Quantum speed limits

A . Farmanian

How fast any quantum system can evolve ?

Why does it matter?

- The maximal rate with which quantum information can be communicated
- The maximal rate with which quantum information can be processed
- The shortest time-scale for quantum optimal control algorithms to converge
- …

Energy-time uncertainty: Emergence of the quantum speed limit

$$\Delta x \, \Delta p \;\; \geq \hbar$$

$$\Delta E \, \Delta t \;\; \geq \hbar$$

W. Heisenberg. "Die physikalischen Prinzipien der Quantumtheorie". 1927.

The uncertainty relation of Mandelstam and Tamm

$$< \psi_i | \psi_f > = 0$$

$$\tau_{QSL} \leq \ time\ of\ evelotion$$

$$\tau_{QSL} = \frac{\pi\hbar}{2\Delta H}$$

$$\Delta H = \sqrt{< H^2 > \ - < H >^2}$$

L. Mandelstam and I. Tamm. The uncertainty relation between energy and time in nonrelativistic quantum mechanics. 1945

The uncertainty relation *interpretation*

Y. Aharonov and D. Bohm. Time in the quantum theory and the uncertainty relation for time and energy. 1961

The uncertainty relation of Mandelstam and Tamm

$$|\psi(0)> = \sum_{0}^{\infty} c_i \,|E_i>$$

$$|\mathrm{S(t)}|^2 = (<\psi(0)|\,\psi(t)>)^2 = 0$$

$$= \sum_{n,n'=0}^{\infty} |C_n|^2 |C_{n'}|^2 e^{i(E_n - E_{n'})t/\hbar} = \sum_{n,n'=0}^{\infty} |C_n|^2 |C_{n'}|^2 \cos\frac{(E_n - E_{n'})t}{\hbar}$$

L. B. Levitin and Y. Toffoli. Fundamental limit on the rate of quantum dynamics: The unified bound is tight. 2009

The uncertainty relation of Mandelstam and Tamm

For $x \geq 0$:

$$\cos x \geq 1 - \frac{4}{\pi^2} x \sin x - \frac{2}{\pi^2} x^2$$

$$|S(t)|^2 \geq 1 - \frac{4}{\pi^2} \sum_{n,n'=0}^{\infty} |C_n|^2 |C_{n'}|^2 \frac{(E_n - E_{n'})t}{\hbar} \sin \frac{(E_n - E_{n'})t}{\hbar} - \frac{2}{\pi^2} \sum_{n,n'=0}^{\infty} |C_n|^2 |C_{n'}|^2 \left(\frac{(E_n - E_{n'})t}{\hbar}\right)^2$$

$$= 1 + \frac{4}{\pi^2} \frac{d|S(t)|^2}{dt} - \frac{1}{\pi^2} \left(\frac{\Delta E}{\hbar/2t}\right)^2$$

$$\frac{d|S(t)|^2}{dt} = 0 \implies 0 \geq 1 - (\Delta E)^2 \frac{4t^2}{\pi^2 \hbar^2} \quad \Rightarrow \quad \tau_{QSL} = \frac{\pi \hbar}{2 \Delta H}$$

The quantum speed limit of Margolus and Levitin

$$|\psi> = \sum_{0}^{\infty} \psi_i |E_i> \quad \& \quad <\psi_i|\psi_f> = 0$$

$$\tau_{QSL} \leq \ time\ of\ evelotion$$

$$\tau_{QSL} = \frac{\pi\hbar}{2 < H >} \qquad < H > = \sum \frac{E_i}{N} \quad if \quad E_{min} = 0$$

N. Margolus and L. B. Levitin. The maximum speed of dynamical evolution.1998

Margolus and Levitin relation

$$S(t) = < \psi(0)| \, \psi(t) >$$

$$\Rightarrow Re\, S(t) = \sum_{n=0}^{\infty} |C_n|^2 \, \cos\frac{E_n t}{\hbar}$$

