به نام خدا

A simple construction of Entanglement Witnesses for arbitrary and different dimensions

Vahid Jannesary

Quick Overview on Entanglement

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

OCTOBER 15, 1935 PHYSICAL REVIEW VOLUME 48

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

N. BOHR, *Institute for Theoretical Physics, University, Copenhagen*

(Received July 13, 1935)

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†

Department of Physics, University of Wisconsin, Madison, Wisconsin

(*Received* 4 *November* 1964)

CHSH inequality

$$|\overline{A \otimes (C + D) + B \otimes (C - D)}| \leq 2$$

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Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model

Reinhard F. Werner*
Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland
(Received 1 May 1989)

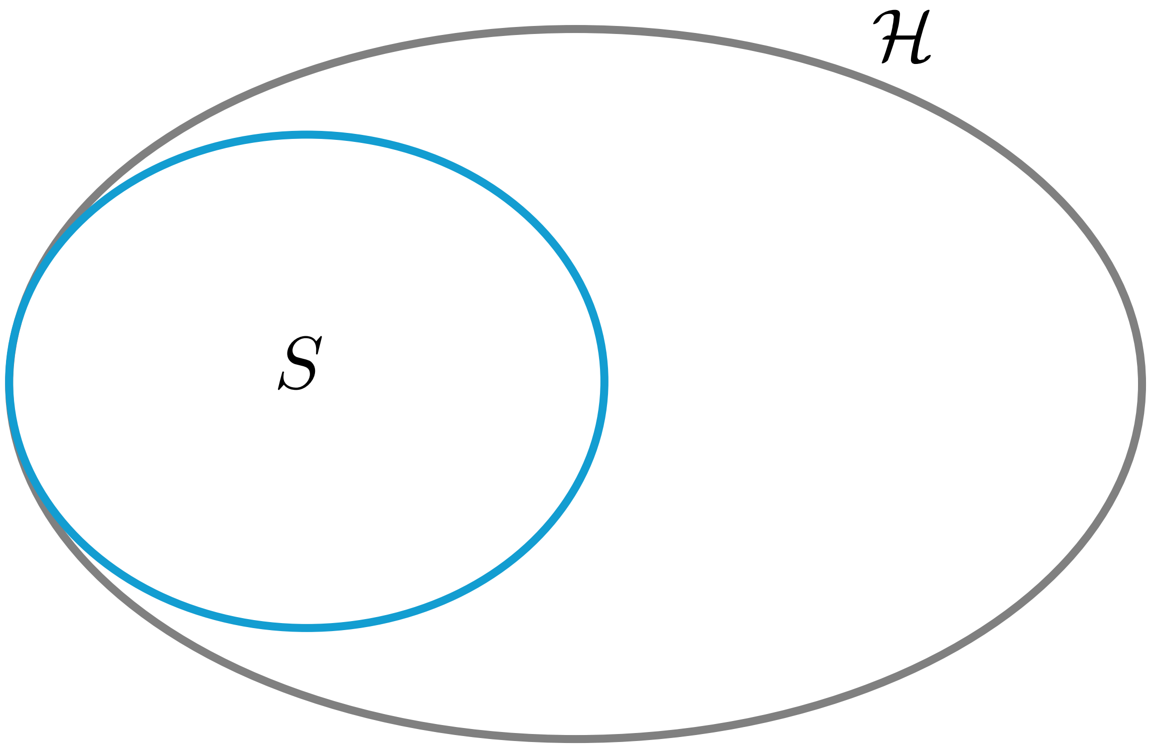
Classically Correlated state:

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

There exist some non-Classically Correlated states that CHSH inequality can not detect them!

Separable States

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$



Peres-Horodecki Criterion (1996)

