# The Landau-Streater Channel as a Noisy Quantum Channel

**Presenter:**
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# The Landau-Streater Channel as a Noisy Quantum Channel

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**Abstract**

In three dimensions, the Landau-Streater channel is nothing but the Werner-Holevo channel. Such a channel has no continuous parameter and hence cannot model an environmental noise. We consider its convex combination with the identity channel, making it suitable as a one-parameter noise model on qutrits. Moreover, whereas the original Werner-Holevo channel exhibits covariance under the complete unitary group $SU(3)$, the extended family maintains covariance only under the group $SO(3)$. This symmetry reduction allows us to investigate its impact on various properties of the original channel. In particular, we examine its influence on the channel's spectrum, divisibility, complementary channel, and exact or approximate degradability, as well as its various kinds of capacities. Specifically, we derive analytical expressions for the one-shot classical capacity and the entanglement-assisted capacity, accompanied by the establishment of lower and upper bounds for the quantum capacity.

**Keywords:** Landau-Streater channel, Classical capacity, Quantum Capacity, Flag-Extension

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# Quantum Channels
## Representations

- Completely Positive Trace-preserving map
- Kraus representation

$$\rho \to \Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$\sum_i K_i^\dagger K_i = I$$

# Quantum Channels

## Representations

- Completely Positive Trace-preserving map
- Choi representation

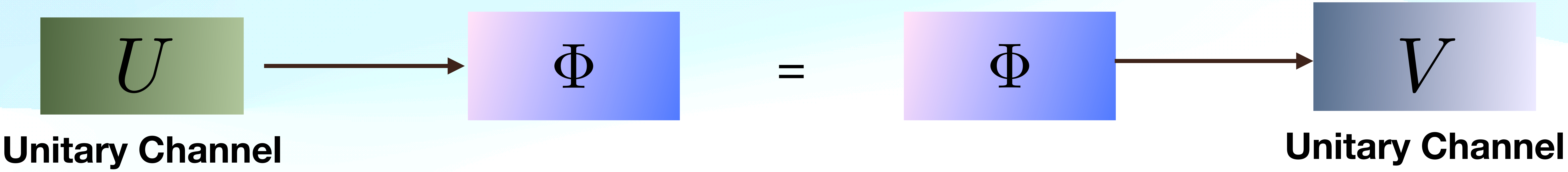
$$J(\Phi) = \sum_{ij} (|i\rangle\langle j|) \otimes (\Phi(|i\rangle\langle j|))$$

$J(\Phi) \geq 0$ ⟶ Completely Positive

$Tr_2(J(\Phi)) = I_1$ ⟶ Trace preserving

# Quantum Channels

## Covariance




$$\Phi(U\rho U^\dagger) = V\Phi(\rho)V^\dagger$$

# Landau-Streater Channel

## Unital Qubit Channels

- d=2:

$$\Phi : \mathcal{H}_2 \to \mathcal{H}_2$$

$$\Phi(I_2) = I_2$$

$$\Rightarrow \quad \Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger$$

- But that's not the case for higher dimensions!

# Landau-Streater Channel

## In three dimensions

$$\Lambda_1(\rho) = \frac{1}{2}\left(J_x \rho J_x + J_y \rho J_y + J_z \rho J_z\right)$$

- SO(3) generators as Kraus operators:

$$J_x = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J_y = -i \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J_z = -i \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Landau-Streater Channel

## in three dimensions is nothing But Werner-Holevo Channel

$$\Lambda_1(\rho) = \Lambda_{WH}(\rho) = \frac{1}{2}(\mathrm{Tr}(\rho)I_3 - \rho^{\mathrm{T}})$$

- Covariance:

$$\Lambda_1(U\rho U^\dagger) = U^*\Lambda_1(\rho)U^T$$

For all $U \in SU(3)$

# Noisy Quantum Channel

$$\Lambda_x(\rho) = (1-x)\rho + x\Lambda_1(\rho)$$

$$\Lambda_x(\rho) = (1-x)\rho + \frac{x}{2}(Tr(\rho)I - \rho^T)$$

- Covariance:

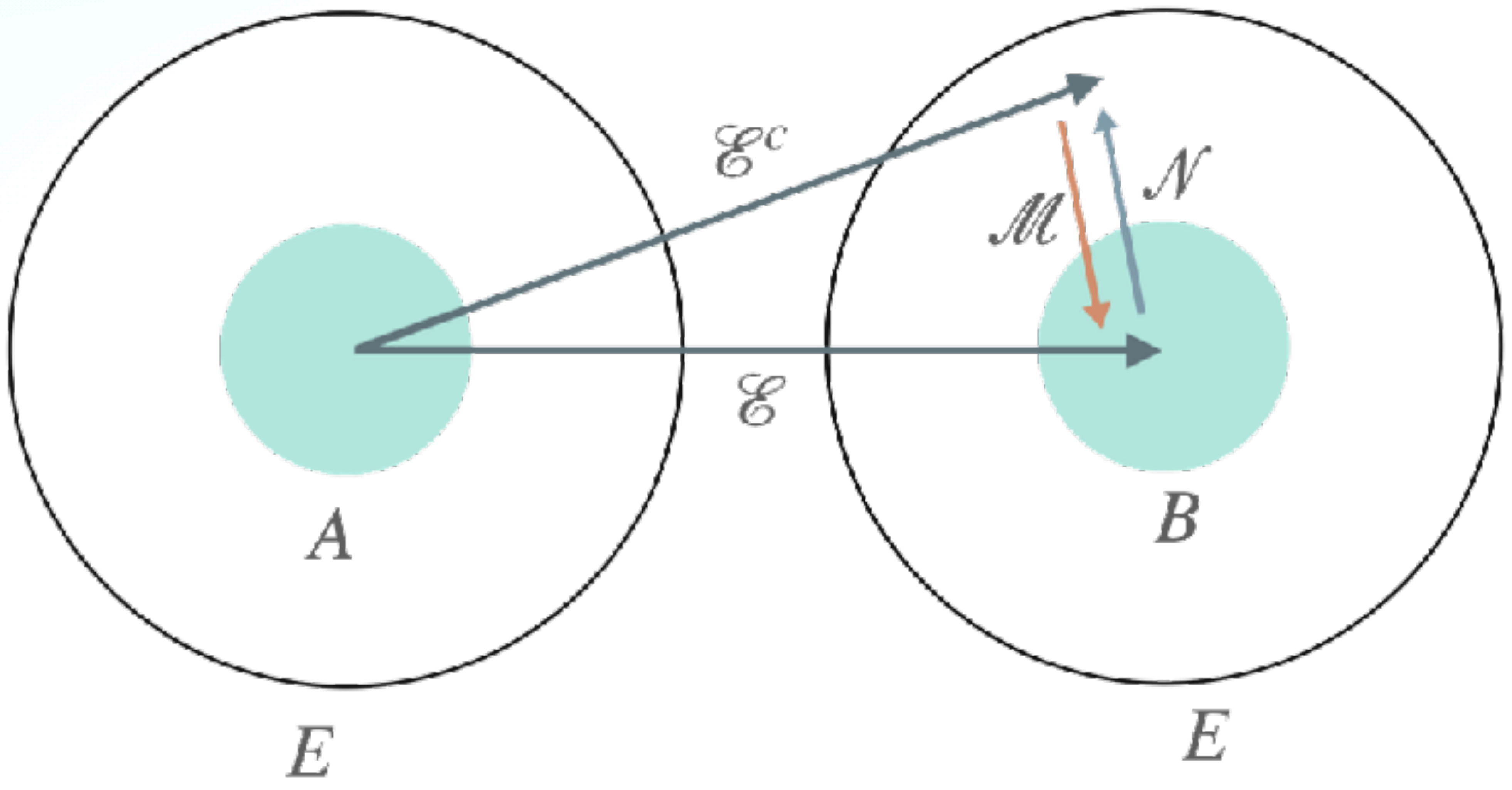
$$\Lambda_x(U\rho U^\dagger) = U\Lambda_x(\rho)U^\dagger \qquad \forall U \in SO(3)$$