For $x \geq 0$:

$$\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$$

$$\Rightarrow Re\, S(t) = \sum_{n=0}^{\infty} |C_n|^2 \, \cos\frac{E_n t}{\hbar} \geq \sum_{n=0}^{\infty} |C_n|^2 \left[1 - \frac{2}{\pi}\left(\frac{E_n t}{\hbar} + \sin\frac{E_n t}{\hbar}\right)\right]$$

$$= 1 - \frac{2Et}{\pi\hbar} + \frac{2}{\pi} \, \mathrm{Im}\, \mathrm{S(t)} \Rightarrow \mathbf{0 \geq 1 - \frac{2Et}{\pi\hbar}}$$

L. B. Levitin and Y. Toffoli. Fundamental limit on the rate of quantum dynamics: The unified bound is tight. 2009

The unified bound is tight

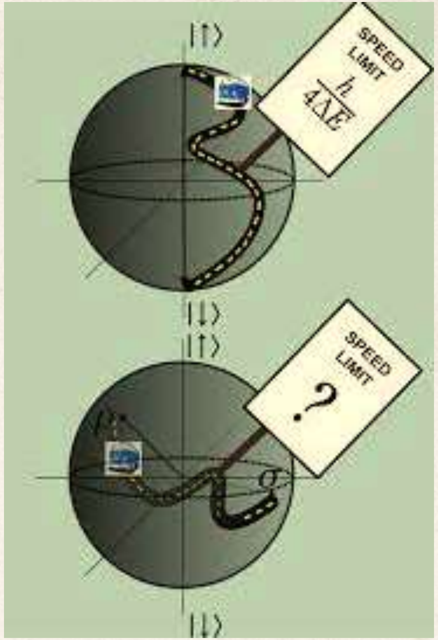
$$\tau_{QSL} = \max\{\frac{\pi\hbar}{2<H>}, \frac{\pi\hbar}{2\Delta H}\}$$

Bound above is attained only by the two-level state :

$$|\psi> = \frac{1}{\sqrt{2}}(|\psi_0> + |\psi_1>)$$

L. B. Levitin and Y. Toffoli. Fundamental limit on the rate of quantum dynamics: The unified bound is tight.2009

Arbitrary angles for pure states



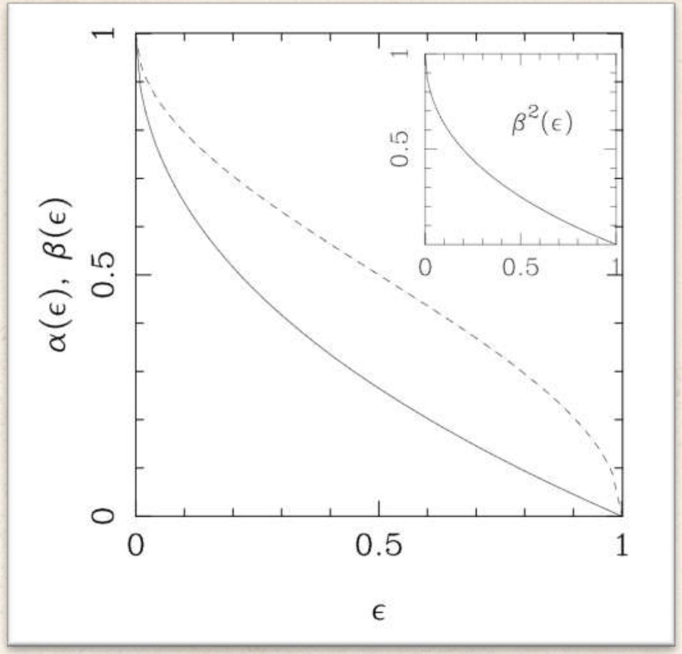
$$\varepsilon(t) = |<\psi(0)|\psi(t)>|^2$$

$$\tau_{QSL} = \max\{\alpha(\varepsilon)\frac{\pi\hbar}{2<H>}, \beta(\varepsilon)\frac{\pi\hbar}{2\Delta H}\}$$