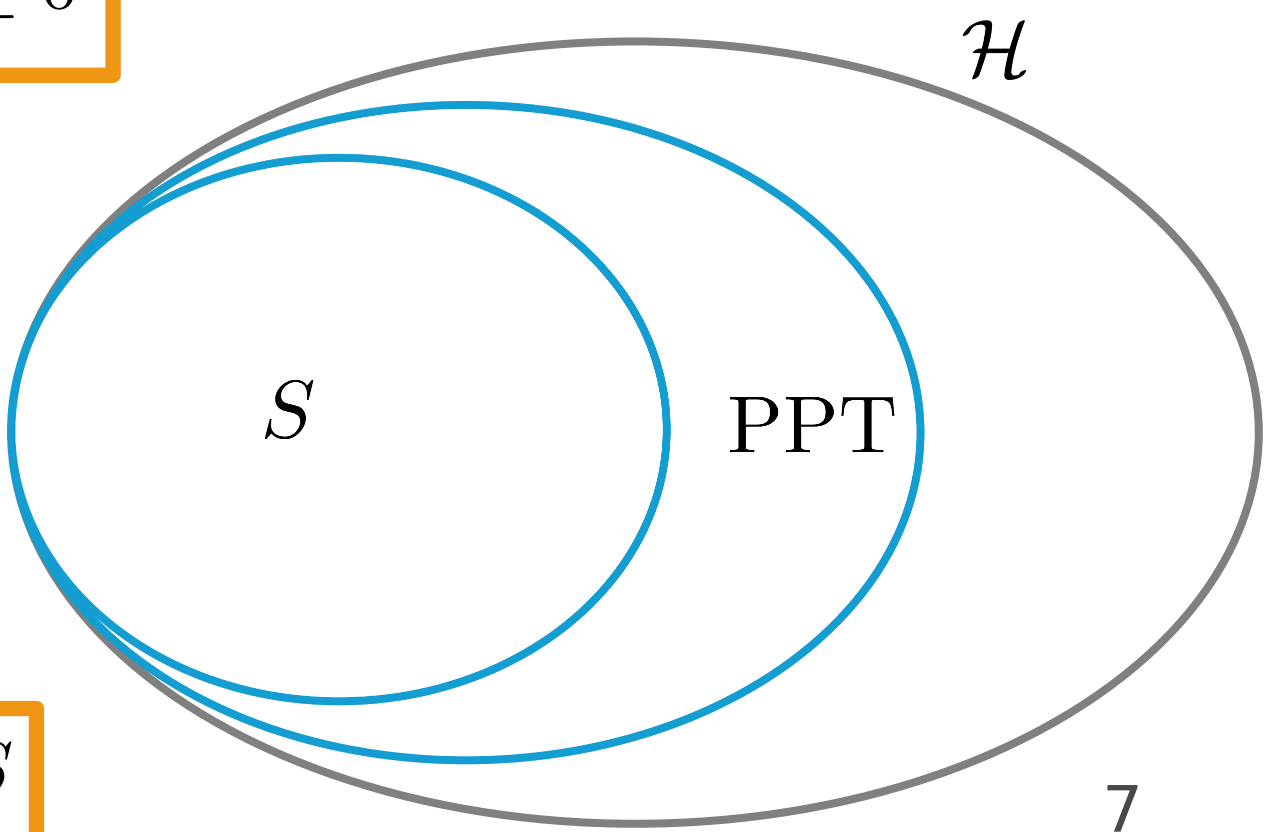
$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \geq 0$$

$$\rho^{\Gamma} = \sum p_i |\psi_i\rangle\langle\psi_i| \otimes \left(|\phi_i\rangle\langle\phi_i|\right)^{T} \geq 0$$

If $\rho^{\Gamma} \ngeq 0 \rightarrow \rho\ is\ Entangled$

PPT state

Are PPT states equal to Separable states?