# Complementary Channel

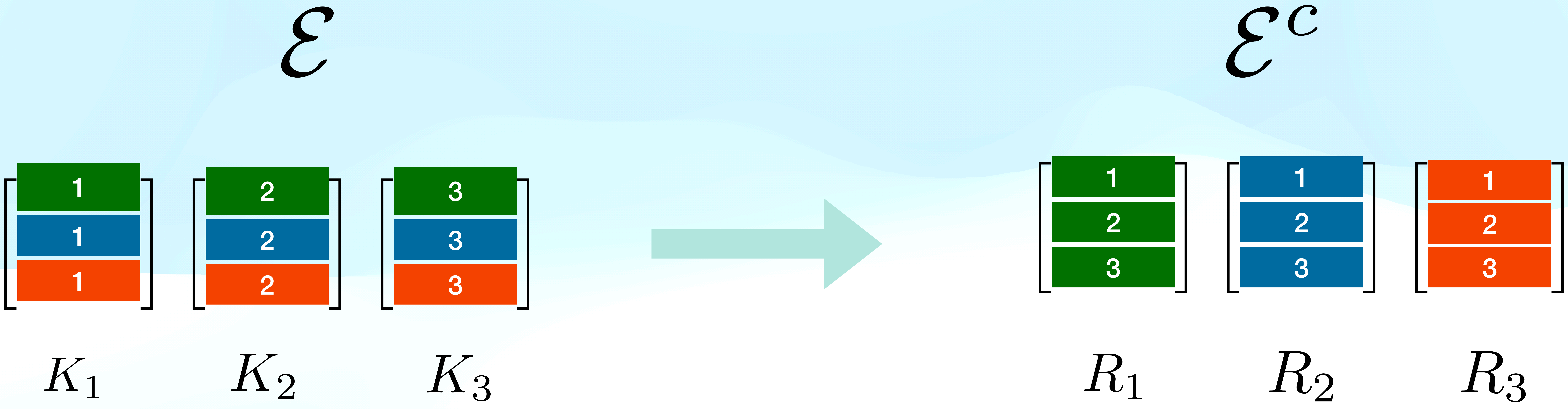
$$\Phi : A \to B$$

$$\Phi(\rho_A) = \mathrm{tr}_{E'}(U(\rho_A \otimes \sigma_E)U^\dagger)$$

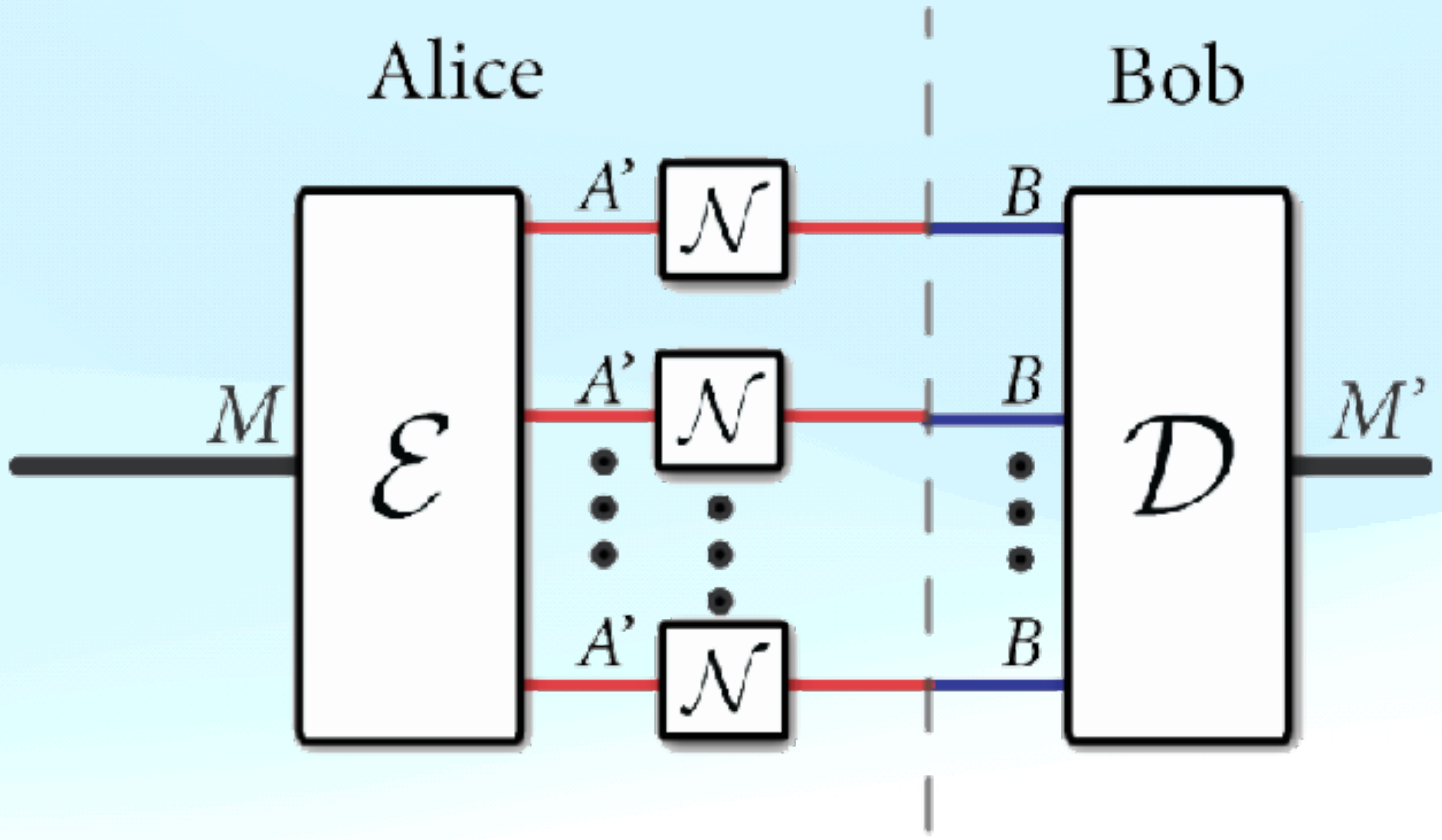
$$\Phi^c(\rho_A) = \mathrm{tr}_B(U(\rho_A \otimes \sigma_E)U^\dagger)$$