-Giovannetti,Lloyd and Maccone, 2003

Numerical value of the functions $\alpha(\varepsilon)$ and $\beta(\varepsilon)$

$$\beta(\varepsilon) = \frac{2}{\pi}\cos^{-1}\sqrt{|\varepsilon|}$$



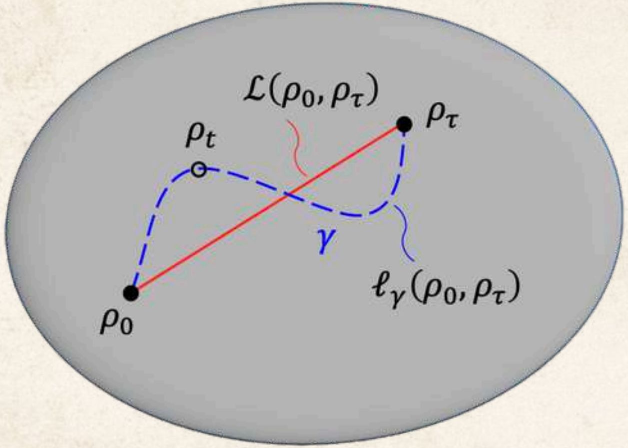
Plot of $\alpha(\varepsilon)$ (continuous line) and $\beta(\varepsilon)$ (dashed line)

bounds for mixed states

The angle between two density operators ρ_0 and ρ_τ :

$$\mathcal{L}(\rho_0\,,\rho_\tau) = \cos^{-1}\sqrt{F(\rho_\tau,\rho_0)}$$

$$F(\rho_\tau,\rho_0) = [tr(\sqrt{\rho_0\,\sqrt{\rho_\tau}\rho_0})]^2$$



Bures ,1969

Mandelstam-Tamm bound for mixed states

Bures metric :

$$d\mathcal{L}^2 = tr\{d\rho\mathcal{R}_\rho^{-1}(d\rho)\}$$

$$\mathcal{R}_\rho^{-1}(O) = \frac{1}{2}\sum_{j,k}\frac{< j|O|K >}{p_j + p_k}|j >< K|$$

$$\frac{\hbar}{\Delta E_\tau}\mathcal{L}(\rho_0\,,\rho_\tau) \leq \tau$$

S. Deffner and E. Lutz. Energy-time uncertainty relation for driven quantum systems. 2013

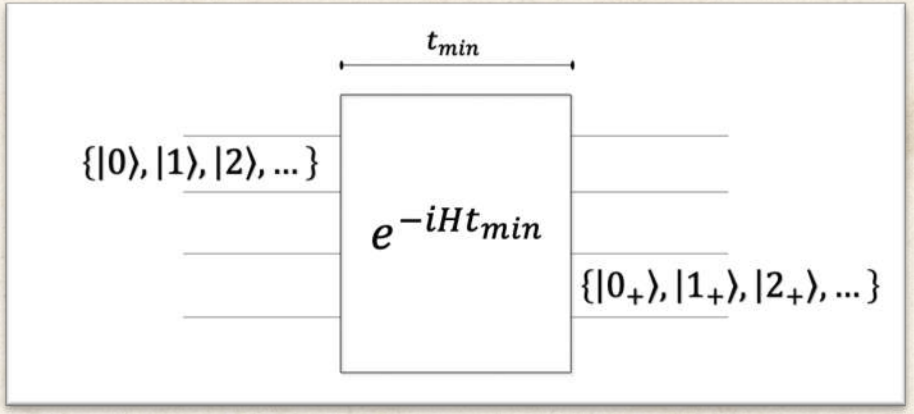
Part 2

(Quantum speed limits for change of basis)