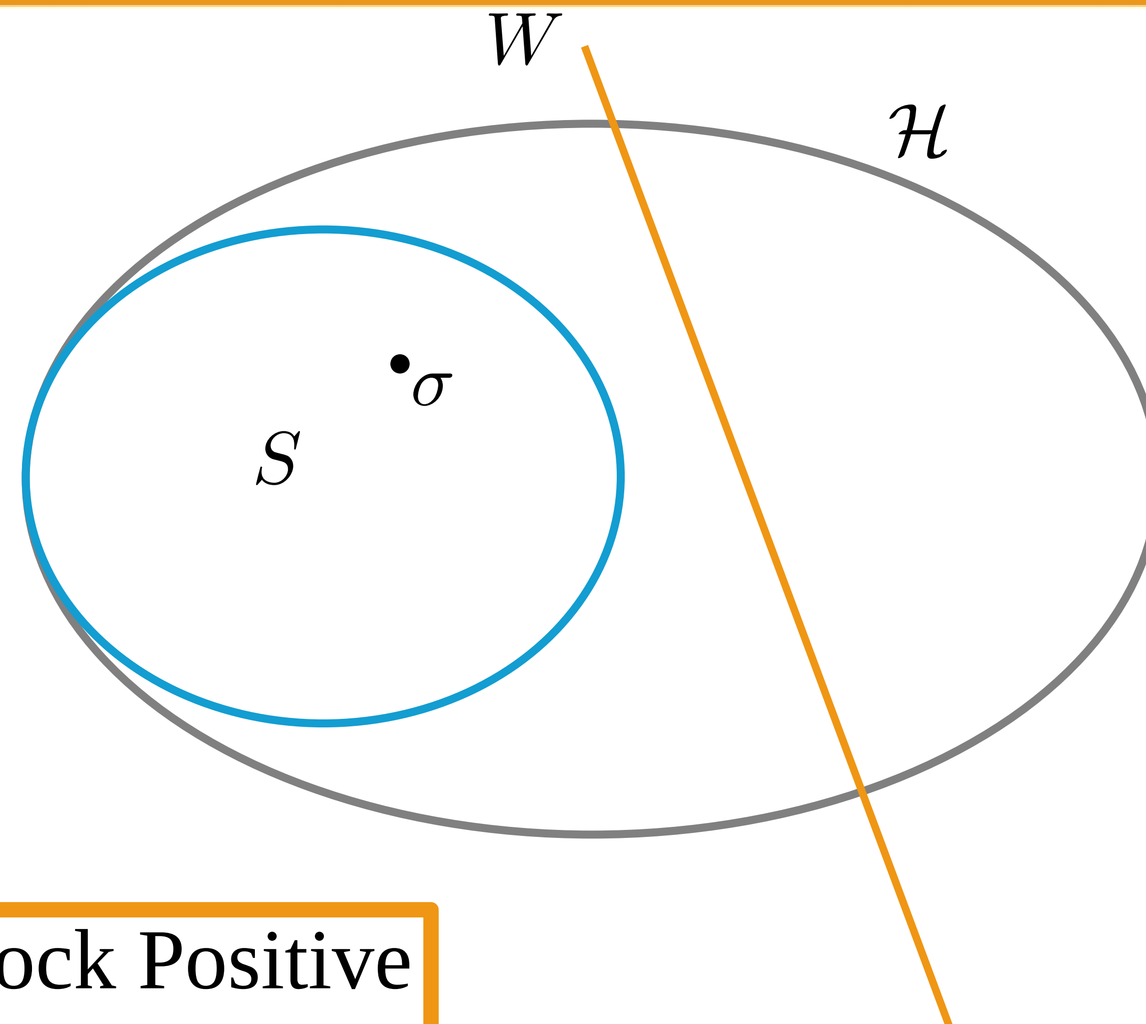


for $D = d_1 \times d_2 \leq 6 : \ PPT = S$

Entanglement Witness

$$Tr(\sigma W) \geq 0, \forall \sigma \in S$$

if $Tr(\rho W) < 0 \to \rho$ is entangled.



$W \in L(H_{AB})$ is EW ⟷ (Iff) → Block Positive → Not Positive

Separability of mixed states: necessary and sufficient conditions, Horodecki family, 1996

Our Method to Construct EW

Overview

There exist a relation between positive maps and EWs

↓

If we can build positive maps, we can build EWs

↓

Is it simple to build Positive maps?