# Kraus Operators of Complementary Channel



# Classical Capacity




$\{1, 2, ..., |\mathcal{M}|\}$

$m \longrightarrow \rho^m_{A'^n}$

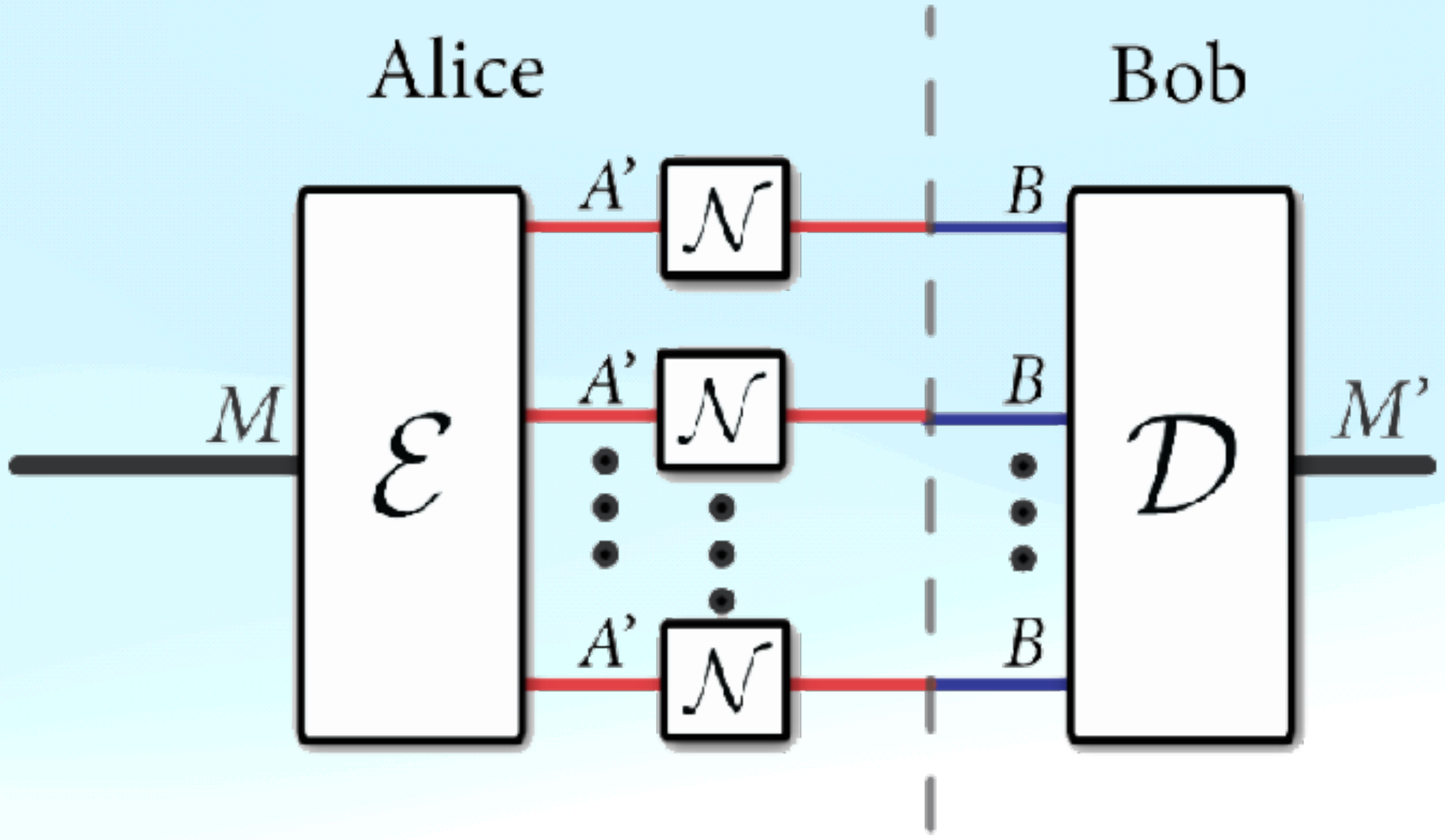
$\Phi^{\otimes n}(\rho^m_{A'^n})$

$\mathcal{P}_m$

Alice

Bob

# Classical Capacity




$$P(M' = m|M = m) = \mathrm{Tr}\big(\mathcal{P}_m \Phi^{\otimes n}(\rho^m_{A'^n})\big)$$

$$\sum_i \mathcal{P}_i = I$$

$$P(\mathrm{error}|M = m) = \mathrm{Tr}\big((I - \mathcal{P}_m)\Phi^{\otimes n}(\rho^m_{A'^n})\big)$$

$$R = \frac{\log |\mathcal{M}|}{n}$$

# Holevo-Schumacher-Westmoreland

$$C_{cl}(\Lambda) = \lim_{n \longrightarrow \infty} \frac{1}{n} \chi^*(\Lambda^{\otimes n}),$$

$$\chi^*(\Lambda) = \max_{p_i, \rho_i} \ S(\sum_i p_i \Lambda(\rho_i)) - \sum_i p_i S(\Lambda(\rho_i))$$

If the channel is irreducibly covariant:

$$\chi^*(\Lambda) = \log d - \min_{\rho} S\left(\Lambda(\rho)\right)$$

$$\min_{\rho} S\left(\Lambda_x(\rho)\right)$$

Because of Concavity it's sufficient to consider pure states.

$$|\Psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

Output entropy is invariant with respect to covariance of quantum channel

$$S(\Lambda_x(U\rho U^\dagger)) = S(\Lambda_x(\rho)) \qquad \forall U \in SO(3)$$

Exploiting Covariance of Channel, we can decrease the parameters of optimization further.

$$|\Psi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} \xrightarrow{R_z} \begin{bmatrix}\psi_1\\ r_2\\ \psi_3\end{bmatrix} \xrightarrow{R_y} \begin{bmatrix}\psi_1\\ r_2\\ r_3\end{bmatrix} \xrightarrow{R_x} \begin{bmatrix}\cos\theta e^{i\phi}\\ \sin\theta\\ 0\end{bmatrix}$$

$r_2 \in R$

$r_2,\ r_3 \in R$

$$R_z = \begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix} \qquad R_y = \begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix} \qquad R_x = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{pmatrix}$$

$$Im(-\psi_1\sin\alpha + \psi_2\cos\alpha) = 0$$

$$-r_2\sin\gamma + r_3\cos\gamma = 0$$

Only two parameters remain.