Mutually unbiased bases

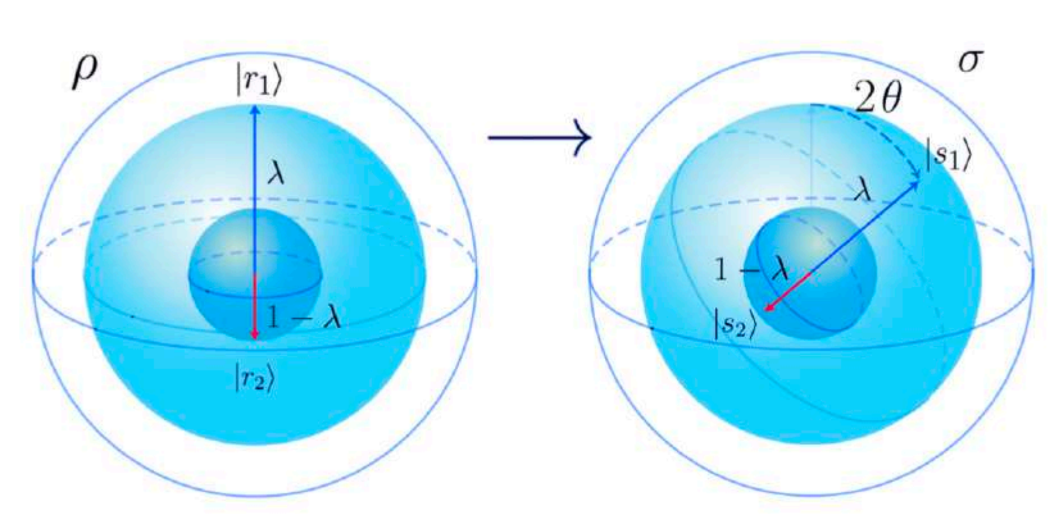
unbiased bases :

$$| < n_i | n'_j > |^2 = \frac{1}{d}$$



Naseri, M.; Macchiavello, C.; Bruß, D.; Horodecki, P.; Streltsov, A. Quantum Speed Limit for Change of Basis. 2022.

quantum speed limit for qubit in a simple case



$$\rho = \lambda|r_1 >< r_1| + (1-\lambda)|\, r_2 >< r_2|$$

$$\sigma = \lambda|s_1 >< s_1| + (1-\lambda)|\, s_2 >< s_2|$$

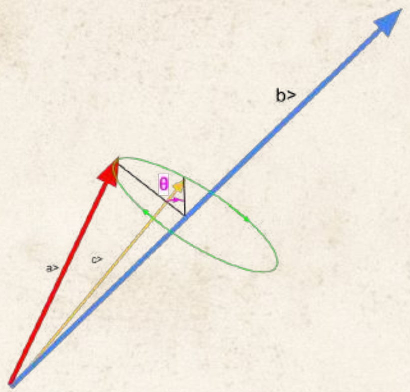
qubit system change of basis

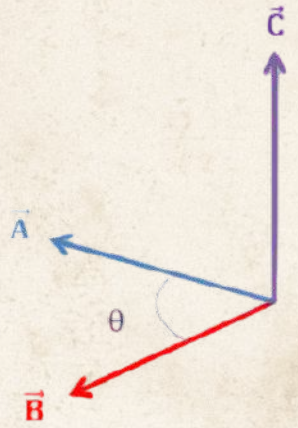
System with 1 qubits :

$$H = E_+|E_+><E_+| + E_-|E_-><E_-|$$

$$\Rightarrow |E_\pm><E_\pm| = \frac{1}{2}(\mathrm{I} \pm n\,.\,\sigma) \Rightarrow U = e^{-iGt}e^{-iEt\,n.\sigma}$$

qubit system change of basis





The unitary can be interpreted as a rotation by an angle $2Et$ about the axis n of the Bloch sphere :

$$\Rightarrow Et \geq \frac{1}{2}\cos^{-1}(r_0.r_1)$$

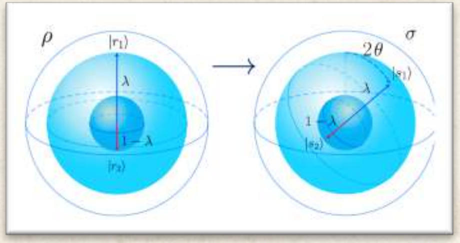
qubit system change of basis

For pure qubit states :

