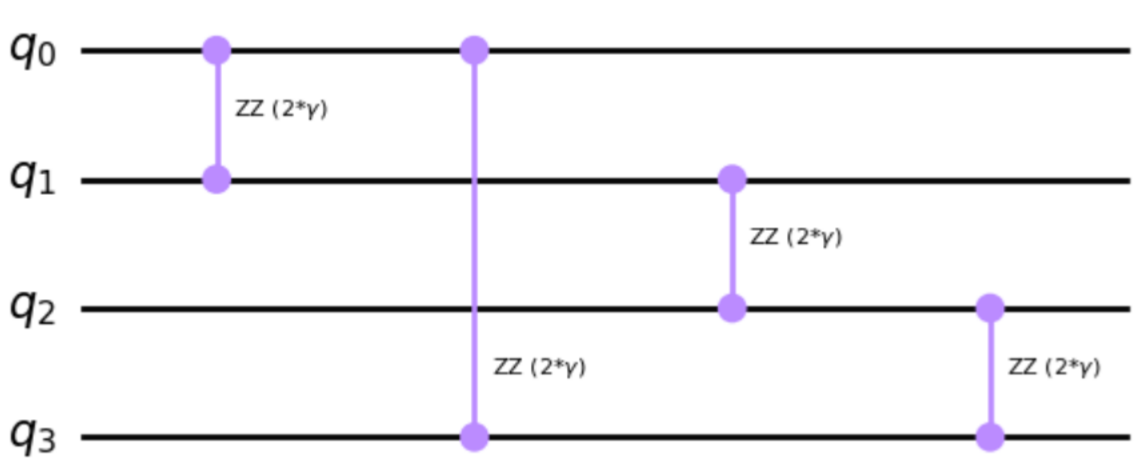


$$H_C = \sigma_z^{(0)}\sigma_z^{(1)} + \sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(2)}\sigma_z^{(3)} + \sigma_z^{(3)}\sigma_z^{(0)}$$

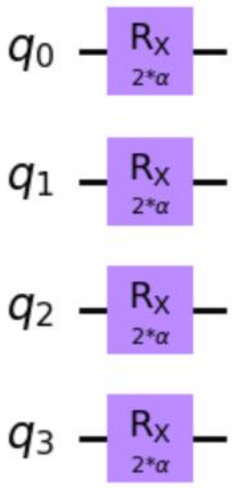
$$\begin{aligned} U_C &= \exp(-i\gamma H_C) \\ &= \exp\left(-i\gamma\sigma_z^{(0)}\sigma_z^{(1)}\right)\exp\left(-i\gamma\sigma_z^{(1)}\sigma_z^{(2)}\right) \times \\ &\quad \exp\left(-i\gamma\sigma_z^{(2)}\sigma_z^{(3)}\right)\exp\left(-i\gamma\sigma_z^{(3)}\sigma_z^{(0)}\right) \end{aligned}$$

Quantum Approximate Optimization Algorithm (QAOA)