- Input State:

$$\begin{bmatrix} \cos\theta e^{i\phi} \\ \sin\theta \\ 0 \end{bmatrix}$$

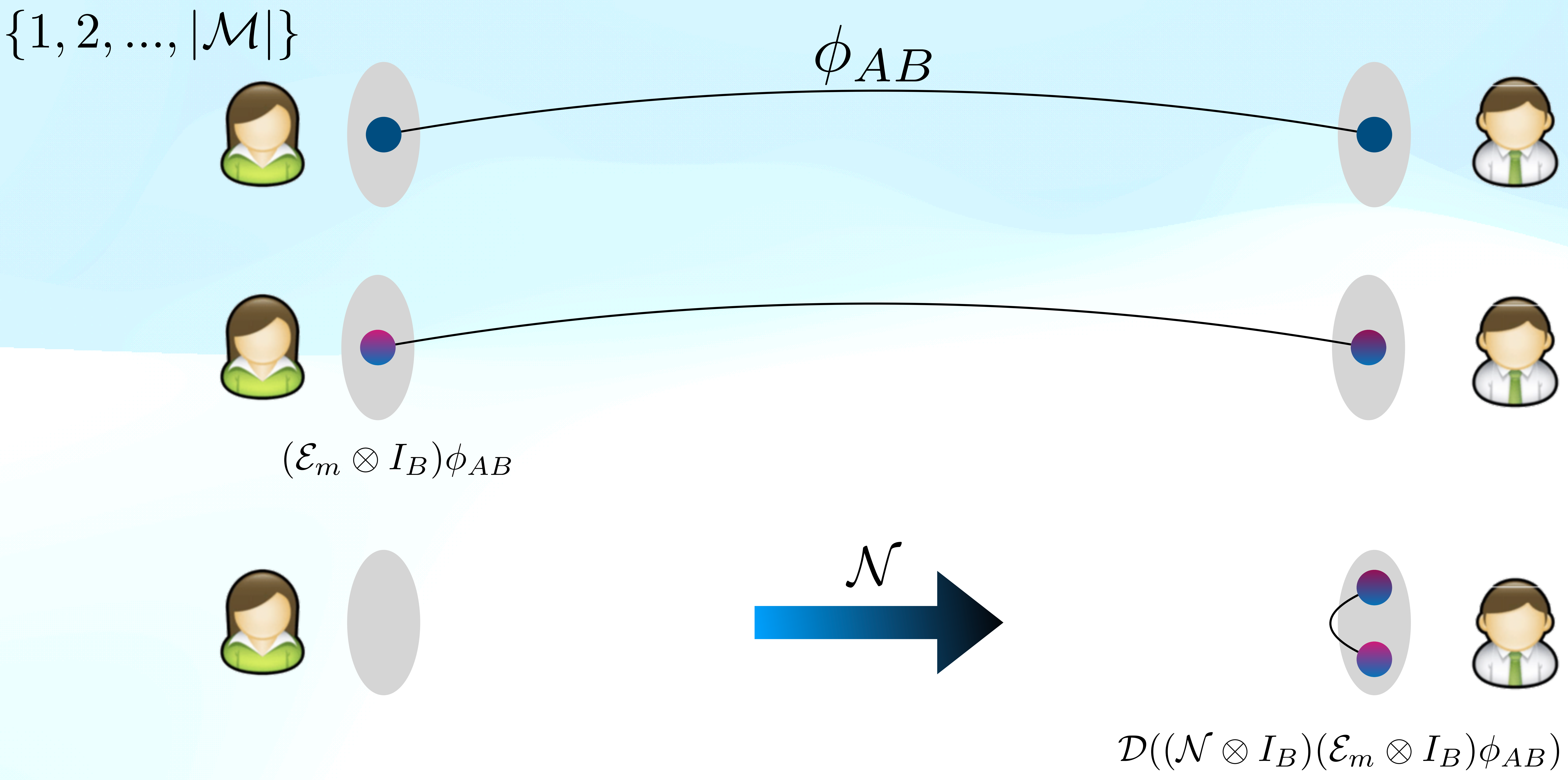
- Parameters:

$$\theta, \phi$$

$$\min_{\phi,\theta} S(\Lambda_x) = -\frac{x}{2}\log\frac{x}{2} - (1-\frac{x}{2})\log\left(1-\frac{x}{2}\right).$$

$$\chi^*(\Lambda_x) = \log_2 3 + \frac{x}{2}\log\frac{x}{2} + (1-\frac{x}{2})\log\left(1-\frac{x}{2}\right)$$