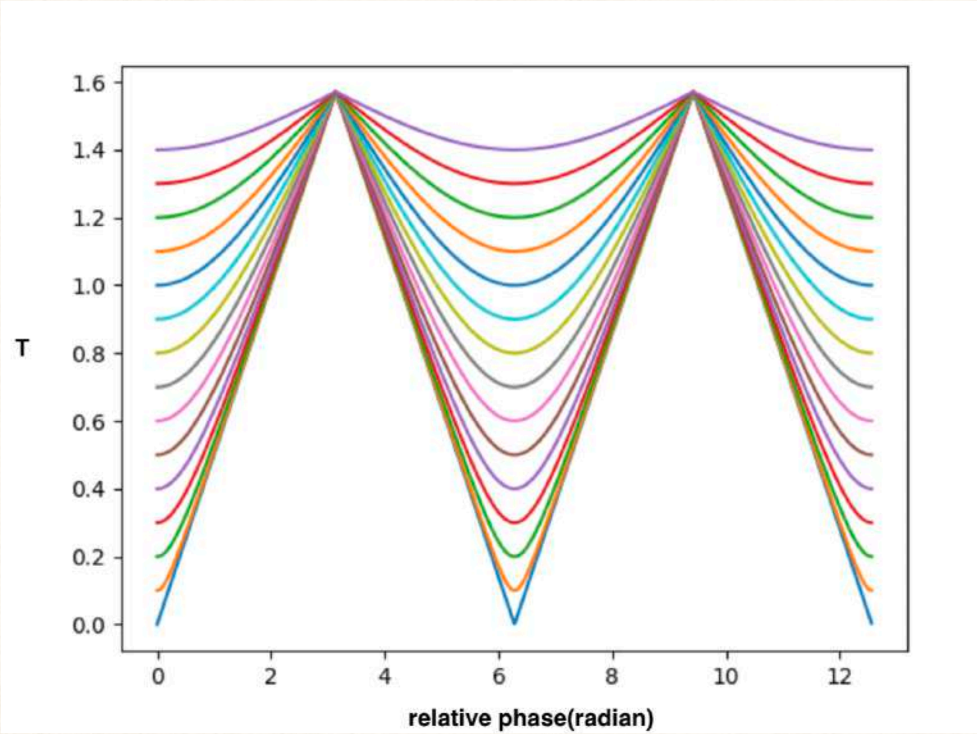
$$Et \geq \frac{1}{2}\cos^{-1}(r_0.r_1) \Rightarrow \mathrm{t} \geq \frac{1}{2E}\cos^{-1}(2\left(<\psi_i|\psi_f>\right)^2 - 1)$$

For a transition $\{|0>, |1>\}$ to an unbiased unbiased qubit basis :

$$\tau_{QSL} \geq \frac{\pi\hbar}{4 < H >}$$



Phase is important!!!



Qutrit MUB formation

$$U = \frac{\sqrt{3}}{3}\begin{pmatrix} e^{i\phi_1} & e^{i(\phi_1-\alpha)} & e^{i(\phi_1-\beta)} \\ e^{i\phi_2} & e^{i(\phi_2-\alpha-\frac{2\pi}{3})} & e^{i(\phi_2-\beta-\frac{4\pi}{3})} \\ e^{i\phi_3} & e^{i(\phi_3-\alpha-\frac{4\pi}{3})} & e^{i(\phi_3-\beta-\frac{2\pi}{3})} \end{pmatrix}$$

$$U = \frac{\sqrt{3}}{3}\begin{pmatrix} e^{i\phi_1} & e^{i(\phi_1-\alpha)} & e^{i(\phi_1-\beta)} \\ e^{i\phi_2} & e^{i(\phi_2-\alpha-\frac{4\pi}{3})} & e^{i(\phi_2-\beta-\frac{2\pi}{3})} \\ e^{i\phi_3} & e^{i(\phi_3-\alpha-\frac{2\pi}{3})} & e^{i(\phi_3-\beta-\frac{4\pi}{3})} \end{pmatrix}$$