Example: Max-Cut Problem



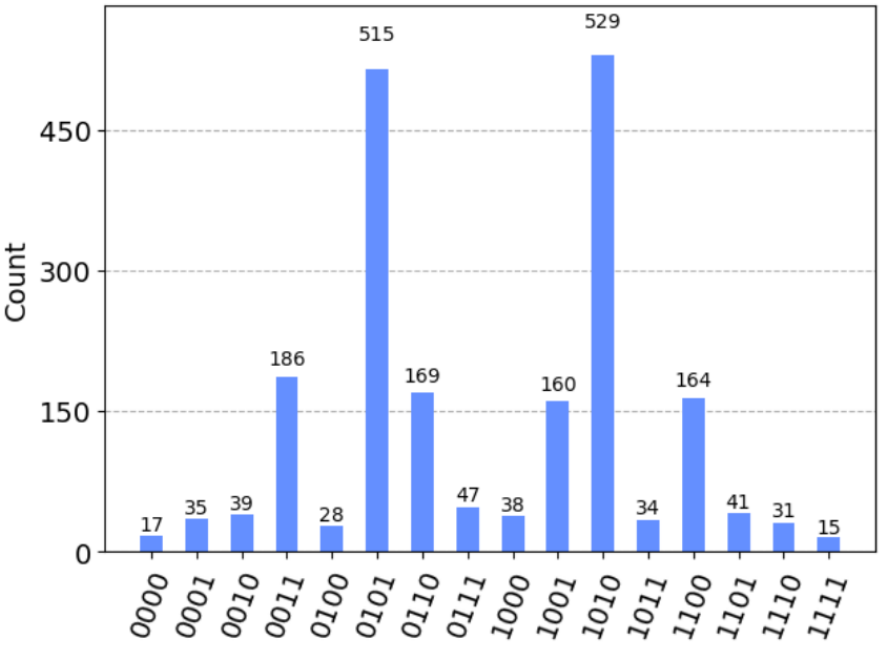
U_C

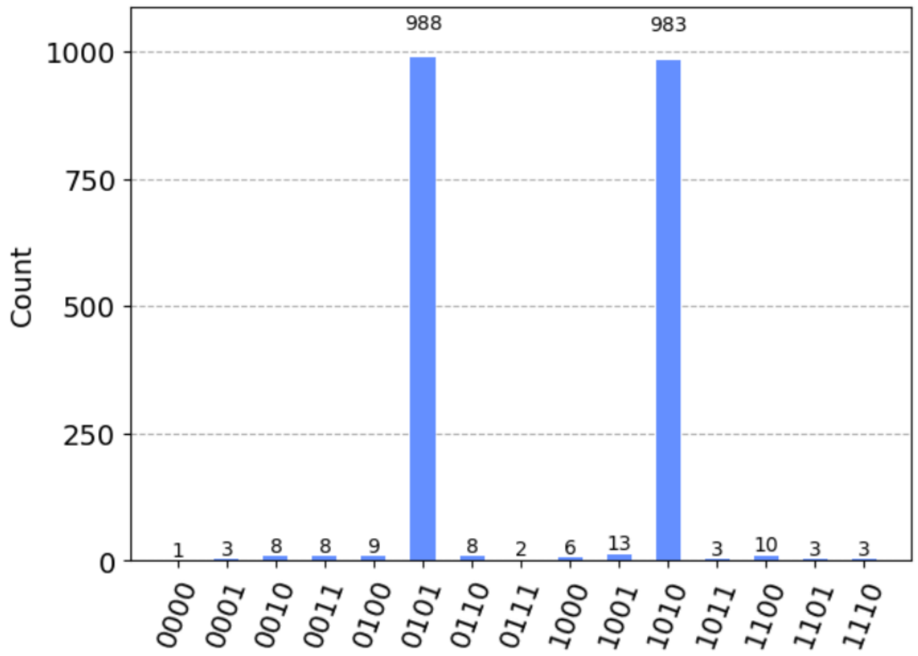


U_M

Quantum Approximate Optimization Algorithm (QAOA)

Example: Max-Cut Problem





Binary Quadratic Model (BQM)

Definition

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

Cost Function: $\boldsymbol{x}^T Q \boldsymbol{x}$

$$Q = \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n} \\ 0 & a_2 & b_{2,3} & \cdots & b_{2,n} \\ 0 & 0 & a_3 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

$x_i \in \{0,1\}$

Qubo Matrix

Binary Quadratic Model (BQM)

Cost Function

$$\boldsymbol{x}^T Q \boldsymbol{x} = \sum_{i} a_i x_i + \sum_{j>i} b_{i,j} x_i x_j$$

D-Wave Quantum Computer

Hamiltonian

$$H = \sum_{i} a_i \sigma_z^{(i)} + \sum_{j>i} b_{ij} \sigma_z^{(i)} \sigma_z^{(j)}$$

D-Wave Quantum Computer

Systems

	D-Wave One	D-Wave Two	D-Wave 2X	D-Wave 2000Q	Advantage	Advantage 2
Release date	2011	2013	2015	2017	2020	2023-2024
Topology	Chimera	Chimera	Chimera	Chimera	Pegasus	Zephyr
Qubits	128	512	1152	2048	5000+	7440
Couplers	352	1,472	3,360	6,016	35,000+	
Active area				$5.5 \times 5.5\ (\mathrm{mm})^2$	$8.4 \times 8.4\ (\mathrm{mm})^2$	
Power consumption (kW)		15.5	25	25	25	

D-Wave Quantum Computer

Systems

	D-Wave One	D-Wave Two	D-Wave 2X	D-Wave 2000Q	Advantage	Advantage 2
Buyers	Lockheed Martin	•Google/NASA/USRA •Lockheed Martin	•Los Alamos National Laboratory •Google/NASA/USRA •Lockheed Martin	•Temporal Defense Systems •Google/NASA/USRA •Los Alamos National Laboratory	•Lockheed Martin •Los Alamos National Laboratory •Jülich Supercomputing Centre	

Travelling Salesman Problem

Quadratic Binary Model: Binary Variables

x_{ij}: City i be visited at time j

$$\boldsymbol{x} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \\ \vdots \\ x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{pmatrix}$$

Travelling Salesman Problem

Quadratic Binary Model: Hamiltonian

$$H = A\sum_{i}\left(1-\sum_{j}x_{i,j}\right)^2 + A\sum_{j}\left(1-\sum_{i}x_{i,j}\right)^2 + B\sum_{uv}w_{u,v}\sum_{i}x_{u,i}\,x_{v,i+1}$$

Feasible Solution

$$A \geq B\ \mathrm{Max}[w_{i,j}]$$

Related Works

- Vargas-Calderón, V., Parra-A, N., Vinck-Posada, H., & González, F. A. (2021). Many-qudit representation for the travelling salesman problem optimisation. *Journal of the Physical Society of Japan*, *90*(11), 114002.
- Carvalho, I. (2022). QUBO formulations for NP-Hard spanning tree problems. *arXiv preprint arXiv:2209.05024*.
- Held, M., & Karp, R. M. (1970). The traveling-salesman problem and minimum spanning trees. *Operations Research*, *18*(6), 1138-1162.

|Thanks for Your Attention⟩