# Entanglement-Assisted Capacity



# Entanglement-Assisted Capacity Theorem

## Bennet-Smolin-Shor-Thapliyal

$$C_{ea}(\Lambda_x) = \max_{\rho} I(\rho, \Lambda)$$

$$I(\rho, \Lambda) := S(\rho) + S(\Lambda_x(\rho)) - S(\rho, \Lambda_x).$$

$$S(\rho, \Lambda_x) = S(\Lambda_x^c(\rho))$$

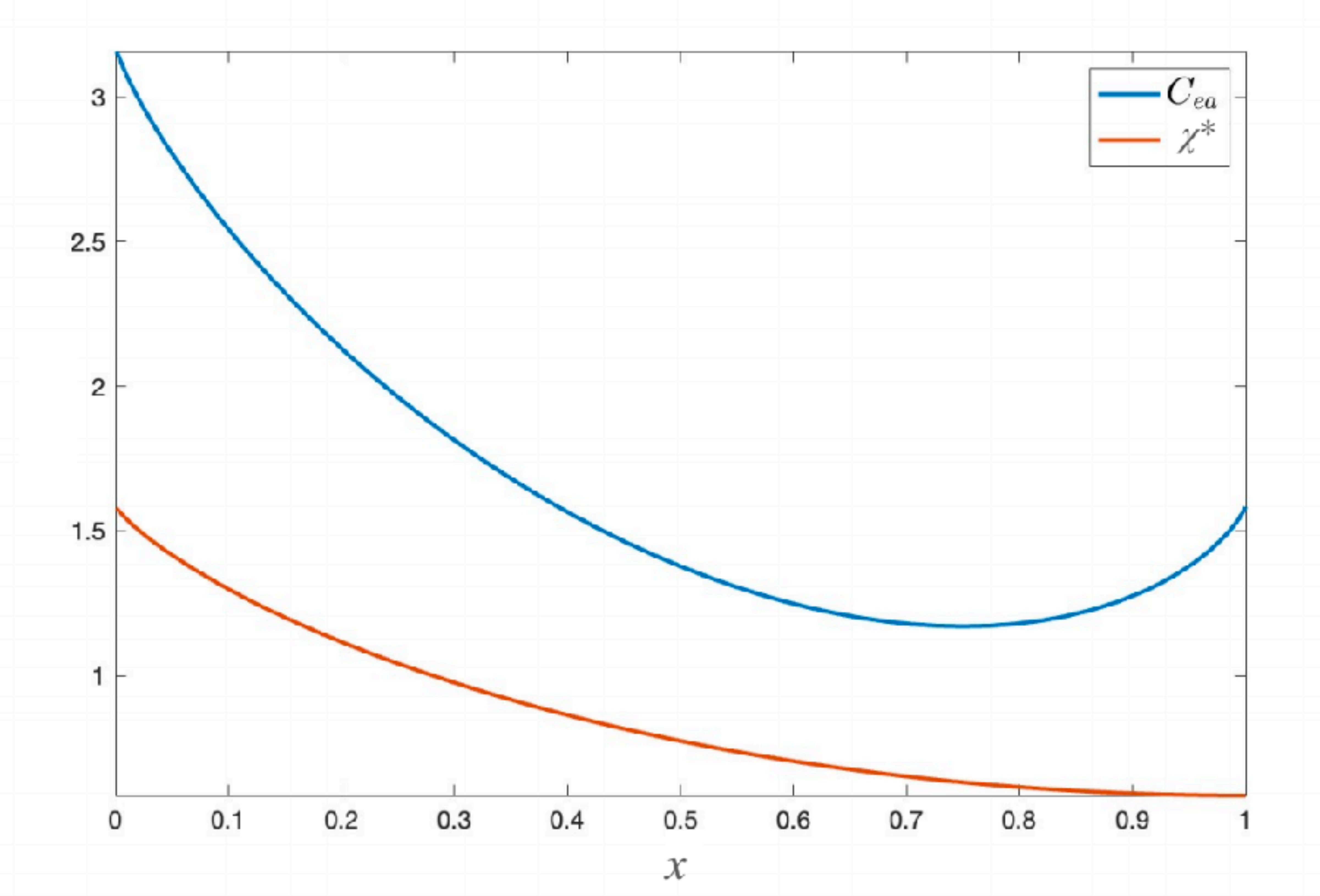
# Entanglement-Assisted Capacity

## The channel

- Channel mutual information is concave
- Channel mutual information is invariant under the channel's covariance
- For irreducible covariant channel is achieved in maximally mixed state[Holevo].

$$C_{ea}(\Lambda_x) = 2S(\frac{I}{3}) - S(\Lambda^c(\frac{I}{3})) = 2\log_2 3 - S(\Lambda^c(\frac{I}{3})).$$

# Classical Capacities



# Quantum Capacity

## Schumacher-nielsen-Lloyd

$$Q_1(\Phi) = \max_{\rho} I_c(\Phi, \rho) = \max_{\rho} S(\Phi(\rho)) - S(\Phi^c(\rho))$$

$$Q(\Phi) = \lim_{n \to \infty} \frac{1}{n} Q_1(\Phi^{\otimes n})$$

# Upper Bounds

- Partial Transposition Bound
- $\epsilon$-degradability and $\epsilon$-Antidegradabilitiy

✓ Flagged Extension Upper bound

✓ Semi-definite programming

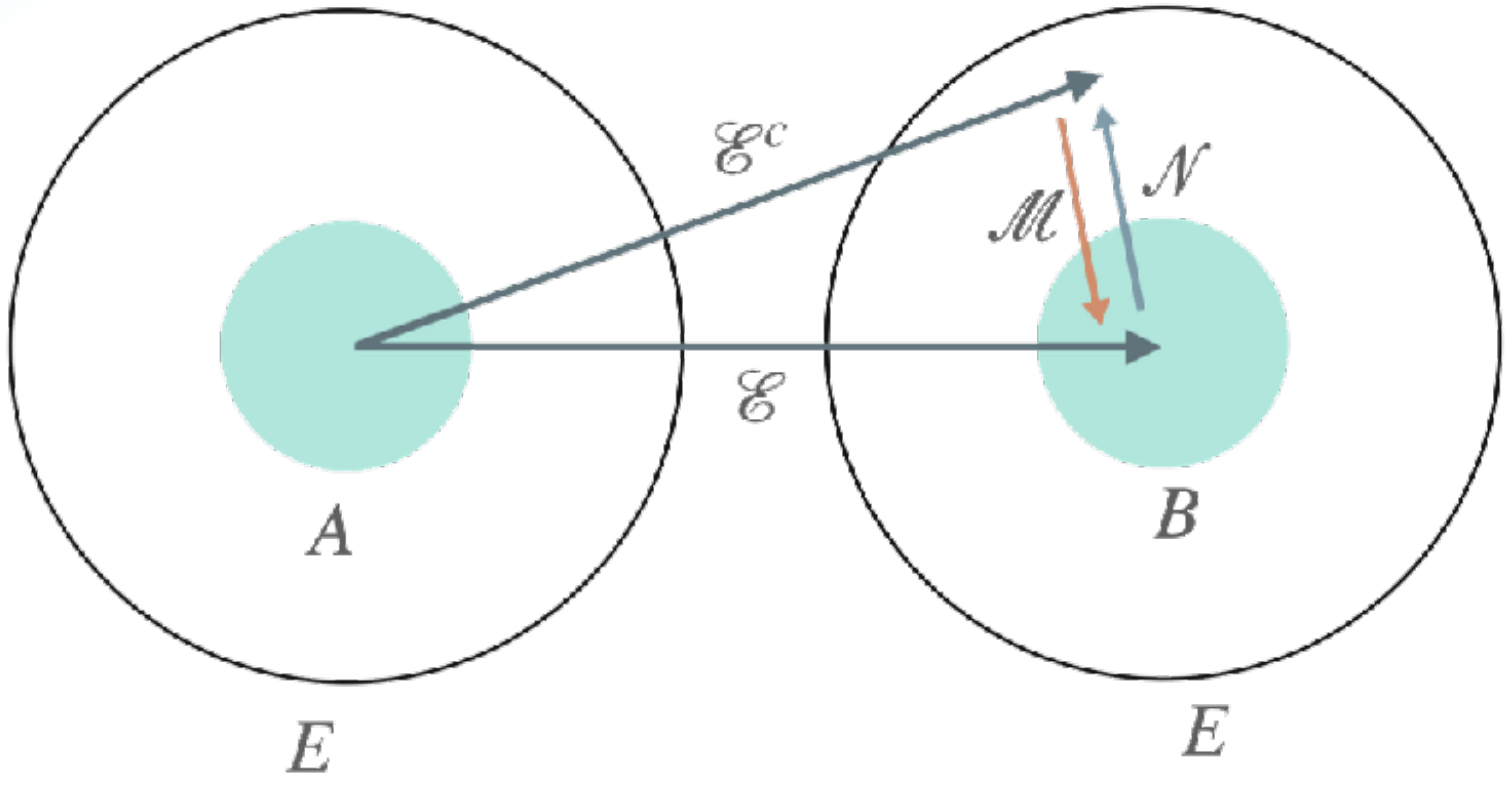
# Degradable and Anti-degradable Channels