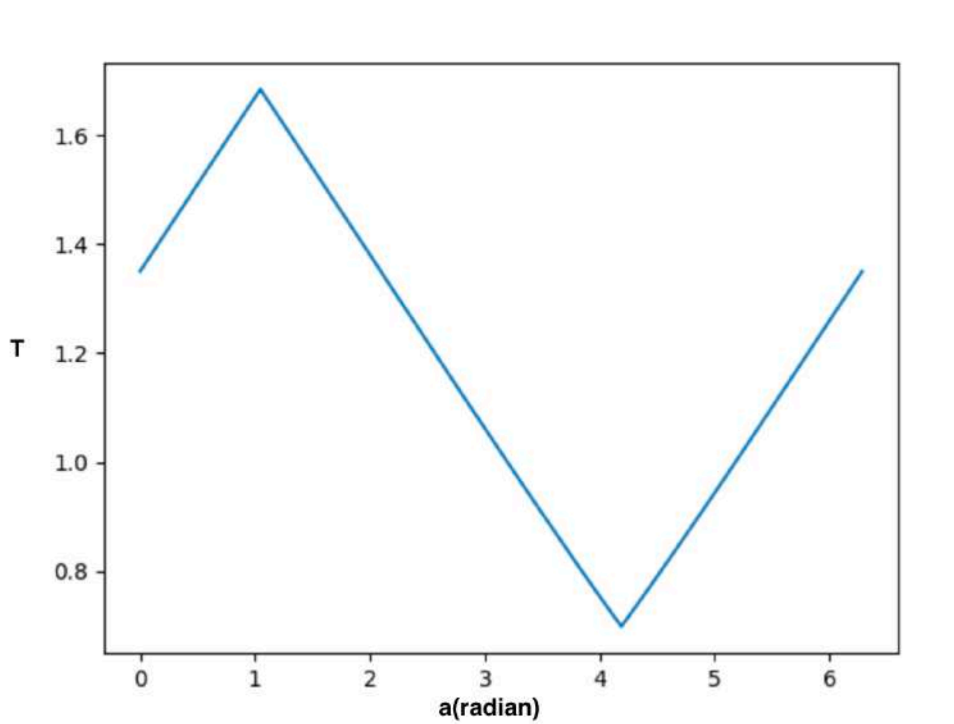
MUB Unitary Energy Class

$$\begin{pmatrix} 1 & e^{ia} & e^{ib} \\ 1 & e^{i(a-\frac{4\pi}{3})} & e^{i(b-\frac{2\pi}{3})} \\ 1 & e^{i(a-\frac{2\pi}{3})} & e^{i(b-\frac{4\pi}{3})} \end{pmatrix}$$

$$\begin{pmatrix} 1 & e^{ia} & e^{ib} \\ 1 & e^{i(a-\frac{2\pi}{3})} & e^{i(b-\frac{4\pi}{3})} \\ 1 & e^{i(a-\frac{4\pi}{3})} & e^{i(b-\frac{2\pi}{3})} \end{pmatrix}$$