From Separability criteria to EWs

$\rho \in D(H_{AB})$ is Separable

Iff

$[I \otimes \Lambda]\rho \geq 0$

$\forall \Lambda \geq 0 : L(H_A) \to L(H_B)$

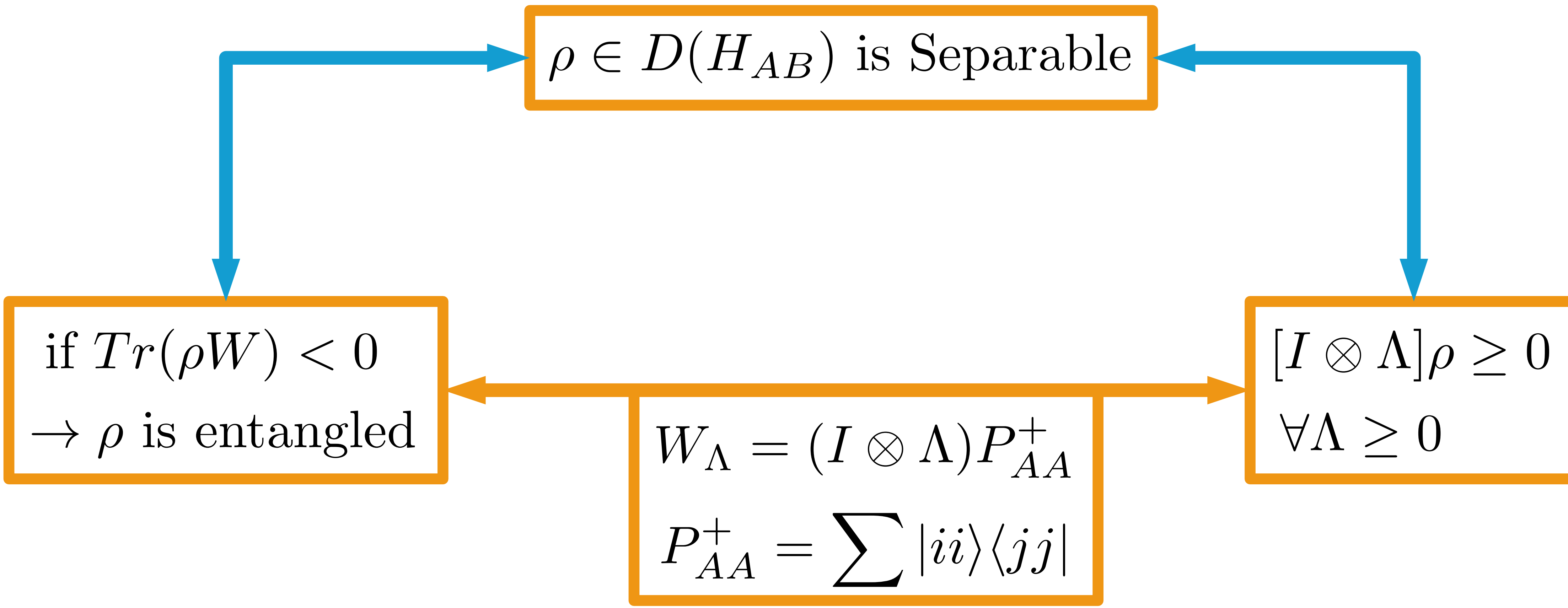
Example

$\Lambda = T :$ If $(I \otimes T)\rho = \rho^{\Gamma} \ngeq 0 \implies \rho$ is Entangled

Peres Criteria

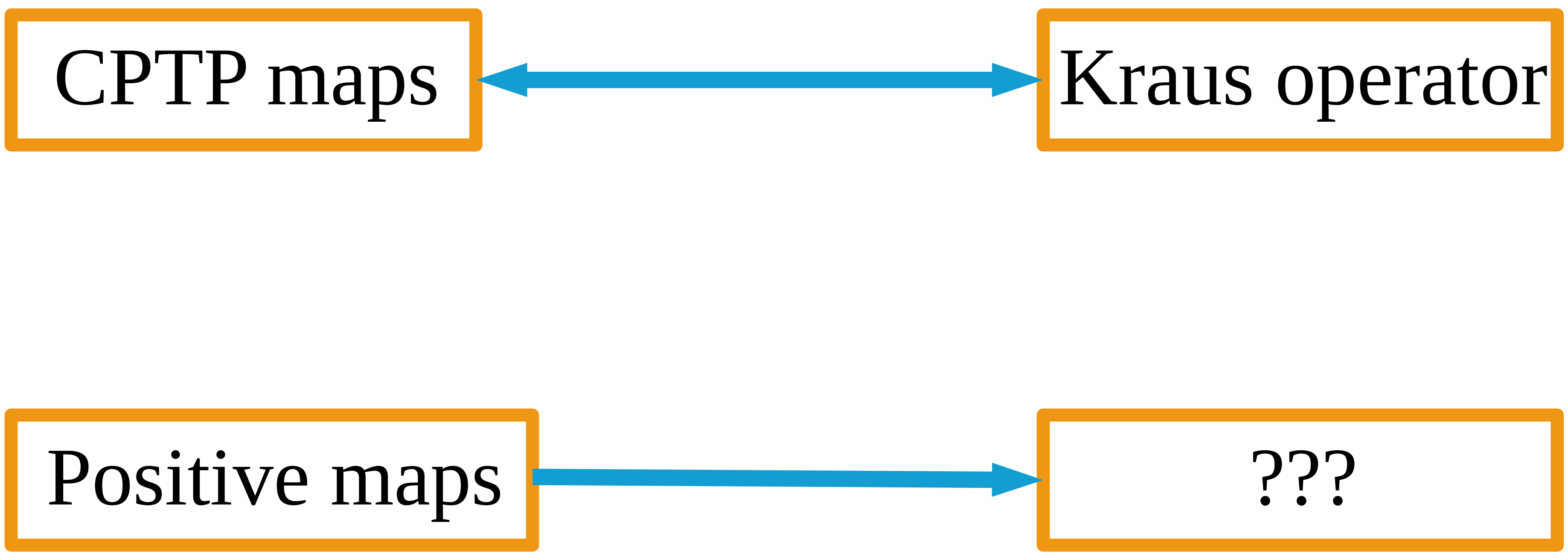
Separability of mixed states: necessary and sufficient conditions, Horodecki family, 1996