- Degradable Channel:

$$\mathcal{N} \circ \mathcal{E}(\rho) = \mathcal{E}^c(\rho) \qquad Q(\mathcal{E}) = Q^1(\mathcal{E})$$

- Anti-Degradable Channel:

$$\mathcal{M} \circ \mathcal{E}^c(\rho) = \mathcal{E}(\rho) \qquad Q(\mathcal{E}) = 0$$



# Degradable and Anti-degradable Channels

$$\Lambda_x(\rho) = (1-x)\rho + x\Lambda_1(\rho)$$

- X=0:

$$\Lambda_0(\rho) = \rho$$

Degradable

- X=1:

$$\Lambda_1(\rho)$$

→ Degradable

→ Anti-Degradable

# Flagged Extension UpperBound

- Flagged Extension Upper Bound

$$\Phi = \sum_i p_i \Phi_i$$

$$Q(\Phi) \leq \sum_i p_i Q(\Phi_i) = \sum_i p_i Q^1(\Phi_i)$$

# Flagged Extension UpperBound

$$\Lambda_x(\rho) = (1-x)\rho + x\Lambda_1(\rho)$$

$$Q(\Lambda_x) \leq (1-x)Q_1(\rho) + xQ_1(\Lambda_1)$$

$\Lambda_1(\rho)$ Anti-Degradable

$$Q(\Lambda_x) \leq (1-x)Q_1(\rho)$$

$$Q(\Lambda_x) \leq (1-x)\log 3$$

→ 1st Upper Bound

# Semi-definite Programming

Maximize $$Tr(AX)$$

Subject to

$$Tr(C_i X) = b_i \qquad 1 \leq i \leq m$$

$$X \geq 0$$

- A is a hermitian matrix

# Semi-definite Upper Bound

$$\Phi : A \to B$$

$$Q(\Phi) \leq Q_\Gamma(\Phi) := \log\left(\max\ \mathrm{Tr}(J(\Phi)R)\right)$$

Subject to:

$$\rho_A, R \geq 0$$

$$Tr(\rho_A) = 1$$

$$-\rho_A \otimes I_B \leq R^{T_B} \leq \rho_A \otimes I_B$$

$R$ is a positive semi-definite matrices in space of $A \otimes B$

# Lower Bound and super additivity

- Super Additivity:

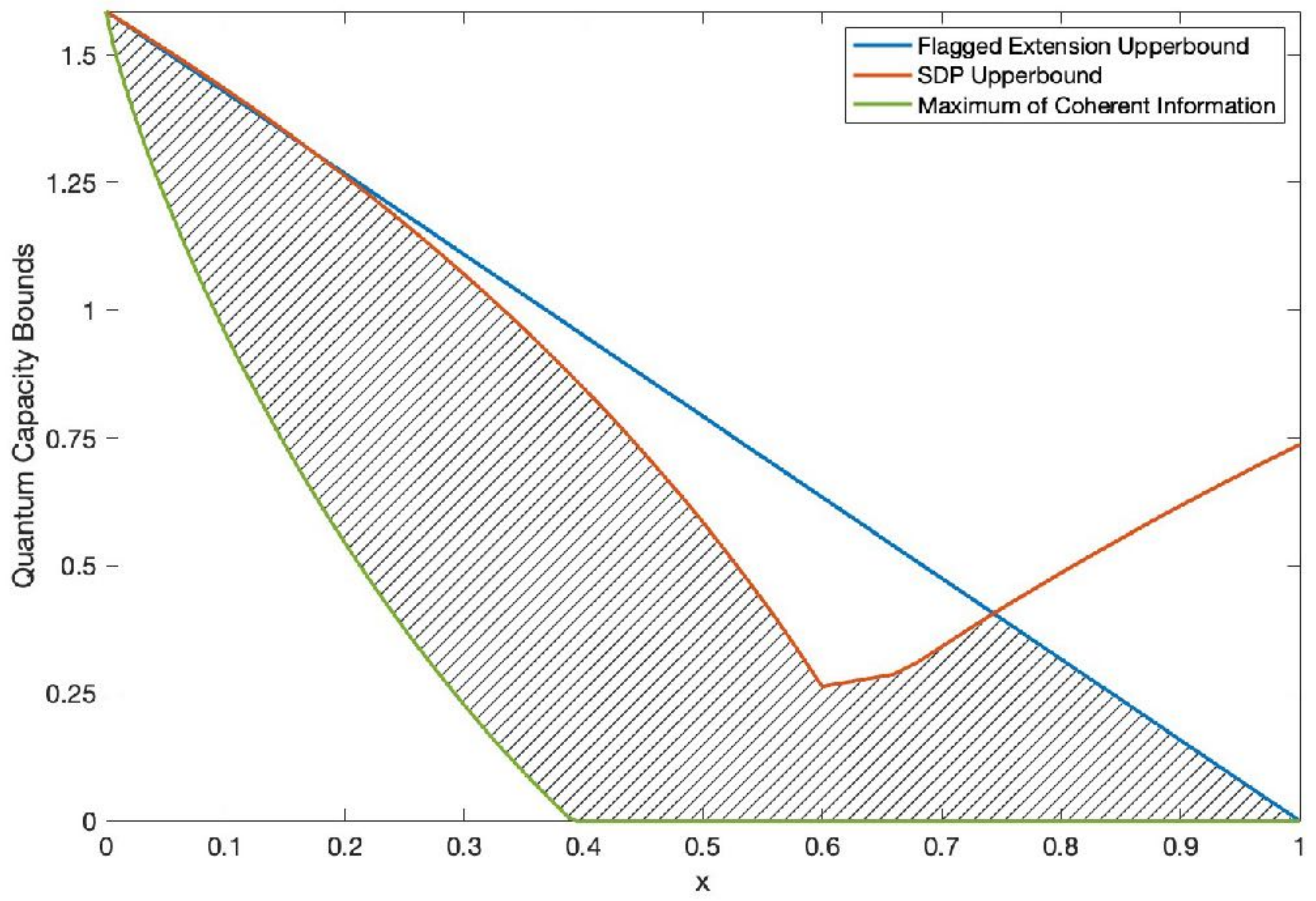
$$Q_1(\Phi_1 \otimes \Phi_2) \geq Q_1(\Phi_1) + Q_1(\Phi_2)$$