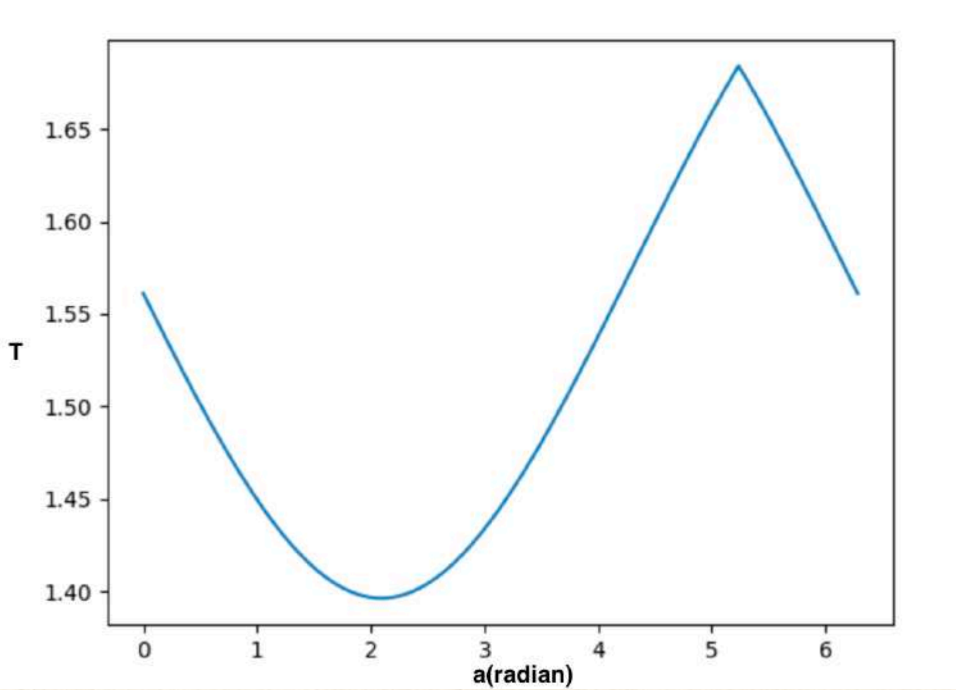
Minimums in MUB Unitary Energy class

$$\begin{pmatrix} 1 & e^{ia} & e^{ib} \\ 1 & e^{i(a-\frac{4\pi}{3})} & e^{i(b-\frac{2\pi}{3})} \\ 1 & e^{i(a-\frac{2\pi}{3})} & e^{i(b-\frac{4\pi}{3})} \end{pmatrix}$$



Minimums in MUB Unitary Energy class

$$\begin{pmatrix} 1 & e^{ia} & e^{ib} \\ 1 & e^{i(a-\frac{2\pi}{3})} & e^{i(b-\frac{4\pi}{3})} \\ 1 & e^{i(a-\frac{4\pi}{3})} & e^{i(b-\frac{2\pi}{3})} \end{pmatrix}$$



MUB Unitary Minimum Energy class

$$|0^+ >= e^{i\phi}(|0 > +e^{i(a-\frac{2\pi}{3})}|1 > +e^{i(b-\frac{2\pi}{3})}|2 >)$$

$$|1^+ >= e^{i\phi}(e^{-ia}|0 > +|1 > +e^{i(b-a-\frac{4\pi}{3})}|2 >)$$

$$|2^+ >= e^{i\phi}(e^{-ib}|0 > +e^{i(a-b-\frac{4\pi}{3})}|1 > +|2 >)$$

$\Rightarrow$ $T = \dfrac{2\pi}{9}$

$$|0^+ >= e^{i\phi}(|0 > +e^{i(a-\frac{4\pi}{3})}|1 > +e^{i(b-\frac{4\pi}{3})}|2 >)$$

$$|1^+ >= e^{i\phi}(e^{-ia}|0 > +|1 > +e^{i(b-a-\frac{2\pi}{3})}|2 >)$$

$$|2^+ >= e^{i\phi}(e^{-ib}|0 > +e^{i(a-b-\frac{2\pi}{3})}|1 > +|2 >)$$

$\Rightarrow$ $T = \dfrac{4\pi}{9}$

systems with d dimension

For system with d dimension :

$$\tau_{QSL} \geq \frac{\pi(d-1)\hbar}{4 < H > d}$$

thank you for your attention