Relation between EW and Positive Maps



Separability of mixed states: necessary and sufficient conditions, Horodecki family, 1996

How to build Positive maps



Mehta's Lemma

A is positive, if

$$\mathrm{Tr}(A^2) \leq \frac{(\mathrm{Tr}A)^2}{D-1}$$

D : Dimension of A

Mehta's Lemma, proof:

$$A \text{ is Positive, if } \mathrm{Tr}(A^2) \leq \frac{(\mathrm{Tr} A)^2}{D-1}$$

Spectrum of $A : (x_1, ... x_D)$

Assume $x_1 < 0$

Using $||A||_1 \leq \sqrt{D}||A||_2$

$$\mathrm{Tr}(A) = \sum_{x=1}^{D} x_i \leq \sum_{x=2}^{D} x_i \leq \sqrt{D-1}\left(\sum_{x=2}^{D} x_i^2\right)^{1/2} < \sqrt{D-1}\sqrt{\mathrm{Tr}(A^2)}$$

Using Mehta's Lemma to construct Positive maps

$$\mathrm{Tr}(A^2) \leq \frac{(\mathrm{Tr} A)^2}{D-1} \qquad A = \Phi(X),\ X = |\psi\rangle\langle\psi|$$

$$\mathrm{Tr}\left(\Phi(X)^2\right) \leq \frac{\left(\mathrm{Tr}\Phi(X)\right)^2}{D-1}$$

$$\Phi(X) \geq 0,\ \forall X = |\psi\rangle\langle\psi|$$

$$\Phi(\rho) = \Phi\left(\sum p_i |\psi_i\rangle\langle\psi_i|\right) = \sum p_i \Phi\left(|\psi_i\rangle\langle\psi_i|\right) \geq 0$$

Constructing Positive maps

$\{\Gamma_i\}$: Orthogonal, Traceless matrices

$$\mathrm{Tr}(\Gamma_i \Gamma_j) = \delta_{i,j}, \quad \Gamma_i = 0$$

X$\in D(H_{d_1})$: Pure state

$$X = \frac{1}{d_1}\mathcal{I}_{d_1} + \sum_{i=1}^{d_1^2-1} x_i \Gamma_i$$

$\Phi : D(H_{d_1}) \rightarrow D(H_{d_2})$: Unital and trace-preserving map:

$$\Phi(\frac{1}{d_1}\mathcal{I}_{d_1}) = \frac{1}{d_2}\mathcal{I}_{d_2} \ , \Phi(\Gamma_i) = \frac{1}{d_2}\sum_{j=1}^{d_2^2-1} \Lambda_{ik}\Omega_k$$

$$X = \frac{1}{d_1}\mathcal{I}_{d_1} + \sum_{i=1}^{d_1^2-1} x_i\Gamma_i, \quad \mathrm{Tr}(\Gamma_i\Gamma_j) = \delta_{i,j}, \quad \Gamma_i = 0.$$