- Lower Bound:

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} Q_1(\Phi^{\otimes n}) \geq Q_1(\Phi)$$

Evaluated numerically by Sampling space of density matrices and a local solver

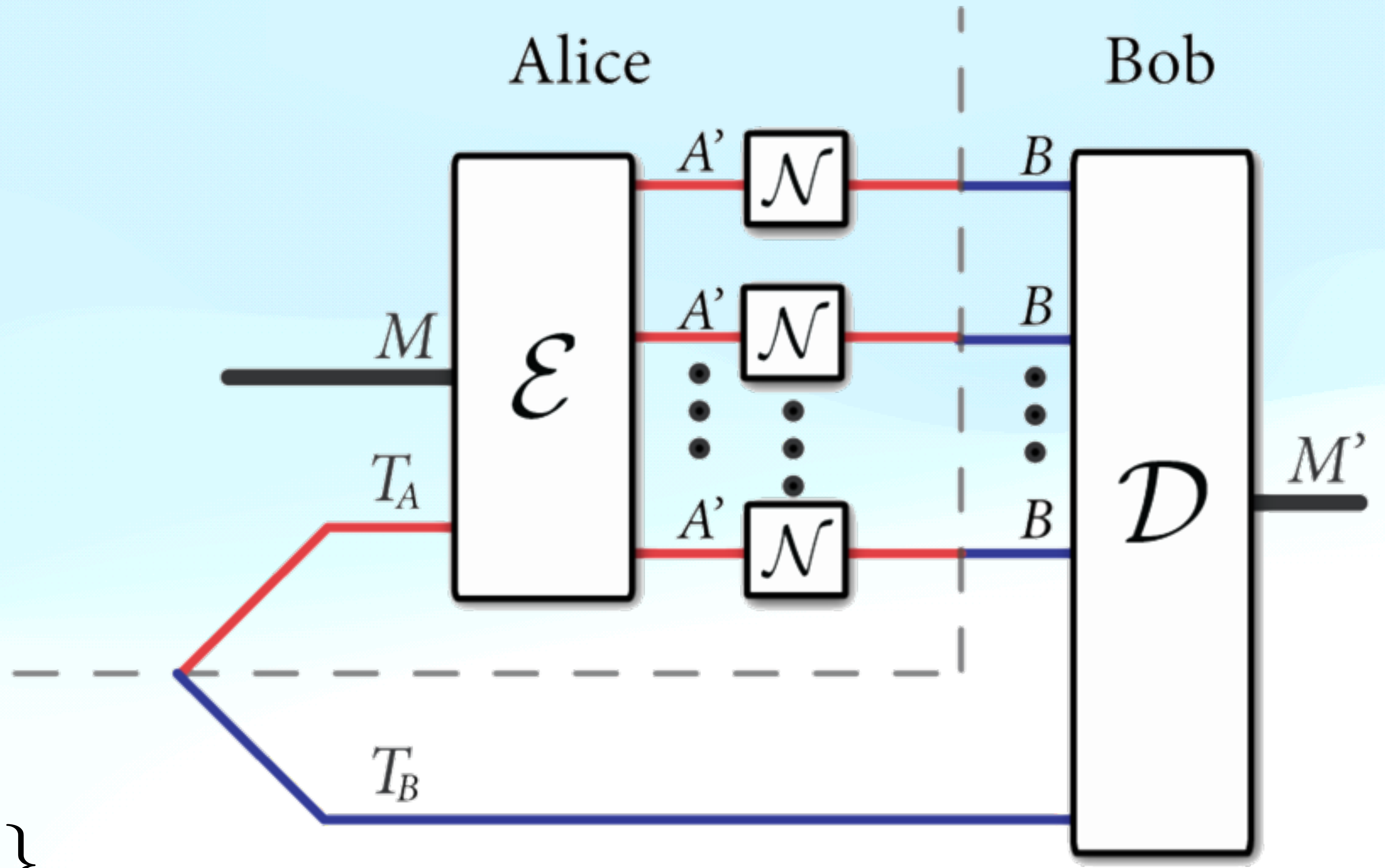
# Bounded Quantum Capacity



# The End

# Thank You for Your Attention

# Entanglement-Assisted Capacity




$\{1, 2, ..., |\mathcal{M}|\}$

$\Psi_{T_A T_B}$

$\mathcal{P}_m$

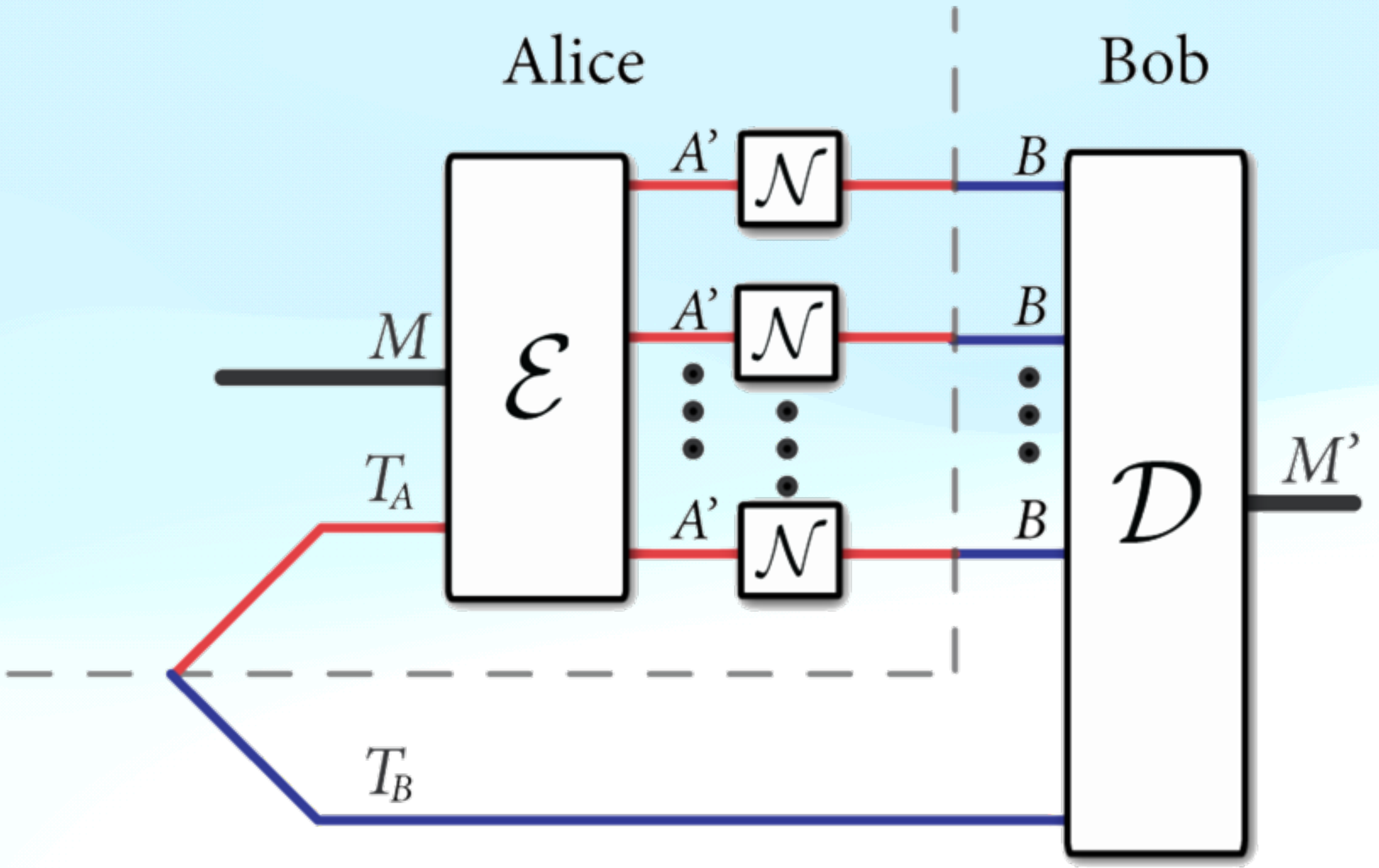
$\mathcal{E}^m_{T_A \to A'^n}(\Psi_{T_A T_B})$