$$\Phi(\frac{1}{d_1}\mathcal{I}_{d_1}) = \frac{1}{d_2}\mathcal{I}_{d_2}\ , \Phi(\Gamma_i) = \frac{1}{d_2}\sum_{j=1}^{d_2^2-1} \Lambda_{ik}\Omega_k$$

$$\Phi(X) = \frac{1}{d_2}\mathcal{I}_{d_2} + \frac{1}{d_2}\sum_{k=1}^{d_2^2-1} x_i\Lambda_{ik}\Omega_k$$

$$\Phi(X) = \frac{1}{d_2}\mathcal{I}_{d_2} + \frac{1}{d_2}\sum_{k=1}^{d_2^2-1} x_i \Lambda_{ik}\Omega_k$$

$$Tr(\Phi(X)^2) \leq \frac{(Tr\Phi(X))^2}{d_2 - 1} = \frac{1}{d_2 - 1}$$

$$\frac{1}{d_2} + \frac{1}{d_2^2}\mathbf{x}^T \Lambda\Lambda^T \mathbf{x} \leq \frac{1}{d_2 - 1}, \forall \mathbf{x}$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_{d_2^2-1})$$

$$\frac{1}{d_2} + \frac{1}{d_2^2}\lambda_{max}(\Lambda\Lambda^T)\mathbf{x}\cdot\mathbf{x} \leq \frac{1}{d_2 - 1}$$

Final Conclusion

Let $\Phi : D(H_{d_1}) \to D(H_{d_2})$ be unital and trace-preserving map:

$$\Phi(\frac{1}{d_1}\mathcal{I}_{d_1}) = \frac{1}{d_2}\mathcal{I}_{d_2} \ , \Phi(\Gamma_i) = \frac{1}{d_2}\sum_{j=1}^{d_2^2-1} \Lambda_{ik}\Omega_k$$

Then it is positive if: $\lambda_{max} \leq \dfrac{d_1 d_2}{(d_1-1)(d_2-1)}$.

where $\lambda_{max} := \lambda_{max}(\Lambda\Lambda^T)$, is the largest eigenvalue of the matrix $\Lambda\Lambda^T$.

The corresponding EW is: $W = \mathcal{I}_{d_1} \otimes \mathcal{I}_{d_2} + \sum_{i,k} \Lambda_{i,k}\Gamma_i \otimes \Omega_k$.

Another Criteria

$$W = I_{d_1} \otimes I_{d_2} + \sum_{i=1}^{d_1^2-1} \sum_{j=1}^{d_2^2-1} \Lambda_{ij} \Gamma_i \otimes \Omega_j .$$

$$Tr(W^2) \leq \frac{(TrW)^2}{d_2 - 1} \longrightarrow W \text{ is positive}$$

$$Tr(W^2) > \frac{Tr(W)^2}{d_1 d_2 - 1} \Longrightarrow \boxed{\frac{d_1 d_2}{d_1 d_2 - 1} < Tr(\Lambda^T \Lambda)}$$

Construction of EW: An example

$$W = P^{\Gamma} = (|\psi\rangle\langle\psi|)^{\Gamma} = \sum a_{\mu\nu}\sigma_{\mu} \otimes \sigma_{\nu}$$ ← Extremal Entanglement Witness

$$M = \{\sigma_X \otimes \sigma_X, \sigma_Y \otimes \sigma_Y, \sigma_Z \otimes \sigma_Z\}$$

$$W = \alpha 1 + \sum a_k 1 \otimes \sigma_k + b_k \sigma_k \otimes 1 + \sum c_k \sigma_k \otimes \sigma_k$$

Optimal entanglement witnesses from limited local measurements, Chruściński et al, 2019

Construction of EW: An example

$$|\varphi\rangle = e^{i\xi_0} m|00> + e^{i\xi_1} n|01> + e^{i\xi_2} q|10> + t|11>$$

$$W = \alpha 1 + \sum a_k 1 \otimes \sigma_k + b_k \sigma_k \otimes 1 + \sum c_k \sigma_k \otimes \sigma_k = |\varphi\rangle\langle\varphi|^\Gamma$$

$$|\varphi_1\rangle = a|\phi^+\rangle + b|\phi^-\rangle; \;\; |\varphi_2\rangle = a|\psi^+\rangle + b|\psi^-\rangle$$

$$|\varphi_3\rangle = a|\phi^+\rangle + b|\psi^+\rangle; \;\; |\varphi_4\rangle = a|\phi^-\rangle + b|\psi^-\rangle$$

$$|\varphi_5\rangle = a|\phi^+\rangle + ib|\psi^-\rangle; \;\; |\varphi_6\rangle = a|\phi^-\rangle + ib|\psi^+\rangle$$