$\mathcal{N}^{\otimes n}(\mathcal{E}^m_{T_A \to A'^n}(\Psi_{T_A T_B}))$

Alice

Bob

# Entanglement-Assisted Capacity

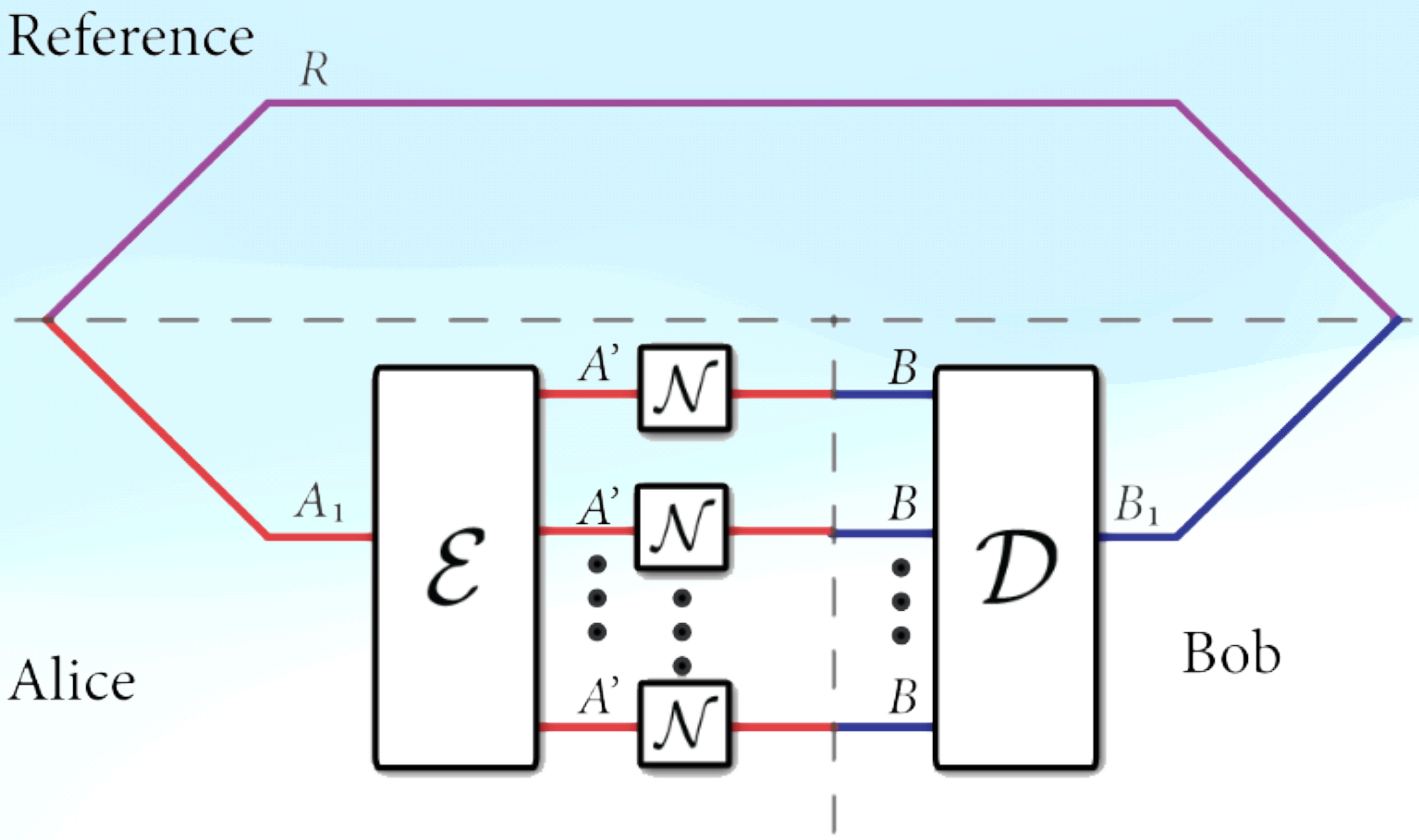



$$R = \frac{\log |\mathcal{M}|}{n}$$

$$P(M' = m|M = m) = \mathrm{Tr}\left[\mathcal{P}_m \mathcal{N}^{\otimes n}(\mathcal{E}^m_{T_A \to A'^n}(\Psi_{T_A T_B}))\right]$$

$$P(\mathrm{error}|M = m) = \mathrm{Tr}\left[(I - \mathcal{P}_m)\mathcal{N}^{\otimes n}(\mathcal{E}^m_{T_A \to A'^n}(\Psi_{T_A T_B}))\right]$$

# Quantum Capacity




$$R = \frac{\log \mathcal{H}_{A_1}}{n}$$

Alice

$\psi_{A_1,R}$

$\mathcal{E}_{A_1 \to A'^n}(\psi_{RA_1})$

$\mathcal{N}^{\otimes n}(\mathcal{E}_{A_1 \to A'^n}(\psi_{RA_1}))$

Bob

$\mathcal{D}(\mathcal{N}^{\otimes n}(\mathcal{E}_{A_1 \to A'^n}(\psi_{RA_1})))$