Optimal entanglement witnesses from limited local measurements, Chruściński et al, 2019

Example 1: qubit-qutrit system

$$M = \{\sigma_1 \otimes J_1, \sigma_2 \otimes J_2, \sigma_3 \otimes J_3\}$$

$$W = \mathcal{I}_2 \otimes \mathcal{I}_3 + a\sigma_1 \otimes J_1 + b\sigma_2 \otimes J_2 + c\sigma_3 \otimes J_3$$

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 1: qubit-qutrit system

$$W = \mathcal{I}_2 \otimes \mathcal{I}_3 + a\sigma_1 \otimes J_1 + b\sigma_2 \otimes J_2 + c\sigma_3 \otimes J_3$$

$$W = \frac{1}{2}\begin{pmatrix} 2 & -ic & 0 & 0 & 0 & b \\ ic & 2 & 0 & 0 & 0 & -ia \\ 0 & 0 & 2 & -b & ia & 0 \\ 0 & 0 & -b & 2 & ic & 0 \\ 0 & 0 & -ia & -ic & 2 & 0 \\ b & ia & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\frac{6}{5} < a^2 + b^2 + c^2$$

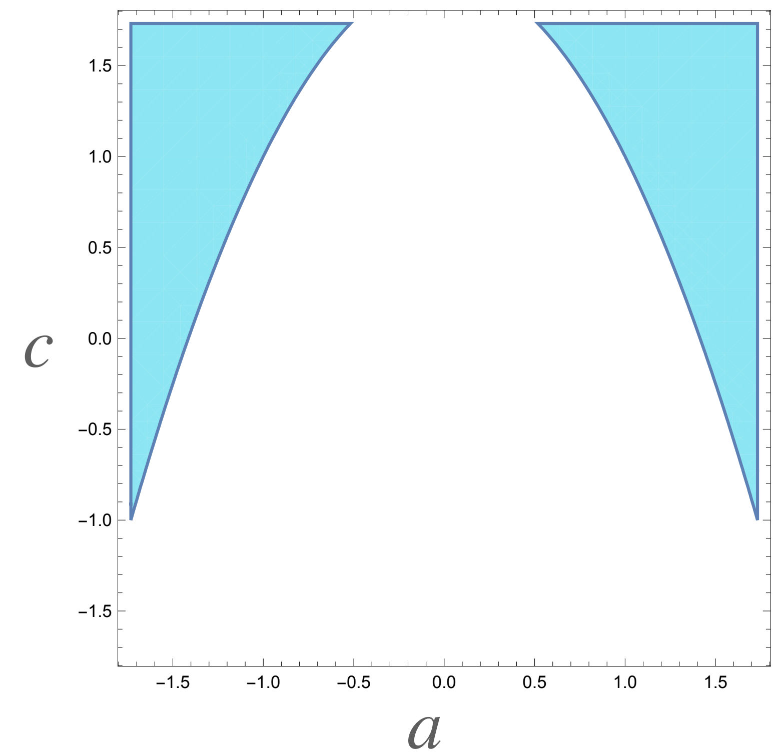
$$a,\ b,\ c < \sqrt{3}$$

Example 1: qubit-qutrit system

Case 1, a=b=c: $\lambda_- = 2(1-a)$ $1 < a \leq \sqrt{3}$

Case 2, c=0: $\lambda_- = 1 - \frac{\sqrt{a^2+b^2}}{2}$ $a^2 + b^2 > 4$

Case 3, a=b: $\lambda_- = \frac{4 - c - \sqrt{8a^2+c^2}}{4}$



Example 2: qubit-qutrit system

$$M = \{\sigma_3 \otimes \Gamma_i\}$$

$$W = I_2 \otimes I_3 + \sigma_3 \otimes \sum_{i=1}^{8} c_i \Gamma_i$$

$$\frac{6}{5} < \sum_{i=1}^{8} c_i^2 \leq 3$$

$$|\lambda_{max}(\sigma_3 \otimes \Gamma)| \leq 1$$

Thanks for your attention